

# GRAPH PRODUCTS OF SPHERES, ASSOCIATIVE GRADED ALGEBRAS AND HILBERT SERIES

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ABSTRACT. Given a finite, simple, vertex-weighted graph, we construct a graded associative (non-commutative) algebra, whose generators correspond to vertices and whose ideal of relations has generators that are graded commutators corresponding to edges. We show that the Hilbert series of this algebra is the inverse of the clique polynomial of the graph. Using this result it is easy to recognize if the ideal is *inert*, from which strong results on the algebra follow. Noncommutative Gröbner bases play an important role in our proof.

There is an interesting application to toric topology. This algebra arises naturally from a partial product of spheres, which is a special case of a generalized moment-angle complex. We apply our result to the loop-space homology of this space.

## 1. INTRODUCTION

This paper connects ideas from algebra and algebraic topology and tries to provide sufficient background to be accessible to both audiences.

Let  $\Gamma$  be a finite simple graph with vertices  $V$  and edges  $E$ , in which each vertex  $i \in V$  is labeled with a positive integer  $p_i$ , called the *weight* of the vertex  $i$ . For  $j \in E$ , let  $a_j, b_j$  denote its endpoints. Let  $c_{i,j}$  be the number of complete subgraphs of  $\Gamma$  with  $i$  vertices whose weights sum to  $j$ . Call

$$c_\Gamma(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j$$

the *clique polynomial* of the weighted graph  $\Gamma$ . It is a polynomial because  $\Gamma$  is finite. Let  $\mathbf{k}$  be a field with characteristic not equal to 2. Let  $[a, b]$  denote the graded commutator  $ab - (-1)^{|a||b|}ba$ . Our main result, which is closely related to a similar result of Cartier and Foata[11], is the following.

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**Theorem 1.1.** *The associative (noncommutative) graded algebra*

$$A(\Gamma) = \mathbf{k}\langle x_i, i \in V \rangle / I(\Gamma),$$

where  $I(\Gamma)$  is the two-sided ideal  $([x_{a_j}, x_{b_j}], j \in E)$ , with  $|x_i| = p_i$  has Hilbert series

$$H_{A(\Gamma)}(z) = [c_\Gamma(z)]^{-1}.$$

We remark that even though  $[x_{a_j}, x_{b_j}]$  may depend on the ordering of the endpoints of the edge  $j \in E$ , the ideal  $I(\Gamma)$  does not.

From this theorem we obtain the following corollary. For  $j \in E$ , let  $q_j = p_{a_j} + p_{b_j}$ . For associative algebras,  $A$  and  $B$ , we write  $A \amalg B$  for their free product. Let  $B(\Gamma)$  denote the subalgebra of the free associative algebra  $\mathbf{k}\langle x_i, i \in V \rangle$  that is generated by  $\{[x_{a_j}, x_{b_j}], j \in E\}$ .

**Corollary 1.2.** *The following are equivalent.*

- (1) *The graph  $\Gamma$  does not contain a triangle, i.e. a 3-cycle.*
- (2)  $\mathbf{k}\langle x_1, \dots, x_n \rangle \cong B(\Gamma) \amalg A(\Gamma)$  (as vector spaces),
- (3)  $H_{A(\Gamma)}(z) = \frac{1}{1 - (z^{p_1} + \dots + z^{p_{|V|}}) + (z^{q_1} + \dots + z^{q_{|E|}})}$ , and
- (4)  $A(\Gamma)$  has global dimension  $\leq 2$ .

If these equivalent conditions are satisfied, we say that the two-sided ideal,  $I(\Gamma)$ , is *inert*. Equivalently, we say that the set  $\{[x_{a_j}, x_{b_j}], j \in E\}$  is inert, or that  $A(\Gamma)$  has a finite inert presentation.

Inert sets play a central role for non-commutative algebra analogous to regular sequences in commutative ring theory. This analogy is made precise by Anick<sup>1</sup> [4]. In general, most problems for finitely presented associative algebras are unsolvable. For example, the word problem is undecidable [30]. However, if the set of relations is inert, then the situation is as simple as it can be, and quite a lot can be said. Unfortunately, it can be very hard to determine whether or not a given ideal is inert. Anick [4] gives some sufficient conditions, but they do not apply to most of the examples considered here. A simple non-trivial example is  $A(\Gamma)$ , where  $\Gamma$  is the pentagon.

This paper gives a large class of associative algebras,  $A(\Gamma)$ , with finite presentations, for which we can easily check inertness. We analyze the case where  $\Gamma$  is the pentagon in detail. Another example, which is easily seen to be inert from its clique polynomial, is  $A(\Gamma)$ , where  $\Gamma$  is the one-skeleton of the dodecahedron. The authors know of no other way to ascertain this fact. Furthermore, we obtain a large class of finitely presented algebras whose presentations are not inert, but for which we

<sup>1</sup>In [4], Anick uses “strongly free” in place of “inert”.

can nevertheless easily calculate their Hilbert series. In addition, we show that these algebraic results have topological versions.

The algebra  $A(\Gamma)$  also arises from a “graph product” of spheres. This is an example of a generalized moment–angle complex, which we now describe.

Let  $K$  be a simplicial complex with vertices  $\{1, \dots, n\}$ . Let  $\underline{X}$  be the collection

$$\underline{X} = \{(X_i, A_i)\}_{i=1}^n,$$

where  $A_i \subseteq X_i$  is a pair of topological spaces.

For each face  $\sigma \in K$ , define

$$X^\sigma = \left\{ (x_1, \dots, x_n) \in \prod_{i=1}^n X_i \mid x_i \in A_i \text{ if } i \notin \sigma \right\} = \prod_{i=1}^n Y_i,$$

$$\text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Then the *generalized moment–angle complex* is given by

$$\underline{X}^K = \cup_{\sigma \in K} X^\sigma.$$

In the case in which  $(X_i, A_i) = (D^2, S^1)$  for all  $i$ , one obtains the moment–angle complex defined by Davis and Januszkiewicz [12], and studied in detail by Buchstaber and Panov [10], Notbohm and Ray [27] and Grbić and Theriault [18]. Davis and Januszkiewicz showed that every smooth projective toric variety is the quotient of a moment–angle complex by the free action of a real torus. The above generalization was defined by Strickland [29] and has been studied by Denham and Suciu [13], Bahri, Bendersky, Cohen and Gitler [1] and Félix and Tanré [16]. This construction is also called a *partial product*, or a *polyhedral product*. The notation  $Z(K, \underline{X})$  is also used for  $\underline{X}^K$ .

We are interested in the case where  $\underline{X}$  is a collection of pointed spheres:

$$\underline{X} = \{(S^{p_i+1}, *)\}_{i=1}^n,$$

where  $*$  is the one–point space with fixed inclusion  $* \hookrightarrow S^{p_i+1}$ , and  $K$  is one–dimensional. That is,  $K$  is a simple graph  $\Gamma$  with  $n$  vertices.

Let  $E = \{1, \dots, m\}$  be the set of edges of  $\Gamma$ . For  $j \in E$ , let  $a_j, b_j$  be the vertices that  $j$  connects. Let  $q_j = p_{a_j} + p_{b_j}$ . Let  $\alpha_j : S^{q_j+1} \rightarrow S^{p_{a_j}+1} \vee S^{p_{b_j}+1}$  denote the top cell attachment of  $S^{p_{a_j}+1} \times S^{p_{b_j}+1}$  (given by the Whitehead product  $[\iota_{a_j}, \iota_{b_j}]_W$  where  $\iota_i$  is the identity map on  $S^{p_i+1}$ ). Then  $\underline{X}^\Gamma$  is given by adjoining  $m$  cells to a wedge of  $n$  spheres:

$$\underline{X}^\Gamma \cong (\vee_{i=1}^n S^{p_i+1}) \cup_f (\vee_{j=1}^m D^{q_j+2}), \text{ where } f = \vee_{j=1}^m \alpha_j.$$

A space with such a construction is called a spherical two–cone. Understanding spaces constructed using such attaching maps is a nontrivial problem first studied by J.H.C. Whitehead [32]. More recent work includes that by Halperin and Lemaire [23, 19], Anick [3], Félix and Thomas [17], and Bubenik [8, 9].

Let  $Y = \underline{X}^\Gamma$ , where  $\underline{X} = \{(S^{p_i+1}, *)\}_{i=1}^n$ . Let  $W = \underline{X}^{\Gamma_0} = \bigvee_{i=1}^n S^{p_i+1}$  and let  $i : W \hookrightarrow Y$  denote the inclusion. Let  $Z = \bigvee_{j=1}^m S^{q_j+1}$ . So  $Y = W \cup_f CZ$ , where  $f = \bigvee_{j=1}^m \alpha_j : Z \rightarrow W$  denotes the attaching map and  $CZ$  denotes the cone on  $Z$ . Let  $B(f)$  denote the image of  $H_*(\Omega f; \mathbf{k}) : H_*(\Omega Z; \mathbf{k}) \rightarrow H_*(\Omega W; \mathbf{k})$ . Let  $I(f)$  denote the 2–sided ideal generated by  $B(f)$ . Let  $A(f) = H_*(\Omega W; \mathbf{k})/I(f)$ ; that is, we have  $A(f) \cong \mathbf{k}\langle x_i, i \in V \rangle / ([x_{a_j}, x_{b_j}], j \in E)$ .

**Theorem 1.3.** *The following are equivalent:*

- (1) *the graph  $\Gamma$  does not contain a 3–cycle,*
- (2)  *$H_*(\Omega i; \mathbf{k}) : H_*(\Omega W; \mathbf{k}) \rightarrow H_*(\Omega Y; \mathbf{k})$  is a surjection,*
- (3)  *$H_*(\Omega Y; \mathbf{k}) \cong A(f)$  (as vector spaces),*
- (4)  *$H_*(\Omega W; \mathbf{k}) \cong B(f) \amalg A(f)$ ,*
- (5)  *$A(f)$  has global dimension  $\leq 2$ , and*
- (6) 
$$H_{A(f)}(z) = \frac{1}{1 - (z^{p_1} + \cdots + z^{p_n}) + (z^{q_1} + \cdots + z^{q_m})}.$$

If these equivalent conditions are satisfied we say that the attaching map  $f$  is *inert*. From these it follows that,

- (4')  *$H_*(\Omega W; \mathbf{k}) \cong B(f) \amalg H_*(\Omega Y; \mathbf{k})$  (as vector spaces),*
- (5')  *$H_*(\Omega Y; \mathbf{k})$  has global dimension  $\leq 2$ , and*
- (6')  *$H_*(\Omega Y; \mathbf{k})$  has Hilbert series  $\frac{1}{1 - (z^{p_1} + \cdots + z^{p_n}) + (z^{q_1} + \cdots + z^{q_m})}$ .*

We have constructed two isomorphic algebras,  $A(\Gamma)$  and  $A(f)$ . Henceforth we denote them by  $A$ . Anick [5] constructed a free graded  $A$ –resolution of  $\mathbf{k}$ . It follows that homology groups  $\text{Tor}_p^A(\mathbf{k}, \mathbf{k})$  are also graded. Let  $\text{Tor}_{p,q}^A(\mathbf{k}, \mathbf{k})$  denote the component of  $\text{Tor}_p^A(\mathbf{k}, \mathbf{k})$  in grading  $q$ . Since  $A$  is finitely generated, it is evident from Anick’s resolution that  $\text{Tor}_{p,q}^A(\mathbf{k}, \mathbf{k})$  is finite dimensional. Thus one can define the *double Poincaré series* of  $A$ ,

$$P_A(y, z) = \sum_{p, q \geq 0} \dim(\text{Tor}_{p,q}^A(\mathbf{k}, \mathbf{k})) y^p z^q.$$

Anick [5, Equation (13)] uses  $\text{Tor}_p^A(\mathbf{k}, \mathbf{k})$  to construct a minimal free  $A$ –resolution of  $\mathbf{k}$ , and calculates that

$$H_A(z) = [P_A(-1, z)]^{-1}.$$

We thus have the following corollary.

**Corollary 1.4.**  $P_A(-1, z) = c_\Gamma(z)$ .

Since this paper contains results that may of interest to both algebraists and topologists, we provide sufficient background for both audiences. We provide some background in Section 2, and show that Corollary 1.2 follows from Theorem 1.1 using results of Anick’s on inert sets. Our proof of Theorem 1.1 uses the theory of noncommutative Gröbner bases, which we introduce in Section 3, and a classical result of the theory of free partially commutative monoids [11], which we recall in Section 4. In Section 5 we prove a generalization of Theorem 1.1, in which we assign weights to both the vertices and the edges of the graph. In Section 6 we apply our results to graph products of spheres using Adams–Hilton models, obtaining Theorem 1.3.

## 2. BACKGROUND

**2.1. Monoids and formal power series.** Let  $\langle x_1, \dots, x_n \rangle$  denote the monoid generated by  $\{x_1, x_2, \dots, x_n\}$  with the empty word 1 as unit.

**Definition 2.1.** A *formal power series* of a monoid  $M$  is an element of the ring,  $\mathbb{Z}\langle\langle M \rangle\rangle$ , of functions (of sets)  $M \rightarrow \mathbb{Z}$ . A formal power series,  $f$ , is often denoted by  $\sum_{w \in M} f(w)w$ .

For example, a formal power series on  $\langle z \rangle$ ,  $f(k) = a_k$ , is denoted  $\sum_{k=0}^{\infty} a_k z^k$ . The set  $\mathbb{Z}\langle\langle M \rangle\rangle$  has the structure of a ring under the usual addition and the non-commutative Cauchy product.

$$(f + g)(w) = f(w) + g(w)$$

$$(f \cdot g)(w) = \sum_{uv=w} f(u)g(v)$$

It has unit given by  $\mathbf{1}(1) = 1$  and  $\mathbf{1}(w) = 0$  if  $w$  is nonempty. A formal power series  $f \in \mathbb{Z}\langle\langle M \rangle\rangle$  is invertible if and only if  $f(1)$  is invertible in  $\mathbb{Z}$ . If  $M = \langle x_1, \dots, x_n \rangle$ , we will write  $\mathbb{Z}\langle\langle M \rangle\rangle = \mathbb{Z}\langle\langle x_1, \dots, x_n \rangle\rangle$ .

**2.2. Algebras and Hilbert series.** In this paper we will always assume  $\mathbf{k}$  is a field with characteristic not equal to 2. Let  $A$  be a (non-negatively) graded associative  $\mathbf{k}$ -algebra. That is,  $A = \bigoplus_{n=0}^{\infty} A_n$ , and for  $x \in A_n, y \in A_m, xy \in A_{n+m}$ . For  $x \in A_n$ , we write  $|x| = n$ . We will always assume that our algebras are non-negatively graded associative  $\mathbf{k}$ -algebras. Ideals will always be assumed to be two-sided.

**Definition 2.2.** The *Hilbert series* of  $A$  is the formal power series in  $\mathbb{Z}\langle\langle z \rangle\rangle$  given by  $H_A(z) = \sum_{i=0}^{\infty} \dim_{\mathbf{k}} A_i z^i$ . If  $\dim(A_0) = 1$ , then  $H_A(z)$  is invertible, and we let  $[H_A(z)]^{-1}$  and  $\frac{1}{H_A(z)}$  denote the inverse of  $H_A(z)$ .

**Definition 2.3.** For  $x, y \in A$ , the *graded commutator* is given by  $[x, y] = xy - (-1)^{|x||y|}yx$ .

**2.3. Graphs and cliques.** Let  $\Gamma$  be a simple graph (i.e. no double edges or loops) with a finite set  $V$  of vertices labeled in  $\mathbb{Z}_{>0}$  and a set  $E$  of edges. For each vertex  $i$  call its label  $p_i$  its *weight*. All graphs we consider are assumed to be finite simple graphs whose vertices are labeled in  $\mathbb{Z}_{>0}$ .

**Definition 2.4.** An  *$i$ -clique of weight  $j$*  of  $\Gamma$  is a complete subgraph of  $\Gamma$  on  $i$  vertices whose weights sum to  $j$ . For  $i, j \geq 0$  let  $c_{i,j}(\Gamma)$  be the number of  $i$ -cliques of weight  $j$  in  $\Gamma$ .

So for all graphs  $c_{0,0}(\Gamma) = 1$ ,  $\sum_j c_{1,j}(\Gamma) = |V|$ ,  $\sum_j c_{2,j}(\Gamma) = |E|$  and for  $i > j$ ,  $c_{i,j}(\Gamma) = 0$ . Also for  $K_n$ , the complete graph on  $n$  vertices,  $\sum_j c_{i,j}(K_n) = \binom{n}{i}$ . We make the following definition.

**Definition 2.5.** The *clique polynomial* of  $\Gamma$  is given by

$$c_\Gamma(z) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i c_{i,j}(\Gamma) z^j.$$

If for all  $i \in V$ ,  $p_i = 1$ , then  $c_{i,j}(\Gamma) = 0$  unless  $i = j$ . So  $c_\Gamma(z) = \sum_{i=0}^{\infty} (-1)^i c_{i,i}(\Gamma) z^i$  in this case.

#### 2.4. Algebras from graphs.

**Definition 2.6.** Given a finite simple weighted graph,  $\Gamma$  as above, let  $\mathbf{k}\langle x_1, \dots, x_{|V|} \rangle$  be the associative graded algebra with the degree of  $x_i$ , denoted  $|x_i|$ , equal to the weight  $p_i$ . For  $j \in E$ , let  $\{a_j, b_j\}$  denote the boundary of  $j$ . Let  $B(\Gamma)$  denote the subalgebra of  $\mathbf{k}\langle x_1, \dots, x_{|V|} \rangle$  generated by  $\{[x_{a_j}, x_{b_j}], j \in E\}$ . Let  $I(\Gamma)$  be the two-sided ideal generated by  $B(\Gamma)$ . That is,

$$I(\Gamma) = (\{[x_{a_j}, x_{b_j}] \mid j \in E\}).$$

Define the graded algebra associated to the graph  $\Gamma$  to be

$$A(\Gamma) = \mathbf{k}\langle x_1, \dots, x_{|V|} \rangle / I(\Gamma).$$

*Remark 2.7.* This can be thought of as a graded version of the graph algebras of [21, 20]. We avoid the convenient terminology “graph algebras” since it has other common usages.

Define the differential graded algebra associated to  $\Gamma$  to be

$$\text{DGA}(\Gamma) = (\mathbf{k}\langle x_1, \dots, x_{|V|}, y_1, \dots, y_{|E|} \rangle, dx_i = 0, dy_j = [x_{a_j}, x_{b_j}]),$$

where  $|x_i| = p_i$  and  $|y_j| = p_{a_j} + p_{b_j} + 1$ . The differential reduces the degree by one.

**Example 2.8.** Let  $\Gamma$  be a pentagon, that is the 5-cycle graph together with weights on its vertices. Then

$$A(\Gamma) = \mathbf{k}\langle x_1, x_2, x_3, x_4, x_5 \rangle / ([x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_5, x_1]), \text{ and}$$

$$\text{DGA}(\Gamma) = (\mathbf{k}\langle x_1, \dots, x_5, y_1, \dots, y_5 \rangle, dx_i = 0, dy_j = [x_j, x_{j+1}]),$$

where  $x_6 = x_1$ .

### 2.5. Inert ideals.

**Lemma 2.9.** *The surjection  $\phi : \mathbf{k}\langle [x_{a_j}, x_{b_j}], j \in E \rangle \rightarrow B(\Gamma)$  from the free associative algebra is a bijection.*

*Proof.* Let  $L(\Gamma)$  denote the Lie subalgebra of the free Lie algebra  $\mathbb{L}\langle x_1, \dots, x_{|V|} \rangle$  generated by  $\{[x_{a_j}, x_{b_j}], j \in E\}$ . Then there is a surjection  $\theta : \mathbb{L}\langle [x_{a_j}, x_{b_j}], j \in E \rangle \rightarrow L(\Gamma)$ . Over a field of characteristic not equal to 2, any subalgebra of a free graded Lie algebra is a free graded Lie algebra [25, Theorem 14.5]. So  $\theta$  is an isomorphism. Let  $U$  denote the universal enveloping algebra functor from graded Lie algebras, to graded associative algebras. Then  $\phi \cong U(\theta)$ . Thus  $\phi$  is also a bijection.  $\square$

**Theorem 2.10** ([4]). *The following are equivalent:*

- (1)  $H(\text{DGA}(\Gamma)) \cong A(\Gamma)$ ,
- (2)  $\mathbf{k}\langle x_1, \dots, x_n \rangle \cong B(\Gamma) \amalg A(\Gamma)$  (as vector spaces),
- (3)  $H_{A(\Gamma)}(z) = \frac{1}{1 - (z^{p_1} + \dots + z^{p_n}) + (z^{q_1} + \dots + z^{q_m})}$ , and
- (4)  $A(\Gamma)$  has global dimension  $\leq 2$ .

If these equivalent conditions are satisfied we say that the set  $\{[x_{a_j}, x_{b_j}], j \in E\}$  is *inert*. Equivalently, one says that  $I(\Gamma)$  is *inert*.

*Proof.* The result follows from combining Lemma 2.9 and several results of Anick's. By Lemma 2.9 and [4, Theorem 2.6], (2) and (3) are equivalent. By Lemma 2.9 and [4, Theorem 2.9], (1) and (2) are equivalent. By Lemma 2.9 and [4, Theorem 2.12(b)], (2) and (4) are equivalent.  $\square$

*Proof of Corollary 1.2.* By Theorem 1.1, it is enough to show that

$$(2.1) \quad 1 - (z^{p_1} + \dots + z^{p_{|V|}}) + (z^{q_1} + \dots + z^{q_{|E|}}) = c_\Gamma(z)$$

if and only if  $\Gamma$  does not contain a triangle. The remainder of Corollary 1.2 then follows from Theorem 2.10.

If there is no triangle in  $\Gamma$ , then there are no cliques with more than 2 vertices. So we have the desired equality

$$c_\Gamma(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j = 1 - \sum_{v \in V} z^{|v|} + \sum_{e \in E} z^{|e|}.$$

On the other hand suppose Equation 2.1 is true. Then

$$\sum_{i=3}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j = 0$$

So

$$(2.2) \quad \sum_{i=3}^{\infty} (-1)^i c_{i,j} = 0 \quad \text{for each } j.$$

Assume  $\Gamma$  contains a triangle. Let  $j_0$  be the minimal weight of a triangle in  $\Gamma$ . Since  $j_0$  is minimal and the weights are positive integers, there cannot be any cliques with more than 3 vertices having weight  $j_0$ , so  $c_{i,j_0} = 0$  for  $i > 3$ . But then by Equation 2.2  $c_{3,j_0} = 0$ . This contradicts the existence of a triangle of weight  $j_0$ . Therefore,  $\Gamma$  does not have any triangles.  $\square$

### 3. NONCOMMUTATIVE GRÖBNER BASES AND HILBERT SERIES

Gröbner bases provide a nice generating set for an ideal in both commutative and noncommutative polynomial rings because they allow us to compute a normal form for elements in the quotient by the ideal. As the reader may not be familiar with noncommutative Gröbner bases, we provide a fairly detailed description here. An excellent resource on noncommutative Gröbner bases is the paper by Ufnarowski [31].

Let  $R = \mathbf{k}\langle x_1, \dots, x_m \rangle$  be a noncommutative polynomial ring. To define the Gröbner basis of a 2-sided ideal of  $R$  we must first define an ordering on the monomials, or words, of  $R$ .

**Definition 3.1.** An *admissible ordering* on  $R = \mathbf{k}\langle x_1, \dots, x_m \rangle$  is a relation  $\geq$  on the words of  $R$  such that

- (1)  $\geq$  is a total ordering on the set of words of  $R$ ,
- (2) for all words  $f, g, h, k$  in  $R$ , if  $f \geq g$  and  $h \geq k$ , then  $fh \geq gk$ , and
- (3) every infinite sequence of words  $w_1 \geq w_2 \geq w_3 \geq \dots$  eventually stabilizes, i.e.  $w_i = w_j$  for all  $i, j > i_0$ .

The familiar lexicographical ordering on a commutative polynomial ring is not an admissible ordering for noncommutative polynomials

since if  $a \geq b$  then  $a \geq ba \geq b^2a \geq \dots$ . The following degree lexicographic ordering, however, is admissible.

Suppose the degree of the variable  $x_i$  is  $d_i$ . We say the *degree* of a word  $w = x_{i_1}x_{i_2} \cdots x_{i_r}$  is  $\deg(w) = d_{i_1} + d_{i_2} + \cdots + d_{i_r}$ .

**Definition 3.2.** The *degree lexicographic ordering*, called *DegLex*, is the ordering such that for words  $w, v$  in  $R$ ,  $w > v$  if

- (1)  $\deg(w) > \deg(v)$ , or
- (2)  $\deg(w) = \deg(v)$  and lexicographically  $w$  comes before  $v$ .

Notice that one has to specify an order on the variables before using DegLex order.

Let  $f = \sum_i c_i w_i$  be an element of  $R$  where  $c_i \in \mathbf{k} - \{0\}$  and  $w_i$  are words in the variables. Given an admissible ordering  $\geq$  on  $R$ , we call the largest (with respect to  $\geq$ ) of the  $w_i$  the *initial term* of  $f$  and we denote it by  $\text{in}(f)$ . For a set  $S \subset R$  we will write  $\text{in}(S)$  for the set of all initial terms of elements of  $S$ .

**Definition 3.3.** Let  $I$  be a two-sided ideal of  $A$ . A subset  $G$  of words of  $I$  is a *Gröbner basis* of  $I$  if the 2-sided ideal generated by  $\text{in}(G)$  is equal to the ideal generated by  $\text{in}(I)$ .

So for every  $f \in I$ , some subword of its initial term  $\text{in}(f)$  is the initial term of an element of  $G$ .

A major difference between the commutative and noncommutative cases is that noncommutative Gröbner bases are often infinite.

To compute Gröbner bases, we use the idea of a *rewriting rule* for the noncommutative polynomial  $f$  with initial term  $w = \text{in}(f)$ . It is the rule that sends  $w$  to  $g$  where  $f$  is proportional to  $w - g$ . We write  $w \rightarrow_f g$ . Note that if the leading coefficient of  $f$  is 1 then  $f = w - g$ .

For a polynomial  $p$  with terms containing  $w$  as subwords, applying the rewriting rule  $w \rightarrow_f g$  to  $p$  means replacing all (successive) occurrences of  $w$  by  $g$ . If the result of this (successive) replacement is  $q$ , then we write  $p \rightarrow_f q$ . Notice that  $q$  is smaller than  $p$  in the ordering.

Let  $f, g \in R$  and let  $u, v$  be their initial terms respectively. A triple of words  $(a, b, c)$  is called a *composition* of  $f$  and  $g$  if  $ab = u$  and  $bc = v$ . If the rewriting rules for  $f$  and  $g$  are  $u \rightarrow_f h$  and  $v \rightarrow_g k$  then the *result of the composition*  $(a, b, c)$  is the difference  $ak - hc$ . There may be multiple compositions for the same pair of polynomials. The trivial one (where  $b = 1$ ) always reduces to zero after rewriting.

Let  $S$  be a set of polynomials in  $R$ , and let  $f, h \in R$ . We say that  $f$  *reduces by  $S$  to  $h$*  if  $h$  can be obtained from  $f$  by applying a sequence of rewriting rules for elements of  $S$ . We write  $f \Rightarrow_S h$ .

**Example 2.8 continued.** Let  $\Gamma$  be the pentagon where each vertex has weight 1. For  $i = 1, \dots, 4$ , let  $g_i = [x_i, x_{i+1}]$  and let  $g_5 = [x_5, x_1]$ . Order  $\mathbf{k}\langle x_1, \dots, x_5 \rangle$  using DegLex order with  $x_1 > x_3 > x_5 > x_2 > x_4$  and  $\deg(x_i) = 1$ .

We have  $\text{in}(g_4) = x_5x_4$  and  $\text{in}(g_5) = x_1x_5$  and rewriting rules  $x_5x_4 \rightarrow_{g_4} -x_4x_5$  and  $x_1x_5 \rightarrow_{g_5} -x_5x_1$ . There are two compositions of  $g_5$  and  $g_4$ :  $(x_1x_5, 1, x_5x_4)$  and  $(x_1, x_5, x_4)$ . The result of the first composition is

$$\begin{aligned} (x_1x_5)(-x_4x_5) - (-x_5x_1)(x_5x_4) &= -x_1x_5x_4x_5 + x_5x_1x_5x_4 \\ &\rightarrow_{g_5} x_5x_1x_4x_5 + x_5x_1x_5x_4 \\ &\rightarrow_{g_4} x_5x_1x_4x_5 - x_5x_1x_4x_5 = 0. \end{aligned}$$

So the result reduces to zero by the set  $S = \{g_1, \dots, g_5\}$ .

The result of the second composition is  $(x_1)(-x_4x_5) - (-x_5x_1)(x_4) = -x_1x_4x_5 + x_5x_1x_4$ , which does not reduce to zero by  $S$ .

The reduction process appears to depend on the order of the rewriting rules, but if the set  $S$  is a Gröbner basis, then there is a unique minimal reduction of  $f$  by  $S$ . In fact, the converse is true as well.

**Theorem 3.4** (Bergman's Diamond Lemma, [7]). *Let  $G$  be a self-reduced set, that is, no element of  $G$  can be further reduced by  $G$ .  $G$  is a Gröbner basis if and only if the results of all possible compositions of elements of  $G$  reduce by  $G$  to zero.*

**Example 2.8 continued.** The set  $S = \{g_1, \dots, g_5\}$  in the pentagon case above is not a Gröbner basis because the result of the composition  $(x_1, x_5, x_4)$  of  $g_4$  and  $g_5$  does not reduce by  $S$  to zero. One can check, however, that

$$G = \{g_1, \dots, g_5\} \cup \{h_k\}_{k=1}^{\infty} \text{ where } h_k = x_1x_4^kx_5 + (-1)^kx_5x_1x_4^k$$

is a Gröbner basis.

Bergman's Diamond Lemma implies a noncommutative analogue of Buchberger's algorithm for finding a Gröbner basis, known as Mora's algorithm [26]:

Let  $G_0$  be a self-reduced generating set for the ideal. Create an ordered list of all the possible compositions of elements of  $G_0$ . Work through the list of compositions one at a time, and if one is found whose result does not reduce by  $G_0$  to zero, then append it to the set of generators and self-reduce to get a new generating set  $G_1$ . Adjust the list of compositions accordingly. If there is a finite Gröbner basis, then eventually the list of compositions will be empty and the final  $G_k$  will be the Gröbner basis. Otherwise there will be an infinite number of larger and larger degree compositions to consider.

Computer programs such as BERGMAN [6] compute the Gröbner basis up to a fixed degree. In the above example, we see a pattern in the new generators and hence guess the form of the Gröbner basis.

We recall a well-known, though not obvious, fact about Hilbert series which is crucial to the proof of Theorem 1.1

**Proposition 3.5.** *Let  $I$  be a 2-sided homogeneous ideal in  $R = \mathbf{k}\langle x_1, \dots, x_m \rangle$ . Then  $H_{R/I}(z) = H_{R/\text{in}(I)}(z)$ .*

*Proof.* We want to show that  $\dim_{\mathbf{k}}(R/I)_n = \dim_{\mathbf{k}}(R/\text{in}(I))_n$  for all  $n \geq 0$ . It is equivalent to show that the dimensions of  $I_n$  and  $(\text{in}(I))_n$  are the same. Suppose  $I_n$  has a vector space basis  $f_1, \dots, f_p$  of monic polynomials. Let  $w_i = \text{in}(f_i)$ . If  $w_i = w_j$  for some  $i \neq j$ , then replace  $f_i$  with  $f_i - f_j$ . Thus, we may assume that  $w_1 > w_2 > \dots > w_p$ . So the  $w_1, \dots, w_p$  are linearly independent. We claim that  $\{w_1, \dots, w_p\}$  is a basis for  $(\text{in}(I))_n$ .

Suppose  $h \in (\text{in}(I))_n$ . Then  $h = \sum_i r_i u_i \text{in}(t_i) v_i$ , where  $r_i \in \mathbf{k}$ ,  $t_i \in I$  and  $u_i, v_i$  words in  $x_1, \dots, x_n$ . Since  $\geq$  is admissible,  $u_i \text{in}(t_i) v_i = \text{in}(u_i t_i v_i)$ . Let  $s_i = u_i t_i v_i$ . Since  $s_i \in I_n$ , it follows that  $s_i$  is a linear combination of  $\{f_1, \dots, f_p\}$ . Therefore  $\text{in}(s_i) = w_j$  for some  $j \in \{1, \dots, p\}$ . Thus  $h$  is a linear combination of  $\{w_1, \dots, w_p\}$ , and so  $\{w_1, \dots, w_p\}$  is a basis for  $(\text{in}(I))_n$ .  $\square$

#### 4. PARTIALLY COMMUTATIVE MONOIDS

A *free partially commutative monoid* is a monoid of the form

$$M = \langle x_1, \dots, x_n, \rangle / \simeq_I,$$

where  $I$  is a set of pairs in  $x_1, \dots, x_n$  and  $\simeq_I$  is the congruence relation generated by setting  $ab = ba$  for all  $\{a, b\} \in I$ . We will let  $w$  denote an element of  $M$  and 1 denote the unit of  $M$ .

Then  $M$  can be represented by a finite simple graph  $\Gamma$  whose vertices correspond to the generators  $x_1, \dots, x_n$  of the monoid, and whose edges correspond to the elements of  $I$ . The monoid  $M$  is also called a trace monoid or a monoid of circuits on a graph. Mazurkiewicz [24] introduced these monoids to the study concurrent systems, where the generators of  $M$  correspond to processes and  $I$  lists processes that are independent. Surveys of the subject include [15], [14] and [22].

Cartier and Foata [11] studied the combinatorics of partially commutative monoids and proved a version of Theorem 1.1, which we will now describe.

Recall from Definition 2.1, that a *formal power series* on  $M$  is a function from  $M$  to  $\mathbb{Z}$ . For example, for  $S \subset M$ , the characteristic

function  $\chi_S$ , given by

$$(4.1) \quad \chi_S(w) = \begin{cases} 1 & \text{if } w \in S \\ 0 & \text{if } w \notin S \end{cases}$$

is a formal power series. An important special case is the characteristic function  $\chi_M$  which is the constant function 1. The set of all power series on  $M$  is a ring denoted by  $\mathbb{Z}\langle\langle M \rangle\rangle$ . The unit of this ring,  $\mathbf{1}$ , is given by  $\chi_{\{1\}}$ . An element  $f \in \mathbb{Z}\langle\langle M \rangle\rangle$  is invertible if and only if  $f(1) = \pm 1$ . For  $f \in \mathbb{Z}\langle\langle M \rangle\rangle$ ,  $f = \sum_{w \in M} f(w)\chi_{\{w\}}$ . As is customary, we will write  $f$  as  $\sum_{w \in M} f(w)w$ .

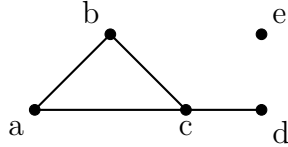
Say that  $Q = \{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \subset \{x_1, \dots, x_n\}$  is a *clique* of  $M$  if the corresponding vertices of  $\Gamma$  form a clique. Since the elements in  $Q$  commute, writing these elements in any order we obtain a representative for the same unique word  $w_Q$  in  $M$ . Therefore for each clique  $Q$ , there is a unique  $[Q] \in \mathbb{Z}\langle\langle M \rangle\rangle$  given by

$$[Q] = \chi_{\{w_Q\}},$$

Let  $\mathcal{Q}$  denote the finite set of all cliques of  $M$  (including the empty clique). Define the *clique polynomial* of  $M$  to be the formal power series

$$(4.2) \quad \mu_M = \sum_{Q \in \mathcal{Q}} (-1)^{|Q|} [Q].$$

**Example 4.1.** Let  $M$  be the monoid generated by  $a, b, c, d, e$  with relations  $ab = ba, ac = ca, bc = cb, cd = dc$ . Then the following graph represents  $M$ .



The clique polynomial of  $M$  is

$$\begin{aligned} \mu_M &= \chi_\emptyset - \chi_{\{a\}} - \chi_{\{b\}} - \chi_{\{c\}} - \chi_{\{d\}} - \chi_{\{e\}} \\ &\quad + \chi_{\{ab\}} + \chi_{\{ac\}} + \chi_{\{bc\}} + \chi_{\{cd\}} - \chi_{\{abc\}} \\ &= 1 - a - b - c - d - e + ab + ac + bc + cd - abc. \end{aligned}$$

**Theorem 4.2** (Cartier and Foata [11]). *The clique polynomial of  $M$  is the inverse of the constant power series  $\chi_M$ .*

We recall Cartier and Foata's elegant proof.

*Proof.* Let  $w \in M$ . We want to show that  $\mu_M \cdot \chi_M = \chi_{\{1\}} = \mathbf{1}$ . That is,

$\mu_M \cdot \chi_M(w) = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$ . By the definition of the Cauchy product in  $\mathbb{Z}\langle\langle M \rangle\rangle$ ,

$$\mu_M \cdot \chi_M(w) = \sum_{\substack{uv=w \\ u \in [Q], Q \in \mathcal{Q}}} (-1)^{|u|}.$$

If  $w = 1$  then  $w$  can be uniquely expanded as  $1 \cdot 1$ , where the first 1 is the word corresponding to the 0-clique. So  $\mu_M \cdot \chi_M(1) = 1$ . Assume  $w \neq 1$ . Let  $S = \{a \in \{x_1, \dots, x_n\} \mid w = aw', w' \in M\}$ . Then  $S \in \mathcal{Q}$ . Let  $m = |S|$ . Therefore,

$$\mu_M \cdot \chi_M(w) = \sum_{T \subset S} (-1)^{|T|} = \sum_{k=0}^m \binom{m}{k} (-1)^k = (1-1)^m = 0. \quad \square$$

## 5. PROOF OF THE MAIN THEOREM

In this section we prove a generalization of Theorem 1.1, in which we assign weights to both the vertices and the edges of the graph.

Let  $\Gamma$  be a finite simple graph with vertices  $V$  and edges  $E$  in which each vertex  $i \in V$  is labeled with a weight  $p_i \in \mathbb{Z}_{>0}$  and each edge  $j \in E$  is labeled with a weight  $q_j \in \mathbb{Z}/2\mathbb{Z}$ . Let  $c_\Gamma(z)$  be the clique polynomial of  $\Gamma$  as in Definition 2.5.

Let  $R = \mathbf{k}\langle x_i, i \in V \rangle$ , with  $|x_i| = p_i$ . For  $j \in E$ , let  $[x_{a_j}, x_{b_j}]$  denote  $x_{a_j}x_{b_j} - (-1)^{q_j}x_{b_j}x_{a_j}$ . Let  $I(\Gamma)$  be the two-sided ideal in  $R$  generated by  $\{[x_{a_j}, x_{b_j}], j \in E\}$ .

**Theorem 5.1.** *The associative (noncommutative) graded algebra*

$$A = R/I(\Gamma)$$

*has Hilbert series*

$$H_A(z) = [c_\Gamma(z)]^{-1}.$$

Theorem 1.1 follows from this theorem by taking  $q_j = p_{a_j}p_{b_j} \pmod 2$ . We start with the case in which all of the edges have weight 0.

**Proposition 5.2.** *The result of Theorem 5.1 holds if  $q_j = 0$  for all  $j \in E$ .*

*Proof.* Let  $M$  be the partially commutative monoid  $\langle x_i, i \in V \rangle / \simeq_E$ . For  $w \in M$ ,  $w \neq 1$ , let  $\text{wgt}(w) = p_{i_1} + \dots + p_{i_l}$  where  $x_{i_1} \dots x_{i_l}$  is some representative of  $w$  and  $\text{wgt}(1) = 0$ . This is well-defined, since for all  $j \in E$ ,  $\text{wgt}(x_{a_j}x_{b_j}) = \text{wgt}(x_{b_j}x_{a_j})$ . Then  $M$  is a basis for  $A$  as a  $\mathbf{k}$ -vector space, and furthermore the product in  $A$  restricts to the composition product on  $M$ .

Define  $\text{Wgt} : \mathbb{Z}\langle\langle M \rangle\rangle \rightarrow \mathbb{Z}\langle\langle z \rangle\rangle$  by

$$\text{Wgt} \left( \sum_{w \in M} f(w)w \right) = \sum_{w \in M} f(w)z^{\text{wgt}(w)}.$$

For  $S \subset M$  let  $\chi_S$  denote the characteristic function, defined in (4.1). Then for  $\chi_M$ , the identity on  $M$ ,

$$\begin{aligned} \text{Wgt}(\chi_M) &= \sum_{k=0}^{\infty} \alpha_k z^k, \text{ where } \alpha_k = \#\{w \mid \text{wgt}(w) = k\} \\ &= H_A(z), \end{aligned}$$

and  $\text{Wgt}(\chi_{\{1\}}) = 1$ .

By Lemma 5.3 below,  $\text{Wgt}$  is a ring homomorphism.

Let  $\mu_M$  be the clique polynomial for  $M$ , defined in (4.2). Then

$$\begin{aligned} \text{Wgt}(\mu_M) &= \sum_{Q \in \mathcal{Q}} (-1)^{|Q|} z^{\text{wgt}([Q])} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i c_{i,j} z^j \\ &= c_{\Gamma}(z), \end{aligned}$$

where  $c_{i,j}$  equals the number of  $i$ -cliques in  $M$  of weight  $j$ .

Then, by Theorem 4.2,  $H_A(z) \cdot c_{\Gamma}(z) = \text{Wgt}(\chi_M) \cdot \text{Wgt}(\mu_M) = \text{Wgt}(\chi_M \cdot \mu_M) = \text{Wgt}(\chi_{\{1\}}) = 1$ .  $\square$

**Lemma 5.3.**  $\text{Wgt} : \mathbb{Z}\langle\langle M \rangle\rangle \rightarrow \mathbb{Z}\langle\langle z \rangle\rangle$  is a ring homomorphism.

*Proof.*

$$\begin{aligned} \text{Wgt}(f + g) &= \sum_{w \in M} (f + g)(w)z^{\text{wgt}(w)} = \\ &= \sum_{w \in M} f(w)z^{\text{wgt}(w)} + \sum_{w \in M} g(w)z^{\text{wgt}(w)} = \text{Wgt}(f) + \text{Wgt}(g). \end{aligned}$$

Let  $M_k$  denote the subset of  $M$  of words with weight  $k$ .

$$\begin{aligned} \text{Wgt}(f \cdot g) &= \sum_{k=0}^{\infty} c_k z^k, \text{ where } c_k = \sum_{w \in M_k} \sum_{uv=w} f(u)g(v) \\ \text{Wgt}(f) \text{Wgt}(g) &= \sum_{k=0}^{\infty} d_k z^k, \text{ where } d_k = \sum_{i+j=k} \sum_{u \in M_i} f(u) \sum_{v \in M_j} g(v). \end{aligned}$$

That  $c_k = d_k$  follows from remarking that

$$\#\{(u, v) \in M \times M \mid uv \in M_k\} = \#\{(u, v) \in M_i \times M_j \mid i + j = k\}. \quad \square$$

To prove Theorem 5.1 it remains to show that the choice of  $q_j \in \mathbb{Z}/2\mathbb{Z}$  does not affect  $H_A(z)$ . We will use noncommutative Gröbner bases to prove this.

Let  $\geq$  be an admissible ordering on  $R = \mathbf{k}\langle x_1, \dots, x_{|V|} \rangle$ . Let  $G$  be a Gröbner basis for  $I(\Gamma)$ . Since  $H_{R/I}(z) = H_{R/\text{in}(I)}(z)$  and  $R/\text{in}(I) = R/(\text{in}(G))$ , it remains to show that one can choose a Gröbner basis for  $I(\Gamma)$  such that  $\text{in}(G)$  does not depend on the  $q_j$ .

For simplicity, enumerate the list  $\{[x_{a_j}, x_{b_j}], j \in E\}$  by  $\{g_1, \dots, g_m\}$ . Say that a rewriting rule  $u \rightarrow_f v$  is a *zig-zag of elementary rewrites* if it can be written as a sequence of rewrites

$$u \rightarrow_{g_{i_1}} u_1 \xleftarrow{g_{i_2}} u_2 \rightarrow_{g_{i_3}} \dots \xleftarrow{g_{i_{n-1}}} u_{n-1} \rightarrow_{g_{i_n}} v.$$

**Lemma 5.4.** *Any zig-zag of elementary rewrites from a word  $w$  to  $\pm w$ , has an even number of  $g_i$  for each  $i$ .*

*Proof.* Fix  $i$  and  $\Phi$ , a zig-zag of elementary rewrites from  $w$  to  $\pm w$ . Assume  $x_{a_i} x_{b_i} \rightarrow_{g_i} (-1)^{q_i} x_{b_i} x_{a_i}$ . Assume  $w = \alpha_1 \cdots \alpha_s$ ,  $\alpha_l \in \{x_1, \dots, x_n\}$ . Let  $\beta_1 \cdots \beta_t$  be the word obtained from  $w$  by deleting all letters except  $x_{a_i}$  and  $x_{b_i}$ . Set  $P(w) = \#\{k \mid \beta_{2k} = x_{a_i}\} \pmod 2$ . Then rewriting using  $g_j$  changes  $P$  if and only if  $i = j$ . Therefore  $\Phi$  contains an even number of  $g_i$ .  $\square$

**Lemma 5.5.** *There does not exist a zig-zag of elementary rewrites from  $w$  to  $-w$ .*

*Proof.* Let  $\Phi$  be a zig-zag of elementary rewrites from  $w$  to  $\pm w$ . Let  $\alpha_i$  be the number of  $g_i$  in  $\Phi$ . Then let  $\alpha = \sum_{i=1}^{|V|} \alpha_i q_i$ . Then  $\Phi$  goes from  $w$  to  $(-1)^\alpha w$ . By the previous lemma,  $\alpha_i$  is even for all  $i$ , so  $(-1)^\alpha = 1$ .  $\square$

The following proposition completes the proof of Theorem 5.1.

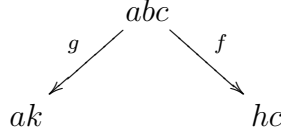
**Proposition 5.6.** *The two-sided ideal  $I(\Gamma)$  has a Gröbner basis  $G$  such that*

- (1) *all of the elements of  $G$  have rewriting rules which are zig-zags of elementary rewrites, and*
- (2)  *$\text{in}(G)$  does not depend on the weights  $\{q_k, k \in E\}$ .*

*Proof.* The proof is by induction on the sets  $G_i$  arising during the application of Mora's algorithm to find the Gröbner basis of  $I(\Gamma)$ . Recall that at each step where we obtain a nonzero result  $r_i$ ,  $G_{i+1}$  is the self reduction of  $G_i \cup r_i$ .

We start with  $G_0 = \{g_1, \dots, g_m\}$ , all of whose elements have elementary rewriting rules and whose initial terms do not depend on the weights  $\{q_k, k \in E\}$ .

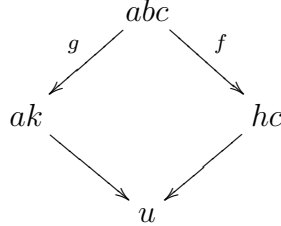
Suppose (1) and (2) of the Proposition are true for  $G_i$  and that  $f, g \in G_i$ . So we have two rewriting rules  $ab \rightarrow_f h$  and  $bc \rightarrow_g k$  both of which are zig-zags of elementary rewriting rules. Then  $f$  and  $g$  have composition  $(a, b, c)$  which has result  $r = ak - hc$ . If  $r \neq 0$ , then we have either  $ak \rightarrow_r hc$  or  $hc \rightarrow_r ak$ . In either case this is a zig-zag of  $f$  and  $g$ ,



which, by induction, is a zig-zag of elementary rewrites.

If  $r \neq 0$  then we add  $r$  to the Gröbner basis. Self-reducing  $G_i \cup r$  may change the elements, but they will still have rewriting rules that are zig-zags of elementary rewrites so all of the elements of  $G_{i+1}$  will have rewriting rules which are zig-zags of elementary rewrites. We remark that  $\text{in}(r)$  does not depend on  $\{q_k, k \in E\}$ .

It remains to show that it cannot be that  $r = 0$  for one choice of  $\{q_k, k \in E\}$ , and  $r \neq 0$  for another choice. Fix a choice of  $\{q_k, k \in E\}$ , and assume that  $r = ak - hc = 0$ . This implies that there are sequences of elementary rewrites from both  $ak$  and  $hc$  to the same word  $u$ .



Composing, we have a zig-zag,  $\Phi$ , of elementary rewrites from  $u$  to  $u$ . Notice that for any zig-zag of elementary rewrites, changing the choice of  $\{q_k, k \in E\}$  only changes the signs of the terms in the zig-zag. By Lemma 5.5, the resulting zig-zag is still from  $u$  to  $u$ . That is, we still have  $r = 0$ .  $\square$

**Example 2.8 continued.** When  $\Gamma$  is the pentagon, the clique polynomial is  $1 - 5z + 5z^2$ . So  $H_{A(\Gamma)}(z) = 1/(1 - 5z + 5z^2)$ . Furthermore by Corollary 1.2,  $I(\Gamma)$  is inert, as is the attaching map in  $(S^{p_i+1}, *)^\Gamma$ , and  $A$  has global dimension 2.

## 6. GRAPH PRODUCTS OF SPHERES AND ADAMS–HILTON MODELS

Given a simply-connected CW complex,  $Y$ , a useful algebraic model is the Adams–Hilton model [2]. It is a free differential graded algebra (DGA),  $\mathbf{AH}(Y)$ , whose algebra generators are in 1–1 correspondence

with the cells of  $Y$ . Furthermore, there is a morphism of DGAs from  $\mathbf{AH}(Y)$  to the singular chain complex on the Moore loops on  $Y$ , that induces an isomorphism  $H\mathbf{AH}(Y) \cong H_*(\Omega Y; \mathbf{k})$ , where  $\Omega Y$  denotes the space of pointed loops on  $Y$ . For a nice summary of the properties of Adams–Hilton models see [28, Theorem 11.10.7].

Let  $\underline{X}^\Gamma$  be the generalized moment–angle complex defined in Section 1 before the statement of Theorem 1.3. Then  $\underline{X}^\Gamma$  has an Adams–Hilton model [2] given by the differential graded algebra

$$\mathbf{AH}(\underline{X}^\Gamma) = (\mathbf{k}\langle x_1, \dots, x_n, y_1, \dots, y_m \rangle, d),$$

$$\text{where } |x_i| = p_i, |y_j| = q_j + 1, \text{ and } dy_j = [x_{a_j}, x_{b_j}].$$

We remark that  $\mathbf{AH}(\underline{X}^\Gamma)$  is a sub–DGA of  $\mathbf{AH}(\underline{X}^K)$  for any simplicial complex  $K$  containing  $\Gamma$ .

If we give  $\mathbf{AH}(\underline{X}^\Gamma)$  a grading by setting each  $x_i$  to have degree 0 and each  $y_j$  to have degree 1, then the degree 0 component of the homology  $H\mathbf{AH}(\underline{X}^\Gamma)$  is

$$A = \mathbf{k}\langle x_1, \dots, x_n \rangle / I, \text{ where } I = ([x_{a_j}, x_{b_j}], j = 1 \dots m).$$

*Proof of Theorem 1.3.* Label the vertices of  $\Gamma$  with  $\{p_i\}$ . Then using notation from Section 2.4,  $\mathbf{AH}(\underline{X}^\Gamma) = \text{DGA}(\Gamma)$ ,  $I = I(\Gamma)$  and  $A = A(\Gamma)$ . Furthermore, using the notation of Theorem 1.3,  $H_*(\Omega Y; \mathbf{k}) \cong A$ ,  $H_*(\Omega W; \mathbf{k}) \cong \mathbf{k}\langle x_1, \dots, x_n \rangle$ , and  $I(f) = I$ . We have a cofibration  $Z \xrightarrow{f} W \xrightarrow{i} Y$ .

Statement (3) from Theorem 1.3 can be rewritten as  $H_*(\Omega Y; \mathbf{k}) \cong H_*(\Omega W; \mathbf{k}) / I(f)$ . By Theorem 2.10, statements (3), (4), (5), and (6) of Theorem 1.3 are equivalent. Statements (1) and (6) of Theorem 1.3 are equivalent by Theorem 1.1. Equivalence with (2) is provided by Félix and Thomas [17, Theorem 1].  $\square$

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