1 Area under a curve

Theorem 1.1 1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Proof. The proof is by induction. We show that the result holds for a base case, and then given that the result holds for n that it must hold for $n+1$.

Consider equation 2. Let $n = 1$, then we have

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1.$$

Now, assume the equation is valid for n and show that this implies it is true for $n+1$.

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Consider equation 3. Let $n = 1$, then we have

$$\sum_{i=1}^{1} i^2 \quad = \quad \frac{1(1+1)(2(1)+1)}{6} = 1,$$

so the equation is valid for n and show that this implies it is true for $n+1$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(2n+3)(n+2)}{6}
\end{aligned}$$

Theorem 1.2

$$\sum_{i=1}^{n} (ca_i + db_i) \quad = \quad c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$$

2 The definite integral

Definition 2.1 *The* **definite integral** *from a to b is*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

for any function f defined on $[a, b]$ for which the limit exists and is the same for any choice of evaluation points, $c_1, c_2, \ldots, c_n$. When the limit exists we say that f is **integrable** *on $[a, b]$.*

Theorem 2.1 *If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.*

Theorem 2.2 *if f and g are integrable on $[a, b]$ and c is any constant, then the following are true*

1. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
2. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Theorem 2.3 *Suppose that $g(x) \leq f(x)$ for all $x \in [a, b]$ and that f and g are integrable on $[a, b]$. Then,*

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx.$$

Theorem 2.4 (Integral mean value theorem) *If f is continuous on $[a, b]$, there is an number $c \in (a, b)$ for which*

$$f(c) \quad = \quad \frac{1}{b-a}\int_a^b f(x)dx.$$

3 Antiderivatives

Definition 3.1 *Given a function $f(x)$ the function $F(x)$ such that $F'(x) = f(x)$ is an* **antiderivative** *of f.*

Theorem 3.1 *Suppose that F and G are both antiderivatives of f on an interval $[a, b]$. Then,*

$$F(x) \quad = \quad G(x) + c,$$

for some constant c.

Proof. Recall that only constant functions have zero derivatives. Define $H(x) = F(x) - G(x)$. Then $H'(x) = F'(x) - G'(x) = 0$, so H must be constant.

Definition 3.2 *Let F be an antiderivative of f. The* **indefinite integral** *of $f(x)$ (with respect to x), is defined by*

$$\int f(x)dx \quad = \quad F(x) + c,$$

where c is an arbitrary constant (the **constant of integration***).*

Theorem 3.2 (Power rule) *For any real number $r \neq -1$,*

$$\int x^r dx \quad = \quad \frac{x^{r+1}}{r+1} + c.$$

Table of integration rules:

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$	$\int \cos x dx = \sin x + c$
$\int \sin x dx = -\cos x + c$	$\int \sec x \tan x dx = \sec x$
$\int \sec^2 x dx = \tan x + c$	$\int \csc x \cot x dx = -\csc x$
$\int \csc^2 x dx = -\cot x + c$	$\int e^{-x} dx = -e^{-x} + c$
$\int e^x dx = e^x + c$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x +$
$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$	$\int \frac{1}{\lvert x \rvert \sqrt{x^2-1}} dx = \sec^{-1} x$

Theorem 3.3 *Suppose that f and g have antiderivatives. Then for any constants a and b,*

$$\int [af(x) + bg(x)]dx \quad = \quad a \int f(x)dx + b \int g(x)dx.$$

Theorem 3.4 *If $\int f(x)dx = F(x) + c$, then for any constants $a \neq 0$,*

$$\int f(ax)dx = \frac{1}{a}F(ax) + c.$$

Theorem 3.5 *For $x \neq 0$, $\frac{d}{dx}\ln|x| = \frac{1}{x}$.*

Theorem 3.6

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c.$$

4 The Fundamental Theorem of Calculus

Theorem 4.1 (The Fundamental Theorem of Calculus, Part I) *If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. First we partition the interval $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for $i =$

$1, 2, \ldots, n$. Note that we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})), \end{aligned}$$

Since F is an antiderivative of f, F is differentiable on (a, b) and continuous on $[a, b]$. By the mean value theorem, we have for each $i = 1, \ldots, n$ that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x.$$

Therefore,

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x,$$

taking limits of both sides yields the theorem.

Definition 4.1 *We will use the notation*

$$F(x) \Big|_a^b = F(b) - F(a).$$

Theorem 4.2 (The Fundamental Theorem of Calculus, Part II) *If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$ on $[a, b]$.*

Proof.

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt] \\
&= \lim_{h\to 0} \frac{1}{h}[\int_x^{x+h} f(t)dt].
\end{aligned}
$$

We use the integral mean value theorem:

$$
\frac{1}{h}\int_x^{x+h} f(t)dt = f(c)
$$

for some c between x and $x+h$. As $h \to 0$, $c \to x$ and since $f(x)$ is continuous, $f(c) \to f(x)$, as required.

Remark 4.1 *The general form of the chain rule when $F(x) = f(u(x))$ is:*

$$
F'(x) = f'(u(x))u'(x).
$$

Therefore, if $F(x) = \int_a^{u(x)} f(t)dt$, then

$$
\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x).
$$

5 Integration by Substitution

$$\frac{d}{dx}[F(u)] \quad = \quad F'(u)\frac{du}{dx} = f(u(x))\frac{du}{dx}.$$

This implies

$$\int f(u)\frac{du}{dx}dx = \int \frac{d}{dx}[F(u)]dx$$
$$= F(u) + c = \int f(u)du.$$

Theorem 5.1 *For any continuous function, f*

$$\int \frac{f'(x)}{f(x)}dx \quad = \quad \ln|f(x)| + c,$$

provided $f(x) \neq 0$.

Substitution in a definite integral:

$$\int_a^b f(u(x))u'(x)dx \quad = \quad \int_{u(a)}^{u(b)} f(u)du.$$

6 Integration by Parts

Recall the product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides of the above equation yields:

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$

Evaluating the integral on the left-hand side and rearranging terms yields

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

We summarize the above in the following integration rule

$$\int udv = uv - \int vdu.$$