Extremal models of concurrent systems, the fundamental bipartite graph, and directed van Kampen theorems

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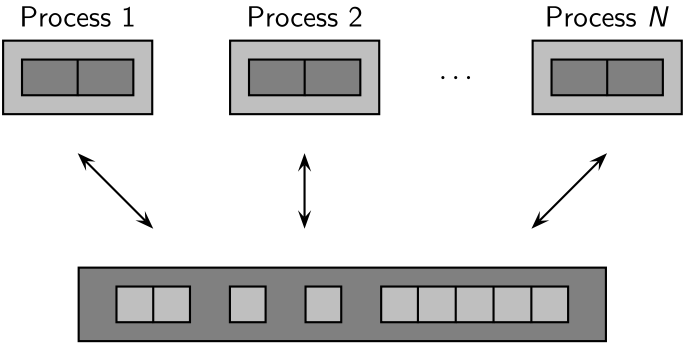
ATMCS III – Paris, France
July 7, 2008

Concurrent parallel computing



Several processes with shared resources

Directed spaces

Definition

- A partial order, $\leq$, is a reflexive, transitive, anti-symmetric relation.
- A po-space is a topological space with a partial order $\leq$.

Definition

A directed map is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

Remark

Subspaces and products of po-spaces inherit a po-space structure.

$\mathbb{R}^n$ as a po-space

Example

$\mathbb{R}$ is a po-space with the usual ordering: $x \leq y \iff y - x$ is non-negative.

Example

$\mathbb{R}^n$ is a po-space with the product order:
$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff x_i \leq y_i$ for $1 \leq i \leq n$.

Notation

Let $\vec{I} = [0,1]$ and $\vec{I}^n = [0,1]^n$ denote the po-spaces whose orderings are induced from $(\mathbb{R}, \leq)$ and $(\mathbb{R}^n, \leq)$.

Directed homeomorphisms

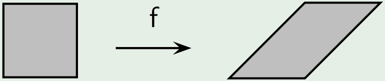
Definition

A directed homeomorphism is a directed map $f : X \to Y$ such that there exists an inverse directed map $g : Y \to X$.

Example

A directed map which is a homeomorphism need not be a directed homeomorphism.

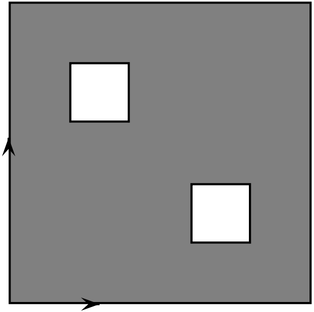
Consider the shear map $(x, y) \mapsto (x + y, y)$.



Directed paths

Definition

A directed path in a pospace X is a directed map $\gamma : \vec{I} \to X$.



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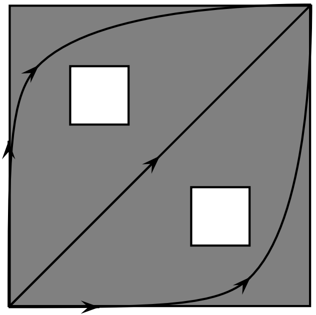

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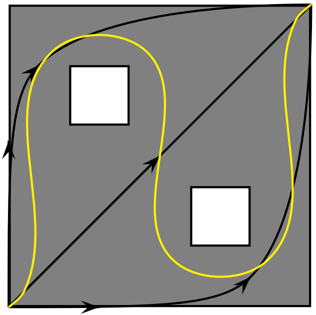


There are paths which are not homotopic to directed paths.

A concurrent system

Example

2 processes using 2 shared resources a and b which can only be used by one process at a time

Notation

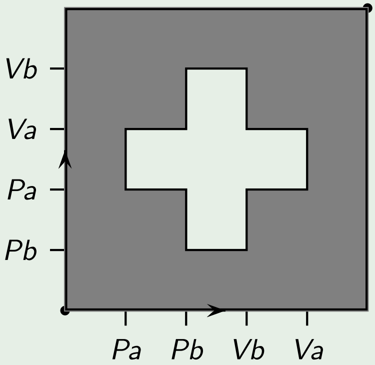
Px - a process locks resource x
Vx - a process releases resource x

Program

The first process:	*Pa*	*Pb*	*Vb*	*Va*
The second process:	*Pb*	*Pa*	*Va*	*Vb*

The Swiss flag

Example



The Swiss flag

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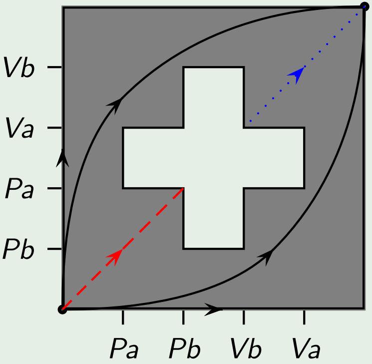

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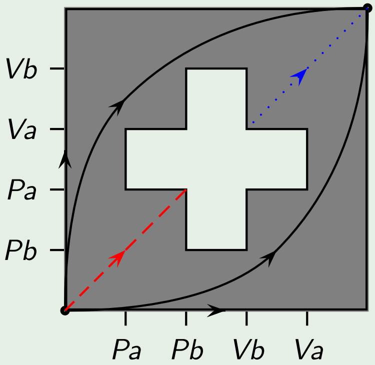
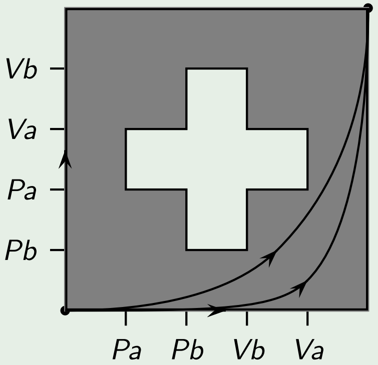


Problem: Uncountably many states and execution paths.

The Swiss flag

Example



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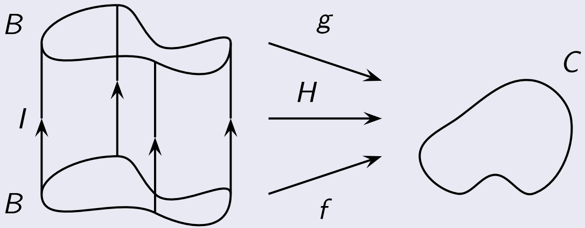
Directed homotopies

Definition

A homotopy between directed maps $f, g : B \to C$ is a directed map $H : B \times \vec{I} \to C$ restricting to f and g. Write $H : f \xrightarrow{\simeq} g$.



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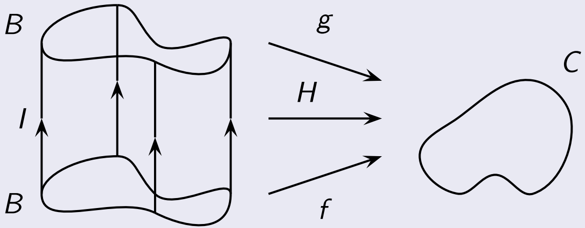


Definition

Directed maps f, g are homotopic if there is a chain of homotopies

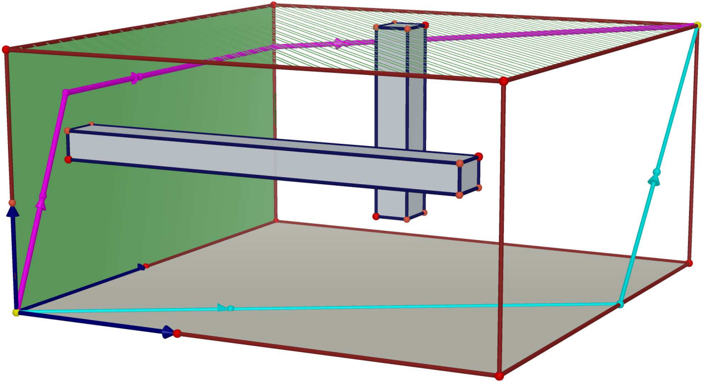
$$f \xrightarrow{\simeq} f_1 \xleftarrow{\simeq} f_2 \xrightarrow{\simeq} \dots \xleftarrow{\simeq} f_n \xrightarrow{\simeq} g.$$

Equivalence classes of directed paths

Definition

Directed paths are homotopy equivalent if they are so relative to their endpoints.

A room with two barriers



Two directed paths which are homotopic as paths, but not as directed paths.

The fundamental group, groupoid, and category

Definition

- For $x \in X$, the fundamental group $\pi_1(X, x)$ is the set of homotopy classes of paths beginning and ending at x.

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- The fundamental groupoid $\pi_1(X)$, is a category with
  - objects: points in X
  - morphisms: homotopy classes of paths

Remark

The existence of composition with associativity and identity is built into the definition of a category.

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Definition

The fundamental category $\vec{\pi}_1(X)$ has

- objects: the points in X
- morphisms: homotopy classes of directed paths

Full subcategories of the fundamental category

Problem

The fundamental category is enormous.

Plan

We would like to derive a "small" category from the fundamental category that still contains useful information.

1. Use the component category. [Fajstrup, Goubault, Haucourt, Raussen]
2. Use full subcategories. [Grandis]

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2. Use full subcategories. [Grandis]

Definition

Given $A \subseteq X$, let $\vec{\pi}_1^X(A)$ have

- objects: points in A
- morphisms: homotopy classes of paths in X

The fundamental bipartite graph

Assume X is a compact pospace.

Definition

Let $\mathrm{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$.
Let $\mathrm{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}$.

The fundamental bipartite graph

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Definition

Let $\mathrm{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$.
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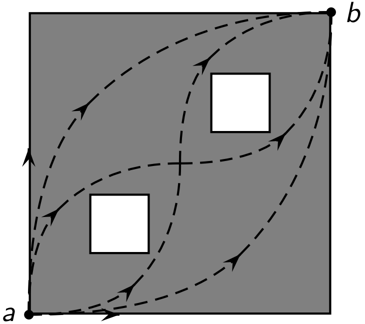
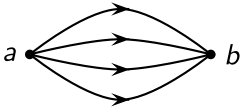
Definition

The fundamental bipartite graph of X is $\vec{\pi}_1^X(\mathrm{Min}\, X \cup \mathrm{Max}\, X)$.

Example of the fundamental bipartite graph

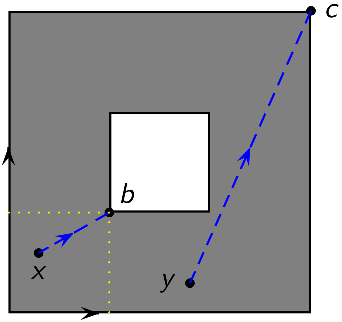




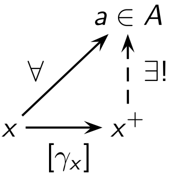
Future retracts

Definition (Grandis, 2005)

A future retract of $\vec{\pi}_1(X)$ moves each $x \in X$ along a directed path in X to a point x^+ which "has the same future".



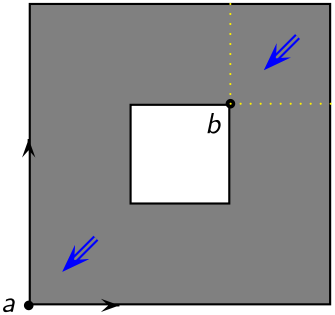
$$P^+ : \vec{\pi}_1(X) \to \vec{\pi}_1^X(A)$$



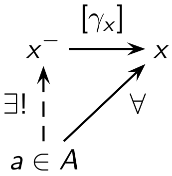
Past retracts

Definition (Grandis, 2005)

A past retract of $\vec{\pi}_1(X)$ moves each $x \in X$ backwards along a directed path in X to a point x^- which "has the same past".



$$P^- : \vec{\pi}_1(X) \to \vec{\pi}_1^X(A)$$



Future retracts and Extremal points

Lemma

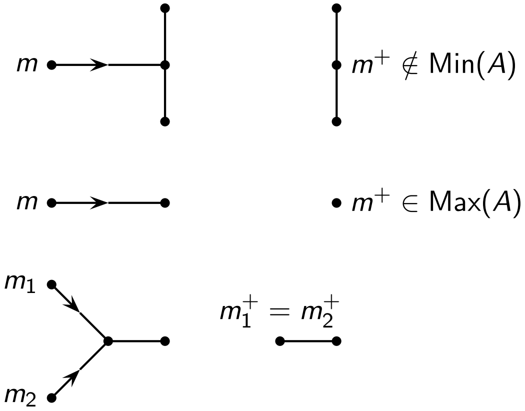
For $M \in \mathrm{Max}(X)$, $M^+ = M$ *and* $P^+ : \mathrm{Max}(X) \xrightarrow{\cong} \mathrm{Max}(A)$.

Future retracts and Extremal points

Lemma

For $M \in \mathrm{Max}(X)$, $M^+ = M$ and $P^+ : \mathrm{Max}(X) \xrightarrow{\cong} \mathrm{Max}(A)$.

However P^+ is not well-behaved with respect to minimal points.



Non-collapsing future retracts

Definition

Call P^+ non-collapsing if none of these three problems occur.

More precisely, $P^+ : \mathrm{Min}'(X) \xrightarrow{\cong} \mathrm{Min}'(A)$ where $\mathrm{Min}'(\cdot) = \mathrm{Min}(\cdot) - \mathrm{Min}(\cdot) \cap \mathrm{Max}(\cdot)$.

Extremal models

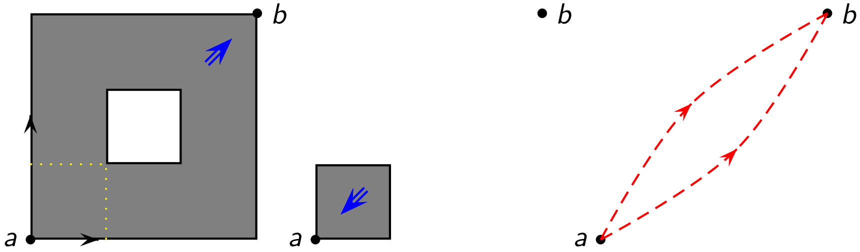
Definition

An extremal model is a sequence of non-collapsing future retracts and non-collapsing past retracts.

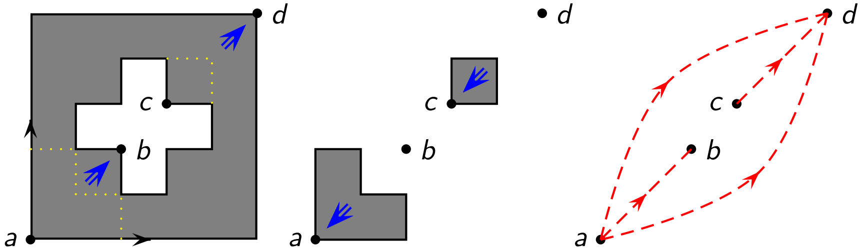
Theorem (B)

An extremal model preserves the fundamental bipartite graph.

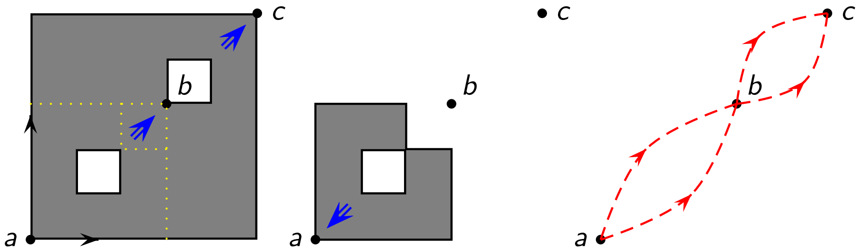
Examples of extremal models (1)



Examples of extremal models (2)



Examples of extremal models (3)



Piece-by-piece simplification

Simple examples, such as the Swiss flag, can easily by analyzed and simplified by hand.

We would like to have algorithms that do the same for much larger, and higher-dimensional examples.

To do so, we would like a way to simplify a small pieces of the fundamental category, and then patch these simplifications together.

Van Kampen Theorem for the fundamental category

Theorem (Grandis 2003, Goubault 2003)

Assume $X = \operatorname{Int}(X_1) \cup \operatorname{Int}(X_2)$ *and let* $X_0 = X_1 \cap X_2$. *Then the pushout of pospaces:*

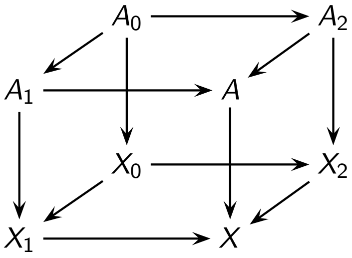
$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \dashrightarrow & X \end{array}$$

induces a pushout of fundamental categories:

$$\begin{array}{ccc} \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_2) & \dashrightarrow & \vec{\pi}_1(X) \end{array}$$

Compatible subspaces

$X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$ and $A = \mathrm{Int}(A_1) \cup \mathrm{Int}(A_2)$.
$X_0 = X_1 \cap X_2$ and $A_0 = A_1 \cap A_2$.
$A_k \subseteq X_k$, $k = 0, 1, 2$.
We have the following pushout in the arrow category of pospaces.



Van Kampen theorem for full subcategories

Theorem (B)

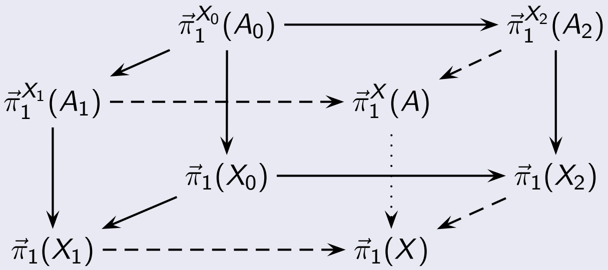
The inclusions in the previous slide induce the following pushout in **Cat**.

$$\begin{array}{ccc} \vec{\pi}_1^{X_0}(A_0) & \longrightarrow & \vec{\pi}_1^{X_2}(A_2) \\ \downarrow & & \downarrow \\ \vec{\pi}_1^{X_1}(A_1) & \longrightarrow & \vec{\pi}_1^{X}(A) \end{array}$$

Van Kampen theorem for full subcategories

Theorem (B)

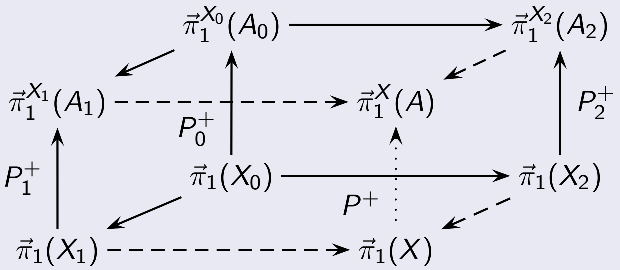
Furthermore, they induce the following pushout in the arrow category on **Cat**.



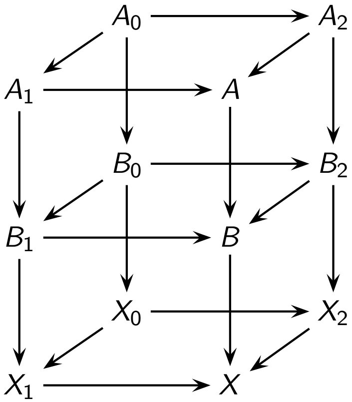
van Kampen theorem for future retracts

Theorem (B)

Given compatible future retracts P_0^+, P_1^+, P_2^+ there is an induced retraction P^+, which is a pushout.



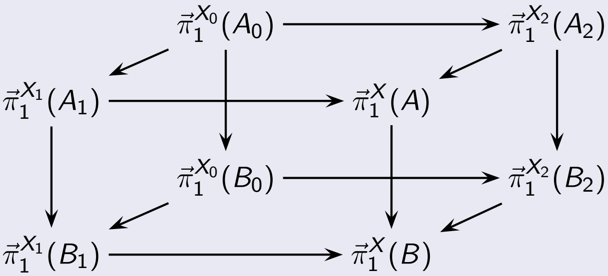
Compatible triples



van Kampen theorem for full subcategories

Theorem (B)

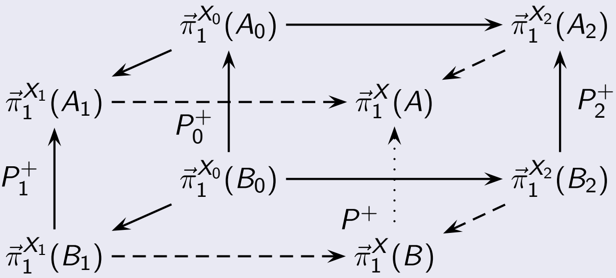
The inclusions in the previous slide induce the following pushout in the arrow category on **Cat***.*



van Kampen theorem for future (past) retracts

Theorem (B)

Given compatible retractions P_0^+, P_1^+, P_2^+, there is an induced retraction P^+, which is a pushout.



van Kampen theorem for extremal models

Theorem (B)

The pushout of compatible extremal models is an extremal model.

Summary

- Po-spaces provide a good mathematical model for concurrent parallel computing.
- The fundamental bipartite graph captures the essential schedules.
- Future and past retracts of the fundamental category which preserve minimal and maximal points can be combined to form an extremal model.
- Extremal models preserve the fundamental bipartite graph.
- Extremal models are amenable to a piece-by-piece analysis.

Open problems

- The fundamental bipartite graph detects deadlock states and captures the essential schedules. Can this be turned into a homology theory?
- When is the component category of the fundamental category isomorphic to an extremal model?