Board Problems from 2/7/06

1.

```
> Int(cos(exp(3*x))*exp(3*x),x=0..Pi);
```

$$\int_0^{\pi} \cos(e^{(3x)})\, e^{(3x)}\, dx$$

Let $u = e^{(3x)}$ then $du = 3\, e^{(3x)}\, dx$.

```
> 1/3*Int(cos(u),u=1..exp(3*Pi))=1/3*int(cos(u),u=1..exp(3*Pi));
```

$$\frac{1}{3}\int_1^{e^{(3\pi)}} \cos(u)\, du = -\frac{1}{3}\sin(1) + \frac{1}{3}\sin(e^{(3\pi)})$$

```
> int(cos(exp(3*x))*exp(3*x),x=0..Pi);
```

$$-\frac{1}{3}\sin(1) + \frac{1}{3}\sin(e^{(3\pi)})$$

```
> evalf(%);
```

$$0.0309553449$$

2.

```
> Int(3*x/sqrt(1-9*x^2),x=0..2);
```

$$\int_0^{2} \frac{3x}{\sqrt{1-9x^2}}\, dx$$

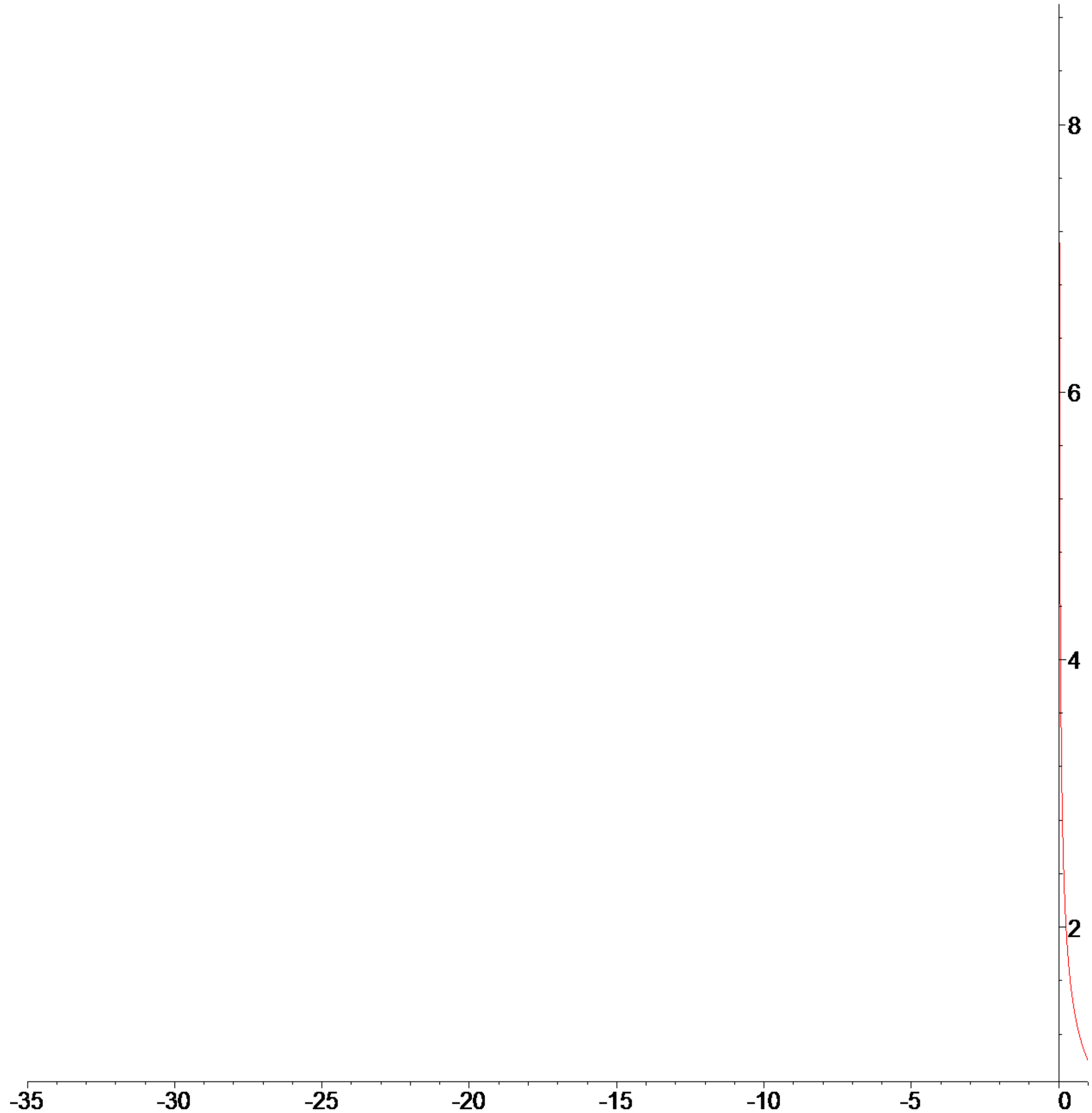
Let $u = 1 - 9x^2$ then $du = -18\, x\, dx$.

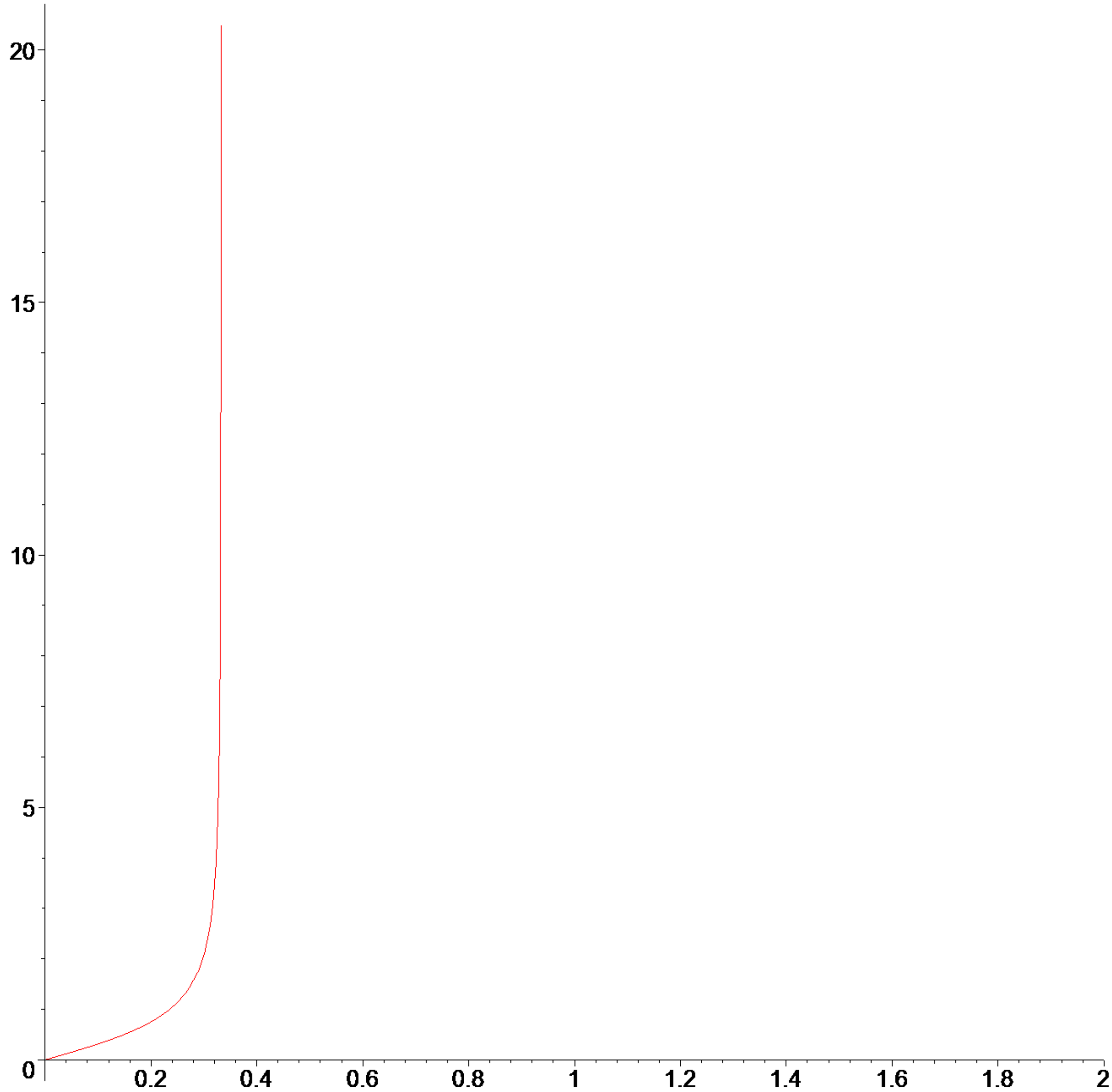
```
> -1/6*Int(1/sqrt(u),u=1..-35);
```

$$-\frac{1}{6}\int_1^{-35} \frac{1}{\sqrt{u}}\, du$$

This integral is improper. Note that when u is 0, the integrand is undefined. Some improper integrals can be evaluated by means of a limiting process that we will discuss when we cover section 4.10. This is not such an integral. Note that in the original integral when x is greater than 1/3, the expression inside the radical is negative. Similarly, when u is less than 0, the expression under the radical is zero. Below the original integrand and the integrand after a change in variables are both plotted. Maple does not plot anything for those values that are not in the domain of the function.

```
> plot(1/sqrt(u),u=1..-35);plot(3*x/sqrt(1-9*x^2),x=0..2);
```


8
6
4
2
-35
-30
-25
-20
-15
-10
-5
0



Suppose that we had been asked to evaluate the same integrand, but with different limits:

```
> Int(3*x/sqrt(1-9*x^2),x=0..1/10);
```

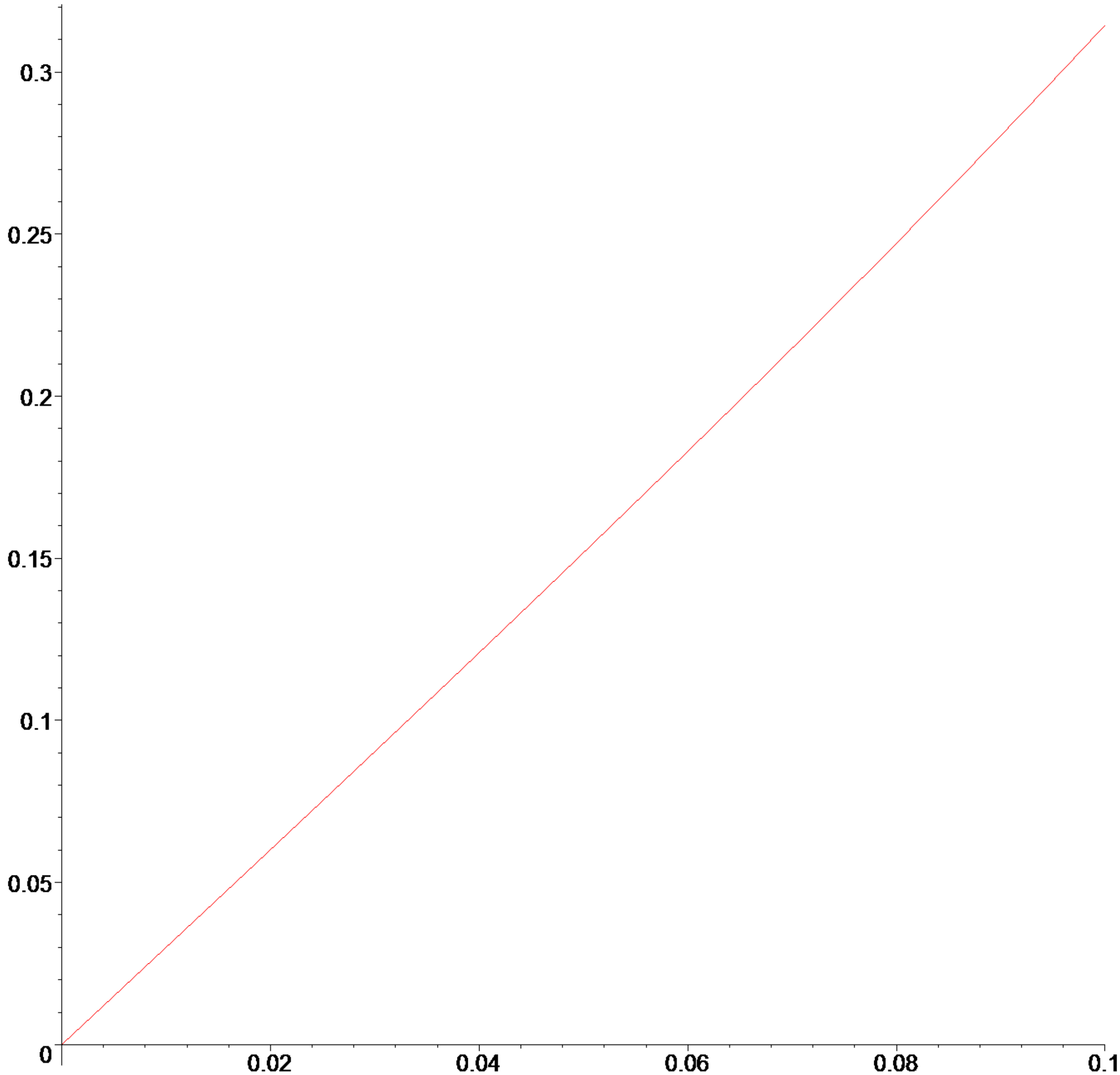
$$\int_0^{1/10} \frac{3\,x}{\sqrt{1-9\,x^2}}\,dx$$

Let $u = 1 - 9\,x^2$ then $du = -18\,x\,dx$.

```
> -1/6*Int(1/sqrt(u),u=1..91/100)=-1/3*(sqrt(91/100)-1);
```

$$-\frac{1}{6}\int_{1}^{\frac{91}{100}} \frac{1}{\sqrt{u}}\, du = \frac{1}{3} - \frac{\sqrt{91}}{30}$$

```
> plot(3*x/sqrt(1-9*x^2),x=0..1/10);
```



3.

```
> Int(ln(3*x+1)/(3*x+1),x=0..5);
```

$$\int_0^5 \frac{\ln(3x+1)}{3x+1}\,dx$$

Let $u=\ln(3x+1)$ then $du=\dfrac{3\,dx}{3x+1}$.

```
> 1/3*Int(u,u=0..ln(16))=1/6*(ln(16)^2-0^2);
```

$$\frac{1}{3}\int_0^{4\ln(2)} u\,du = \frac{8}{3}\ln(2)^2$$

```
> int(ln(3*x+1)/(3*x+1),x=0..5);
```

$$\frac{8}{3}\ln(2)^2$$

```
> evalf(%);
```

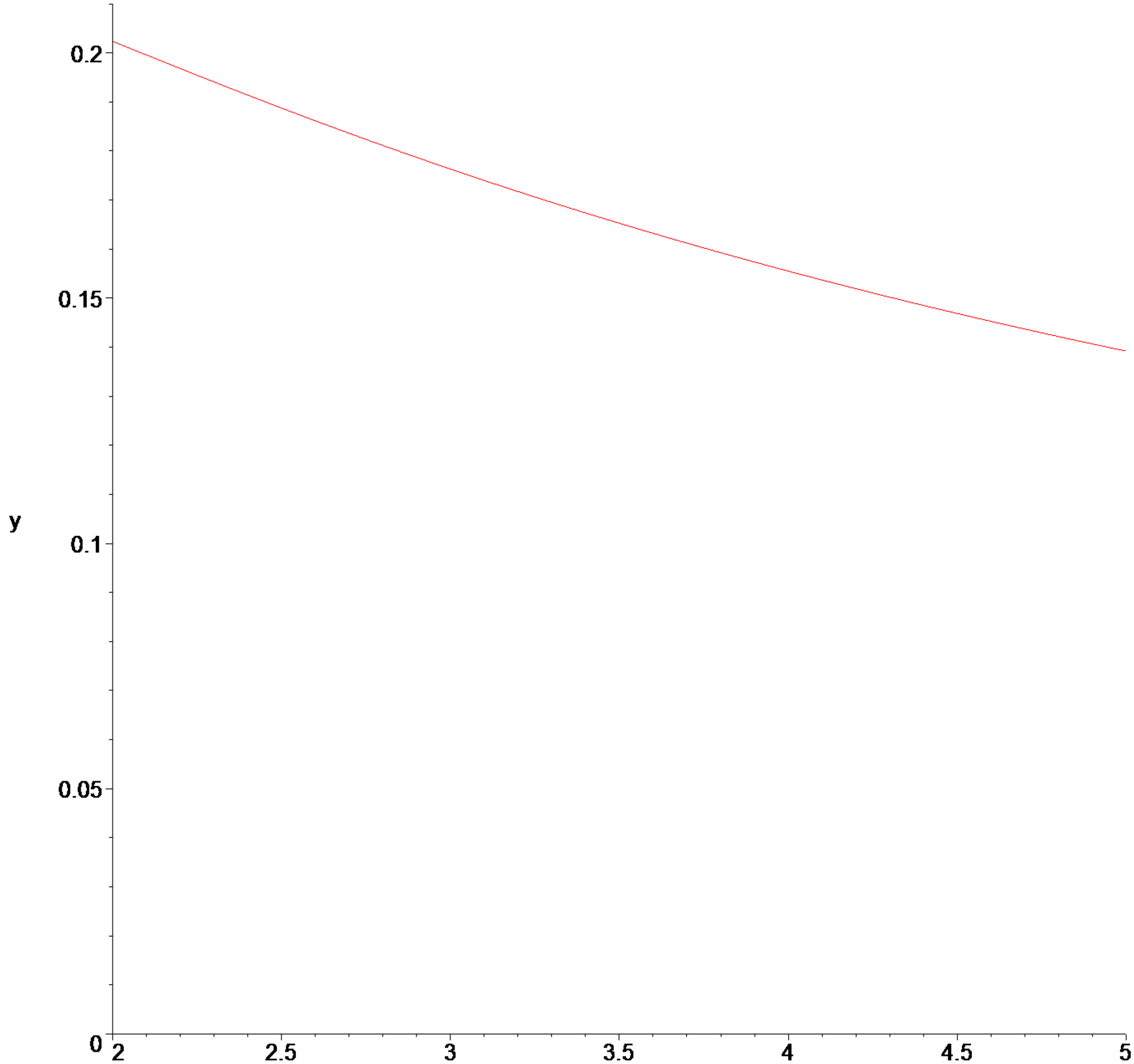
$$1.281208037$$

4.

```
> Int(ln(3*x^2+2*x+1)/(6*x+2),x=2..5);
```

$$\int_2^5 \frac{\ln(3x^2+2x+1)}{6x+2}\,dx$$

```
> plot(ln(3*x^2+2*x+1)/(6*x+2),x=2..5,y=0..0.21);(subs(x=2,ln(3*x^2+
  2*x+1)/(6*x+2))+subs(x=5,ln(3*x^2+2*x+1)/(6*x+2)))*3/2;
```



$$\frac{3}{28}\ln(17)+\frac{3}{64}\ln(86)$$

```
> evalf(%);
```

$$0.5123561020$$

Let $u = 3\,x^2 + 2\,x + 1$, then $du = (6\,x + 2)\,dx$.

```
> solve(u = 3*x^2+2*x+1,x);
```

$$-\frac{1}{3}+\frac{\sqrt{-2+3\,u}}{3},\ -\frac{1}{3}-\frac{\sqrt{-2+3\,u}}{3}$$

```
> du/(6*x+2)^2 = dx;
```

$$\frac{du}{(6x+2)^2} = dx$$

```
> subs(x=-1/3+1/3*(-2+3*u)^(1/2),%);
```

$$\frac{du}{4(-2+3u)} = dx$$

```
> Int(1/4*ln(u)/(-2+3*u),u = 17 .. 86);
```

$$\int_{17}^{86} \frac{1}{4}\frac{\ln(u)}{-2+3u}\,du$$

Let $v = -2+3u$, then $dv = 3\,du$.

```
> Int(1/12*ln(1/3*v+2/3)/v,v = 49 ..
  256)=Int(1/12*(ln(v+2)-ln(3))/v,v = 49 ..
  256);Int(1/12*(ln(v+2))/v,v = 49 .. 256)-Int(1/12*(ln(3))/v,v = 49
  .. 256)=Int(1/12*(ln(v+2))/v,v = 49 .. 256)-int(1/12*(ln(3))/v,v =
  49 .. 256);
```

$$\int_{49}^{256} \frac{1}{12}\frac{\ln\left(\frac{v}{3}+\frac{2}{3}\right)}{v}\,dv = \int_{49}^{256} \frac{1}{12}\frac{\ln(v+2)-\ln(3)}{v}\,dv$$

$$\int_{49}^{256} \frac{1}{12}\frac{\ln(v+2)}{v}\,dv - \int_{49}^{256} \frac{1}{12}\frac{\ln(3)}{v}\,dv = \int_{49}^{256} \frac{1}{12}\frac{\ln(v+2)}{v}\,dv + \frac{1}{6}\ln(7)\ln(3) - \frac{2}{3}\ln(2)\ln(3)$$

Consider the unevaluated integral: $\int_{49}^{256} \frac{1\ln(v+2)}{12\,v}\,dv$. Let $w = \frac{v+2}{2}$, then $dw = \frac{dv}{2}$.

```
> 1/12*Int(ln(2*w)/(w-1),w = 51/2 .. 129)=1/12*Int(ln(2)/(w-1),w =
  51/2 .. 129)+1/12*Int(ln(w)/(w-1),w = 51/2 ..
  129);1/12*int(ln(2)/(w-1),w = 51/2 .. 129)+1/12*Int(ln(w)/(w-1),w
  = 51/2 .. 129)=1/12*int(ln(2)/(w-1),w = 51/2 ..
  129)+1/12*int(ln(w)/(w-1),w = 51/2 .. 129);
```

$$\frac{1}{12}\int_{51/2}^{129} \frac{\ln(2w)}{w-1}\,dw = \frac{1}{12}\int_{51/2}^{129} \frac{\ln(2)}{w-1}\,dw + \frac{1}{12}\int_{51/2}^{129} \frac{\ln(w)}{w-1}\,dw$$

$$-\frac{1}{6}\ln(2)\ln(7) + \frac{2}{3}\ln(2)^2 + \frac{1}{12}\int_{51/2}^{129} \frac{\ln(w)}{w-1}\,dw =$$

$$-\frac{1}{6}\ln(2)\ln(7)+\frac{2}{3}\ln(2)^2+\frac{1}{12}\operatorname{dilog}\left(\frac{51}{2}\right)-\frac{1}{12}\operatorname{dilog}(129)$$

where the dilog function is $\operatorname{dilog}(t)=\int_1^t \frac{\ln(s)}{1-s}\,ds$. Therefore,

```
> Int(ln(3*x^2+2*x+1)/(6*x+2),x=2..5)=1/6*ln(7)*ln(3)-2/3*ln(2)*ln(3
)-1/6*ln(2)*ln(7)+2/3*ln(2)^2+1/12*dilog(51/2)-1/12*dilog(129);
```

$$\int_2^5 \frac{\ln(3x^2+2x+1)}{6x+2}\,dx =$$

$$\frac{1}{6}\ln(7)\ln(3)-\frac{2}{3}\ln(2)\ln(3)-\frac{1}{6}\ln(2)\ln(7)+\frac{2}{3}\ln(2)^2+\frac{1}{12}\operatorname{dilog}\left(\frac{51}{2}\right)-\frac{1}{12}\operatorname{dilog}(129)$$

```
> evalf(%);
```

$$0.5014646064 = 0.5014646056$$

```
> Digits:=10;
```

$$Digits := 10$$

5.

```
> Int(x*exp(3*x),x=0..2);
```

$$\int_0^2 x\,\mathbf{e}^{(3x)}\,dx$$

Let $u=x$, $dv=\mathbf{e}^{(3x)}\,dx$ then $du=dx$ and $v=\frac{\mathbf{e}^{(3x)}}{3}$.

```
> subs(x=2,x*exp(3*x)/3)-subs(x=0,x*exp(3*x)/3)-Int(exp(3*x)/3,x=0..
2);
```

$$\frac{2}{3}\mathbf{e}^6-\int_0^2 \frac{1}{3}\mathbf{e}^{(3x)}\,dx$$

```
> 2/3*exp(6)-subs(x=2,1/9*exp(3*x))+subs(x=0,1/9*exp(3*x));
```

$$\frac{5}{9}\mathbf{e}^6+\frac{1}{9}\mathbf{e}^0$$

```
> simplify(%);
```

$$\frac{1}{9}+\frac{5}{9}\mathbf{e}^6$$

6.

```
> Int(x*cos(x^2),x=Pi/2..3*Pi/2);
```

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos(x^2)\,dx$$

Let $u = x^2$ then $du = 2\,x\,dx$.

```
> 1/2*Int(cos(u),u=Pi^2/4..9*Pi^2/4)=1/2*(sin(9*Pi^2/4)-sin(Pi^2/4))
  ;
```

$$\frac{1}{2}\int_{\frac{\pi^2}{4}}^{\frac{9\pi^2}{4}} \cos(u)\,du = \frac{1}{2}\sin\left(\frac{9\pi^2}{4}\right) - \frac{1}{2}\sin\left(\frac{\pi^2}{4}\right)$$

```
> evalf(%);
```

$$-0.4190320320 = -0.4190320354$$

```
>
>
```