1 Area under a curve

**Theorem 1.1** \[ \sum_{i=1}^{n} c = cn \]

2. \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

3. \[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

**Proof.** The proof is by induction. We show that the result holds for a base case, and then given that the result holds for \( n \) that it must hold for \( n + 1 \).

Consider equation 2. Let \( n = 1 \), then we have

\[
\sum_{i=1}^{1} i = \frac{1(1 + 1)}{2} = 1.
\]

Now, assume the equation is valid for \( n \) and show that this implies it is true for \( n + 1 \).

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) = \frac{n(n+1)}{2} + (n + 1) = \frac{n(n+1) + 2(n + 1)}{2} = \frac{(n + 1)(n + 2)}{2}.
\]
Consider equation 3. Let $n = 1$, then we have
\[
\sum_{i=1}^{1} i^2 = \frac{1(1 + 1)(2(1) + 1)}{6} = 1,
\]
so the equation is valid for $n$ and show that this implies it is true for $n + 1$.
\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2
\]
\[
= \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2
\]
\[
= \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6}
\]
\[
= \frac{(n + 1)(n(2n + 1) + 6(n + 1))}{6}
\]
\[
= \frac{(n + 1)(2n^2 + 7n + 6)}{6}
\]
\[
= \frac{(n + 1)(2n + 3)(n + 2)}{6}
\]

Theorem 1.2
\[
\sum_{i=1}^{n} (ca_i + db_i) = c \sum_{i=1}^{n} a_i + d \sum_{i=1}^{n} b_i
\]
2 The definite integral

Definition 2.1 The definite integral from \( a \) to \( b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x,
\]

for any function \( f \) defined on \([a, b]\) for which the limit exists and is the same for any choice of evaluation points, \( c_1, c_2, \ldots, c_n \). When the limit exists we say that \( f \) is integrable on \([a, b]\).

Theorem 2.1 If \( f \) is continuous on the interval \([a, b]\), then \( f \) is integrable on \([a, b]\).

Theorem 2.2 if \( f \) and \( g \) are integrable on \([a, b]\) and \( c \) is any constant, then the following are true

1. \( \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \).
2. \( \int_{a}^{b} [f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \).
3. \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \).
4. \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \).
Theorem 2.3 Suppose that \( g(x) \leq f(x) \) for all \( x \in [a, b] \) and that \( f \) and \( g \) are integrable on \([a, b]\). Then,

\[
\int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx.
\]

Theorem 2.4 (Integral mean value theorem) If \( f \) is continuous on \([a, b]\), there is an number \( c \in (a, b) \) for which

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

3 Antiderivatives

Definition 3.1 Given a function \( f(x) \) the function \( F(x) \) such that \( F'(x) = f(x) \) is an antiderivative of \( f \).

Theorem 3.1 Suppose that \( F \) and \( G \) are both antiderivatives of \( f \) on an interval \([a, b]\). Then,

\[
F(x) = G(x) + c,
\]

for some constant \( c \).

Proof. Recall that only constant functions have zero derivatives. Define \( H(x) = F(x) - G(x) \). Then \( H'(x) = F'(x) - G'(x) = 0 \), so \( H \) must be constant.
Definition 3.2  Let $F$ be an antiderivative of $f$. The indefinite integral of $f(x)$ (with respect to $x$), is defined by

$$\int f(x)\,dx = F(x) + c,$$

where $c$ is an arbitrary constant (the constant of integration).

Theorem 3.2 (Power rule)  For any real number $r \neq -1$,

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c.$$

Table of integration rules:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$
- $\int \cos x\,dx = \sin x + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \csc^2 x\,dx = -\cot x + c$
- $\int e^x\,dx = e^x + c$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$
- $\int \frac{1}{x}\,dx = \ln |x| + c$

Theorem 3.3  Suppose that $f$ and $g$ have antiderivatives. Then for any constants $a$ and $b$,

$$\int [a\,f(x) + b\,g(x)]\,dx = a \int f(x)\,dx + b \int g(x)\,dx.$$
Theorem 3.4  If $\int f(x) \, dx = F(x) + c$, then for any constants $a \neq 0$,

$$\int f(ax) \, dx = \frac{1}{a} F(ax) + c.$$  

Theorem 3.5  For $x \neq 0$, $\frac{d}{dx} \ln |x| = \frac{1}{x}$.

Theorem 3.6

$$\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c.$$  

4  The Fundamental Theorem of Calculus

Theorem 4.1 (The Fundamental Theorem of Calculus, Part I)

If $f$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$  

Proof. First we partition the interval $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for $i =$
1, 2, . . . , n. Note that we can write

\[ F(b) - F(a) = F(x_n) - F(x_0) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) , \]

Since \( F \) is an antiderivative of \( f \), \( F \) is differentiable on \((a, b)\) and continuous on \([a, b]\). By the mean value theorem, we have for each \( i = 1, \ldots, n \) that

\[ F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x. \]

Therefore,

\[ F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x, \]

taking limits of both sides yields the theorem.

**Definition 4.1** We will use the notation

\[ F(x) \bigg|^{b}_{a} = F(b) - F(a). \]

**Theorem 4.2 (The Fundamental Theorem of Calculus, Part II)**

If \( f \) is continuous on \([a, b]\) and \( F(x) = \int_{a}^{x} f(t)dt \), then \( F'(x) = f(x) \) on \([a, b]\).
Proof.

\[ F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ \int_{x}^{x+h} f(t)dt - \int_{x}^{x} f(t)dt \right] \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ \int_{x}^{x+h} f(t)dt \right]. \]

We use the integral mean value theorem:

\[ \frac{1}{h} \int_{x}^{x+h} f(t)dt = f(c) \]

for some \( c \) between \( x \) and \( x + h \). As \( h \to 0 \), \( c \to x \) and since \( f(x) \) is continuous, \( f(c) \to f(x) \), as required.

Remark 4.1 The general form of the chain rule when \( F(x) = f(u(x)) \) is:

\[ F'(x) = f'(u(x))u'(x). \]

Therefore, if \( F(x) = \int_{a}^{u(x)} f(t)dt \), then

\[ \frac{d}{dx} \int_{a}^{u(x)} f(t)dt = f(u(x))u'(x). \]
5 Integration by Substitution

\[ \frac{d}{dx}[F(u)] = F'(u) \frac{du}{dx} = f(u(x)) \frac{du}{dx}. \]

This implies

\[ \int f(u) \frac{du}{dx} \, dx = \int \frac{d}{dx}[F(u)] \, dx = F(u) + c = \int f(u) \, du. \]

**Theorem 5.1** For any continuous function, \( f \)

\[ \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c, \]

provided \( f(x) \neq 0 \).

Substitution in a definite integral:

\[ \int_{a}^{b} f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du. \]


6 Integration by Parts

Recall the product rule

\[ \frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \]

Integrating both sides of the above equation yields:

\[ \int \frac{d}{dx} [f(x)g(x)] \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx. \]

Evaluating the integral on the left-hand side and rearranging terms yields

\[ \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx. \]

We summarize the above in the following integration rule

\[ \int u dv = uv - \int v du. \]