Towards a model category for local po-spaces

A framework for a homotopy theory of concurrency

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Outline

1. Introduction - Background, Categorical Framework
2. A basic model category
3. Towards a better model category - via localization
1. Introduction

Concurrency
We would like to understand systems in which processes run concurrently.

Example:
2 processes using 2 shared resources $a$ and $b$

Notation
$P_x$ - a process locks resource $x$
$V_x$ - a process releases resource $x$
The swiss flag
A sub-po-space of the swiss flag
Goal

Develop a framework for concurrency where equivalences are accounted for.
Motivation for studying local po-spaces

Theorem (Fajstrup-Goubault-Raussen): For concurrency, instead of studying HDA/cubical complexes, one can study local po-spaces.
Definitions

**Top** - a category with

**objects**: subspaces of $\mathbb{R}^n$ for some $n$

**morphisms**: continuous maps

**po-space**: an object $M$ of **Top** together with a *partial order* (reflexive, transitive, anti-symmetric relation) which is a closed subset of $M \times M$

**order atlas**: an open cover of po-spaces with compatible partial orders

order atlases are **equivalent** if they have a common refinement
**Definition of LoPospc**

**LoPospc** - a category with

**objects:** \((M, \bar{U}), M \in \text{Ob } \textbf{Top}, \bar{U}\) is an equivalence class of order atlases

**morphisms:** continuous maps which respect the orders
Remark: There are induced orderings on products and subspaces.

Example:

\[(x, y), (x', y') \in I \times I, \text{ where } I = ([0, 1], \leq)\]

\[(x, y) \leq (x', y') \iff x \leq x' \text{ and } y \leq y'\]

So \((0, \frac{1}{3})\) and \((\frac{2}{3}, 0)\) are not comparable.
**Top and LoPospc**

**Remark:** Top and LoPospc are small categories.

**Lemma:** There are adjoint functors

\[ F : \text{Top} \dashv \text{LoPospc} : U. \]
Model categories

Recall: A model category has 3 special classes of morphisms:

WE, COF, FIB

and satisfies a list of axioms (M1, . . . , M5).

Goal: Construct a model category for LoPospc.
Problem

Remark: By itself, LoPospc cannot have a model structure since it does not contain pushouts.

Example: The pushout of $\tilde{S} \times \tilde{I}$ by collapsing top is not a po-space
Solution

Need to enlarge the category LoPospc.
Enlarging a Category

Let $\mathbf{C}$ be a small category.

$\text{Pre}(\mathbf{C}) = \text{Set}^{\mathbf{C}^{\text{op}}}$ called the presheaves on $\mathbf{C}$

**Remark:** $\mathbf{C}$ embeds into $\text{Pre}(\mathbf{C})$

(via the Yoneda embedding $y : \mathbf{C} \to \text{Pre}(\mathbf{C})$, $y(\alpha) = \mathbf{C}(\dashv, \alpha)$)

$s\text{Pre}(\mathbf{C}) = \text{sSet}^{\mathbf{C}^{\text{op}}}$ called the simplicial presheaves on $\mathbf{C}$

**Remark:** $\mathbf{C}$ embeds into $s\text{Pre}(\mathbf{C})$

(via the Yoneda embedding $\bar{y} : \mathbf{C} \to s\text{Pre}(\mathbf{C})$)
A model structure

Theorem (Jardine): Under condition $Q$, $sPre(C)$ has a (proper, simplicial) model structure such that

- $\text{COF} = \text{monomorphisms}$, and
- $\text{WE} = \text{‘stalkwise equivalences’}$.
Outline for Section 2

- define $Q$
- show $\text{LoPospc}$ satisfies $Q$
- define stalkwise equivalence
- for $\varphi \in \text{Mor} \text{LoPospc}$ determine when $\tilde{y}(\varphi)$ is a stalkwise equivalence
2. A basic model category

\( \mathcal{C} \) satisfies \( Q \) if

1. \( \mathcal{C} \) is a site, and
2. \( \text{Shv}(\mathcal{C}) \) has enough points

A Grothendieck topology \( J \) assigns to each \( M \in \text{Ob} \mathcal{C} \) a collection of covering families \( \{U_i \to M\} \in \text{Mor} \mathcal{C} \) satisfying

1. it contains isomorphism
2. a transitivity condition, and
3. a stability condition
Condition $Q_1$

**Example:** $C = \text{Top}$ and $M \in \text{Ob Top}$

Let $J(M) = \{\text{open covers of } M\}$

(actually a basis for a Grothendieck topology)

A *site* is a small category with a Grothendieck topology.

**Example:** $\text{LoPospc}$ is a site. So it satisfies $Q_1$. 
Condition \( Q2 \)

\( Q2 \)

Given a site \((C, J)\) a \textit{sheaf} is a presheaf \( P : C^{\text{op}} \to \text{Set} \) such that for each covering family, each compatible family of elements of \( P \) has a unique amalgamation.

\textbf{Example:} \( X \in \text{Ob Top}, C = O(X) \) the open subspaces of \( X \)

The presheaf of continuous function is a sheaf. \( \text{Shv}(C, J) \) is a subcategory of \( \text{Pre}(C) \).
Points

A point on Shv($\mathcal{C}, J$) is a pair of adjoint functors

\[ p^* : \text{Shv}(\mathcal{C}, J) \rightleftarrows \text{Set} : p_* \]

such that $p_*$ preserves finite limits.

Shv($\mathcal{C}, J$) has enough points if given

\[ f \neq g : P \to Q \in \text{Shv}(\mathcal{C}, J) \]

then there is a point $p^*$ such that

\[ p^* f \neq p^* g : p^* P \to p^* Q \in \text{Set}. \]
Points in LoPospc

Let $\mathbf{C} = \text{LoPospc}$. Let $Z \in \text{Ob } \mathbf{C}$ and let $x \in Z$. Define

$$p^*_x : \text{Pre}(\mathbf{C}) = \text{Set}^{\mathbf{C}^{\text{op}}} \to \text{Set}$$

$$F \mapsto \text{colim}_{x \in \text{L}_{\text{open}} \subseteq Z} F(L)$$

$$\begin{array}{ccc}
\text{Pre}(\mathbf{C}) & \xrightarrow{p^*_x} & \text{Set} \\
\downarrow a & & \uparrow i \\
\text{Shv}(\mathbf{C}) & & \\
\end{array}$$

**Proposition (B):** $p^*_x$ descends to a point on $\text{Shv}(\mathbf{C})$. 
Points in LoPospc (continued)

**Theorem (B):** These points provide enough points for $\text{Shv}(C)$.

So **LoPospc** satisfies $Q2$. 
Stalks

Let \( p^* \) be a point in \( \text{Shv}(\text{LoPospc}) \).
Let \( \alpha \in \text{sPre} \) \( \text{LoPospc} = \text{sSet}^{\text{LoPospc}^{\text{op}}} \).
The stalk of \( \alpha \) at \( p^* \) is given by

\[
\alpha_p = \{ p^* a(\alpha_n) \}_{n \geq 0}
\]

Remark: \( \alpha_n \in \text{Pre}(\text{LoPospc}) \),
\( a(\alpha_n) \in \text{Shv}(\text{LoPospc}) \), \( p^* a(\alpha_n) \in \text{Set} \).

Say that \( f : P \to Q \in \text{sPre} \) \( \text{LoPospc} \) is a stalkwise equivalence if for all points \( p^* \in \text{Shv}(\text{LoPospc}) \),
\( f_p : P_p \to Q_p \in \text{sSet} \) is a weak equivalence.
**Stalkwise equiv. in LoPospc**

**Recall:** Let $\varphi : X \to Y \in \text{Mor} \ LoPospc$. Then $\bar{y}(\varphi) : \bar{y}(X) \to \bar{y}(Y) \in \text{Mor} \ sPre(LoPospc)$.

**Theorem (B):** $\bar{y}(\varphi)$ is a stalkwise equivalence if and only if $\varphi$ is an isomorphism.

**Summary:** LoPospc includes into a model category such that

- $\text{COF} \cap \text{LoPospc} = \{ \text{monomorphisms} \}$
- $\text{WE} \cap \text{LoPospc} = \{ \text{isomorphisms} \}$
3. Localization

We need to introduce non-trivial equivalences.

Example: For \textbf{Top} one localizes with respect to the maps

\[ X \times I \xrightarrow{\text{proj}} X, \]

where \( I \) is the unit interval \([0, 1]\).

Let \( \mathcal{I} = \{ \tilde{y}(X \times I \to X) | X \in \text{Ob Top} \} \).

Then \( \text{sPre(Top)}/\mathcal{I} \) produces the usual homotopy theory on \textbf{Top}. 
Localization for LoPospc

Question: What is a good $\mathcal{I}$ for LoPospc?

Ideas:

- $\mathcal{I}$ as above
- $\mathcal{I}$ as above but with $\tilde{I}$ instead of $\mathcal{I}$
- dihomotopy equivalences
- d-homotopy equivalence
Problems

With each of these we have the following weak equivalences:

\[ \sim \quad \ast \quad \text{and} \]

\[ \sim \quad \ast \quad \text{but} \quad \text{and} \quad \sim \]

are not the same in concurrency.

**Recall:** The model structure on $\text{sPre LoPospc}$ is proper.

Thus so is the model structure on $\text{sPre(LoPospc)}/\mathcal{I}$. 
Left proper

In a left proper model category the pushout of a weak equivalence over a cofibration is a weak equivalence.

Thus $A \to X \in \mathcal{I}$ and $A \to B \in \text{COF}$ implies that $B \to Y \in \text{WE}$ in $s\text{Pre}(\text{LoPospc})/\mathcal{I}$. 
Problem

Assume some map \( \vec{I} \times \vec{I} \to \vec{I} \in \mathcal{I} \).
Choose two points \( x, y \in \vec{I} \times \vec{I} \) such that \( x \not\leq y \) and \( y \not\leq x \).
Possible Solution

Use \textit{context}.
Work with $A \downarrow \text{LoPospc}$ instead of $\text{LoPospc}$, where $A$ depends on the pastings one wants to consider.
Example

\[ A = \{x, y\} \].

Then \( A \downarrow \text{LoPospc} \) is the category of local po-spaces with two marked points.

Furthermore,

\[
\begin{array}{c}
\text{is not a dihomotopy equivalence, but}
\end{array}
\]

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Another example

Also $\text{y}$

$x \rightarrow \ast \rightarrow \text{x=y}$

dihomotopy equivalence.

is not a
Context for the Swiss flag

Let $A = \{a, b, c, d\}$. Then

is a d-homotopy equivalence in $A \downarrow \text{LoPospc}$. 
Proposal

Work with

\( \bar{y}(A) \downarrow \text{sPre}(\text{LoPospc}) \)

which inherits a model structure from \( \text{sPre}(\text{LoPospc}) \). Then localize

\[ (\bar{y}(A) \downarrow \text{sPre}(\text{LoPospc})) / \mathcal{I} \]

where

\[ \mathcal{I} = \{ \bar{y}(\text{d-homotopy equivalences}) \} \]

or

\[ \mathcal{I} = \{ \bar{y}(\text{dihomotopy equivalences}) \}. \]