A mathematical model for parallel computing

Peter Bubenik

Cleveland State University
http://academic.csuohio.edu/bubenik_p/

August 15, 2007. University of Guelph
The end of Moore’s Law?

A mathematical model for parallel computing
Example: Multi-core processors

Intel

AMD

A mathematical model for parallel computing
Example: Internet database
Classical non-parallel computing

A process with its own private resources
Concurrent parallel computing

Several processes with shared resources
A concurrent system

Example

2 processes using 2 shared resources $a$ and $b$ which can only be used by one process at a time

Notation

$P_x$ - a process locks resource $x$
$V_x$ - a process releases resource $x$

Program

The first process: $Pa$ $Pb$ $Vb$ $Va$
The second process: $Pb$ $Pa$ $Va$ $Vb$
A mathematical model

Concurrent systems can be modeled by subspaces of $\mathbb{R}^n$ together with a partial order.

**Definition**

A **po-space** is a topological space $U$ with a partial order $\leq$.

**Example**
The Swiss flag

Example

Peter Bubenik
A mathematical model for parallel computing
The Swiss flag

Example

A mathematical model for parallel computing
Problem: The state space is infinite.
The essential schedules of the Swiss flag

Example

This is a sub-po-space of the Swiss flag.
Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis. This will make the analysis of large programs tractable.
Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis. This will make the analysis of large programs tractable.

Idea

*Use algebraic topology.*
Undirected equivalences

Definition

Given continuous maps $f, g : B \to C$, 

\begin{align*}
B & \xrightarrow{g} C \\
B & \xrightarrow{f} C
\end{align*}
Definition

Given continuous maps $f, g : B \rightarrow C$,
Definition

Given continuous maps $f, g : B \to C$,

A homotopy between $f$ and $g$ is a continuous map $H : B \times I \to C$ restricting to $f$ and $g$. This is an equivalence relation. Write $H : f \sim g$. 

Peter Bubenik

A mathematical model for parallel computing
Undirected equivalences

**Definition**

Spaces $B$, $C$ are **homotopy equivalent** if there are maps $f : B \leftrightarrow C : g$ such that

$$g \circ f \simeq \text{Id}_B \quad \text{and} \quad f \circ g \simeq \text{Id}_C.$$
**Definition**

- A **po-space** is a topological space $U$ with a partial order $\leq$ which is a closed subset of $U \times U$.
- A **directed map (dimap)** is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

**Remark**

*Subspaces and products of po-spaces inherit a po-space structure.*
**Definition**

A **directed homotopy (dihomotopy)** between dimaps $f, g : B \to C$ is a dimap $H : B \times \overrightarrow{I} \to C$ restricting to $f$ and $g$. Write $H : f \to g$.

\[ B \xrightarrow{\overrightarrow{I}} C \]
Directed equivalences

Definition

Write $f \simeq g$ if there is a chain of dihomotopies

$$f \rightarrow f_1 \leftarrow f_2 \rightarrow \ldots \leftarrow f_n \rightarrow g.$$ 

Po-Spaces $B, C$ are **dihomotopy equivalent** if there are dimaps $f : B \leftrightarrow C : g$ such that

$$g \circ f \simeq \text{Id}_B \quad \text{and} \quad f \circ g \simeq \text{Id}_C.$$
A problem

Recall

We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

However all of the following spaces are dihomotopy equivalent.

Thus, a stronger notion of equivalence is needed.
**One solution**

**Idea (B, 2004)**

*Instead of working with just po-spaces work with po-spaces together with context.*

**Definition**

Choose a po-space $A$ (called the context). Consider po-spaces $B$ together with a dimap $i_B : A \to B$ and consider morphisms which are dimaps such that

$$f(i_B(a)) = i_C(a) \text{ for all } a \in A$$
A **dihomotopy** between $f, g : B \to C$ in the context of $A$ is a dihomotopy $H : f \to g \text{ rel } A$. 

---

**Definition**

- A **dihomotopy** between $f, g : B \to C$ in the context of $A$ is a dihomotopy $H : f \to g \text{ rel } A$. 

![Diagram](image)
Equivalences using context

Definition

- Write $f \simeq g$ if there is a chain of dihomotopies
  
  $f \rightarrow f_1 \leftarrow f_2 \rightarrow \ldots \leftarrow f_n \rightarrow g$.

- $i_B : A \rightarrow B$, $i_C : A \rightarrow C$ are dihomotopy equivalent if there are dimaps
  
  $A \xrightarrow{i_B} B \xleftarrow{g} C \xrightarrow{i_C}$

  such that $g \circ f \simeq \text{Id}_B$ and $f \circ g \simeq \text{Id}_C$. 
Let $A = \{x, y\}$ with $x \leq y$. Then po-spaces under the context $A$ are just po-spaces with two marked points, one of which is after the other.

In this category is not a dihomotopy equivalence since there is no dimap in the reverse direction.
In the same context (of two marked points) the dimaps $x \xrightarrow{\rightarrow} y \mapsto \max(x, y)$

$(t, t) \leftarrow t$

$(x, y) \leftrightarrow \max(x, y)$

give a dihomotopy equivalence.
Another equivalence

$I \times I$ with a square removed and two marked points

is dihomotopy equivalent to its boundary.
Let $A = \{a, b, c, d\}$. Then the inclusion

is a dihomotopy equivalence in the context of the four marked points.
Sketch of the proof
Sketch of the proof
Definition

Let $x, y \in$ the po-space $B$.

- A dipath is a dimap $\vec{I} \to B$.
- Dipaths are dihomotopy equivalent if they are so in the context of their endpoints.
- Let $\vec{\pi}_1(B)(x, y)$ be the set of dihomotopy equivalence classes of dipaths from $x$ to $y$. 
Proposition (B, 2004)

Given a dimap \( f : B \rightarrow C \) respecting the context and \( x, y \in A \) there is an induced map

\[
\bar{\pi}_1(f)(x, y) : \bar{\pi}_1(B)(x_B, y_B) \rightarrow \bar{\pi}_1(C)(x_C, y_C).
\]

If \( f \) is a dihomotopy equivalence then it is an isomorphism.
In the context of its four endpoints

the left hand po-space is not dihomotopy equivalent to $\vec{I}$.
We would like to find equivalent po-spaces to the following examples by analyzing them piece-by-piece.
Equivalences of pieces

Peter Bubenik
A mathematical model for parallel computing
Equivalences of pieces

Peter Bubenik
A mathematical model for parallel computing
Patching the pieces together

A mathematical model for parallel computing
A more general model

We would like to model execution loops such as

These cannot be modeled by po-spaces.

However they can be modeled by local po-spaces.
Local po-spaces

Definition

- An order atlas is an open cover of po-spaces with compatible partial orders.
- A local po-space is a topological space together with an order atlas.
- A morphism of local po-spaces is a continuous map which respect the orders.
Just as with po-spaces, we can define local po-spaces under some context $A$, and we consider morphisms which respect the context.

We can also define dihomotopy equivalences using context exactly the same way as with po-spaces.
A powerful framework for studying equivalences is given by model categories.

**Definition**

A model category is a category and with three distinguished classes of morphisms: weak equivalences, cofibrations, and fibrations satisfying five simple axioms.

The structure of a model category allows one to apply the machinery of homotopy theory.
Theorem (B-Worytkiewicz, 2006)

The category of local po-spaces under a context $A$ embeds into a model category such that

- the weak equivalences are the dihomotopy equivalences, and
- the cofibrations are the monomorphisms
- pushouts of weak equivalence with cofibrations are weak equivalences
Po-spaces and local po-spaces provide a good mathematical model for concurrent parallel computing.

Using directed homotopies one can hope to cope with the “state space explosion” and find the essential schedules.

Relative directed homotopies allow a piece-by-piece analysis of the model.

All of the above can be studied in the framework of model categories.
L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss the flag, in any dimension.

E. Goubault and E. Haucourt (2005) gathered the dihomotopy classes into “components” to develop a static analyzer of concurrent parallel programs.
Open problems

- How can the new applications of directed homotopy theory to parallel computing be used to give new algorithms?
- What are the connections between context and components of the fundamental category?
- Can the idea of context be used to develop a static analyzer?
- How can higher directed homotopy be used to dramatically simplify the state space?
- Is there a directed homology theory that detects deadlock states or counts essential schedules?
Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$

and glue them together along the yellow lines.
Non-discrete context

Take $\vec{I} \times \vec{I}$ and $I \times I$

and glue them together along the yellow lines.