A mathematical model for parallel computing

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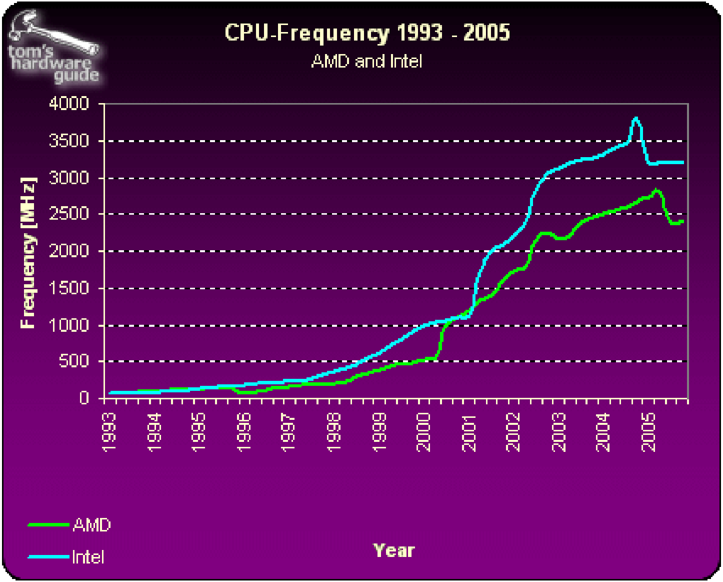
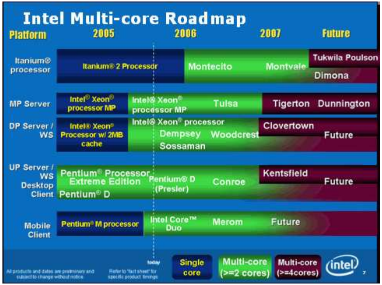
The end of Moore's Law?

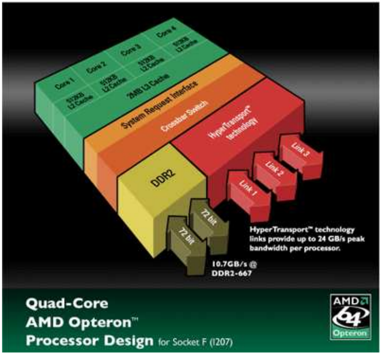


Example: Multi-core processors

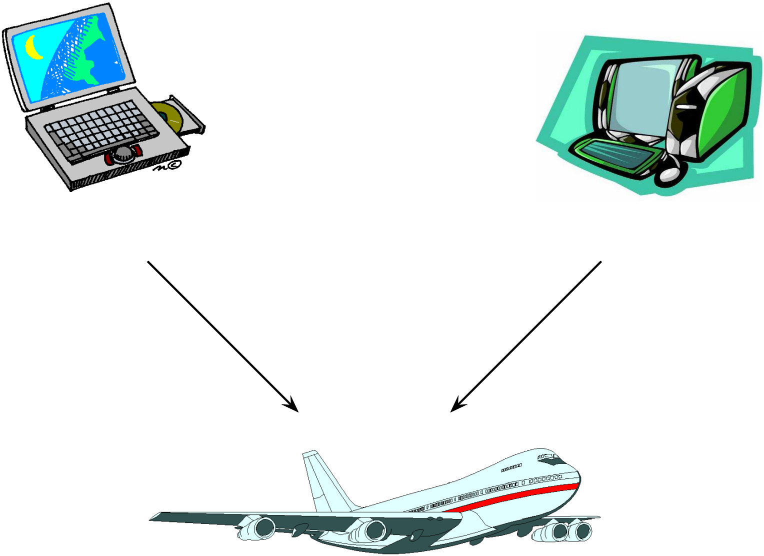
Intel



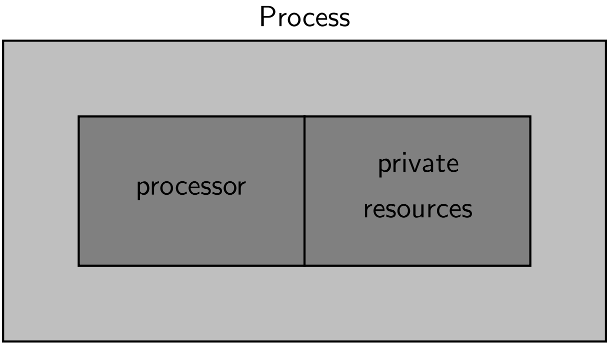
AMD



Example: Internet database

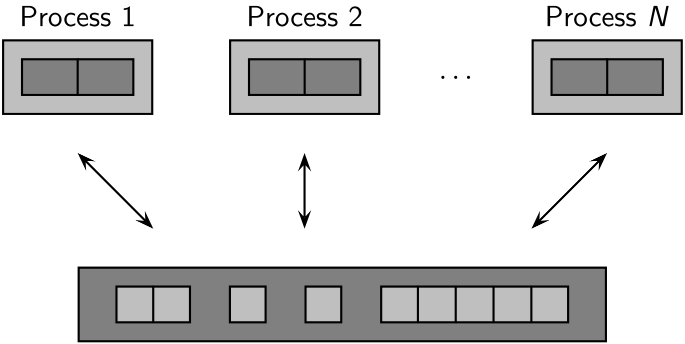


Classical non-parallel computing



A process with its own private resources

Concurrent parallel computing



Several processes with shared resources

A concurrent system

Example

2 processes using 2 shared resources a and b which can only be used by one process at a time

Notation

Px - a process locks resource x
Vx - a process releases resource x

Program

The first process:	*Pa*	*Pb*	*Vb*	*Va*
The second process:	*Pb*	*Pa*	*Va*	*Vb*

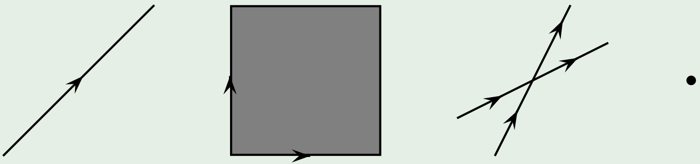
A mathematical model

Concurrent systems can be modeled by subspaces of $\mathbb{R}^n$ together with a partial order.

Definition

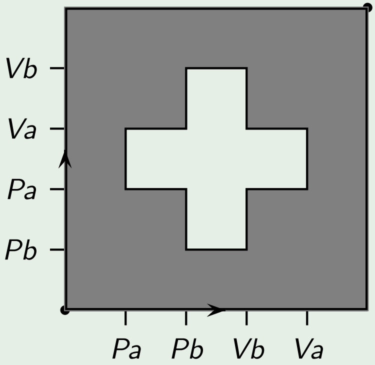
A po-space is a topological space U with a partial order $\leq$.

Example



The Swiss flag

Example



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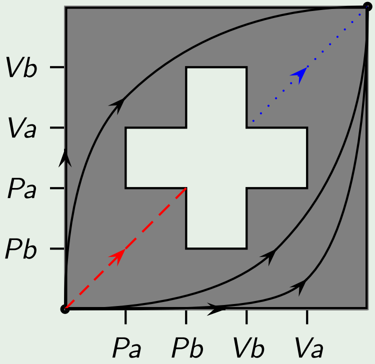
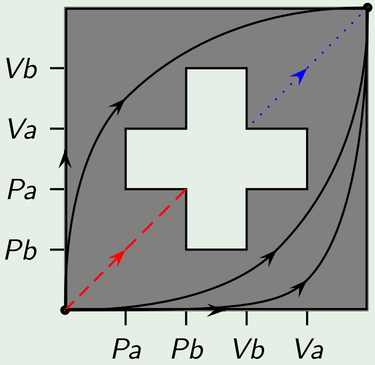


The Swiss flag

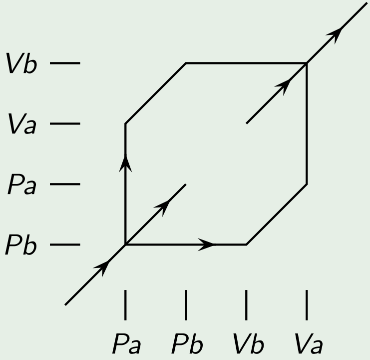
Example



Problem: The state space is infinite.

The essential schedules of the Swiss flag

Example



This is a sub-po-space of the Swiss flag.

Goal

Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis. This will make the analysis of large programs tractable.

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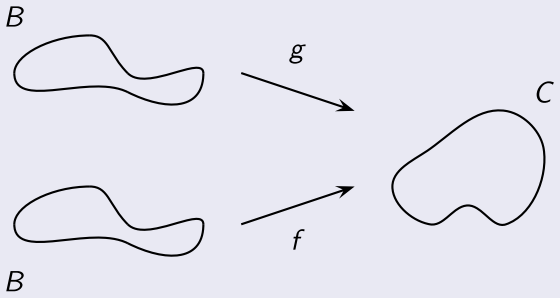
Idea

Use algebraic topology.

Undirected equivalences

Definition

Given continuous maps $f, g : B \to C$,



Undirected equivalences

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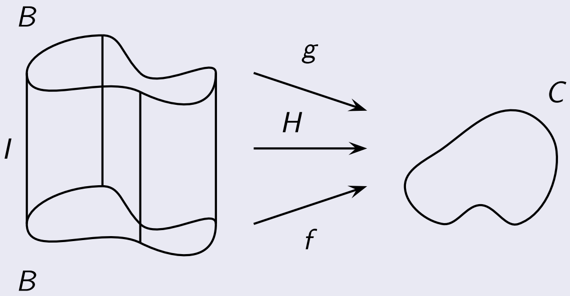

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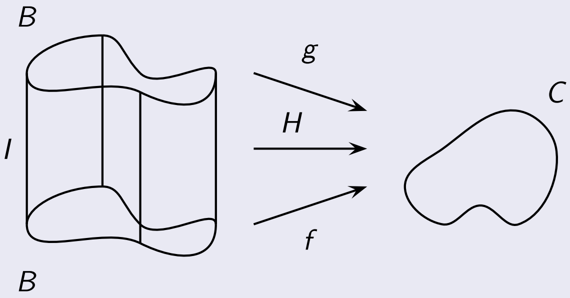


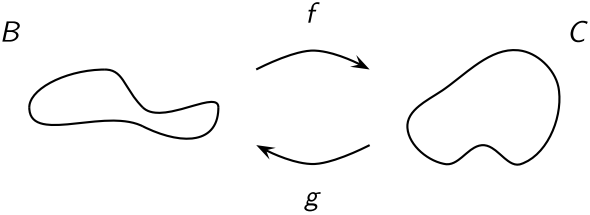
a homotopy between f and g is a continuous map $H : B \times I \to C$ restricting to f and g. This is an equivalence relation. Write $H : f \xrightarrow{\simeq} g$.

Undirected equivalences

Definition

Spaces B, C are homotopy equivalent if there are maps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \text{ and } f \circ g \simeq \mathrm{Id}_C .$$



Directed spaces

Definition

- A po-space is a topological space U with a partial order $\leq$ which is a closed subset of $U \times U$.
- A directed map (dimap) is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

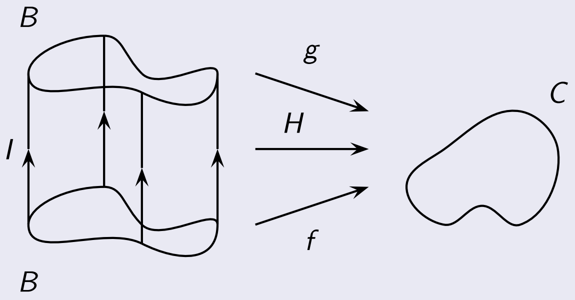
Remark

Subspaces and products of po-spaces inherit a po-space structure.

Directed equivalences

Definition

- A directed homotopy (dihomotopy) between dimaps $f, g : B \to C$ is a dimap $H : B \times \vec{I} \to C$ restricting to f and g. Write $H : f \to g$.



Directed equivalences

Definition

- Write $f \simeq g$ if there is a chain of dihomotopies

$$f \rightarrow f_1 \leftarrow f_2 \rightarrow \ldots \leftarrow f_n \rightarrow g\,.$$

- Po-Spaces B, C are dihomotopy equivalent if there are dimaps $f : B \leftrightarrows C : g$ such that

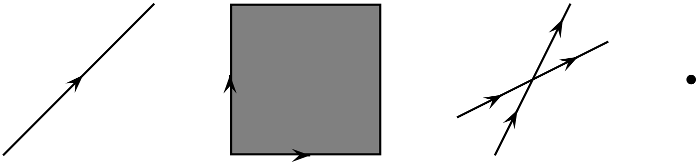
$$g \circ f \simeq \mathrm{Id}_B \;\text{ and }\; f \circ g \simeq \mathrm{Id}_C\,.$$

A problem

Recall

We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

However all of the following spaces are dihomotopy equivalent.



Thus, a stronger notion of equivalence is needed.

One solution

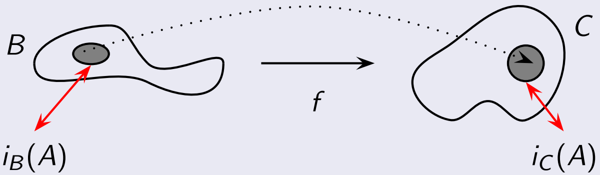
Idea (B, 2004)

Instead of working with just po-spaces work with po-spaces together with context.

Definition

Choose a po-space A (called the context). Consider po-spaces B together with a dimap $i_B : A \to B$ and consider morphisms which are dimaps such that

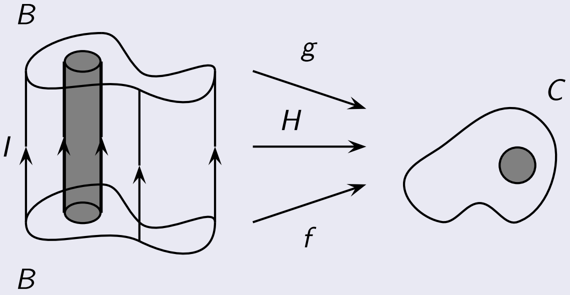
$$f(i_B(a)) = i_C(a) \text{ for all } a \in A$$



Equivalences using context

Definition

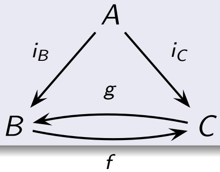
- A dihomotopy between $f, g : B \to C$ in the context of A is a dihomotopy $H : f \to g$ rel A.



Equivalences using context

Definition

- Write $f \simeq g$ if there is a chain of dihomotopies $f \to f_1 \leftarrow f_2 \to \ldots \leftarrow f_n \to g$.
- $i_B : A \to B, i_C : A \to C$ are dihomotopy equivalent if there are dimaps



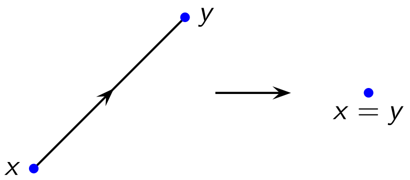
such that $g \circ f \simeq \mathrm{Id}_B$

and $f \circ g \simeq \mathrm{Id}_C$.

Example

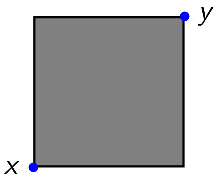
Let $A = \{x, y\}$ with $x \leq y$.
Then po-spaces under the context A are just po-spaces with two marked points, one of which is after the other.

In this category



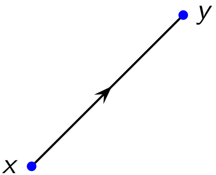
is not a dihomotopy equivalence since there is no dimap in the reverse direction.

In the same context (of two marked points) the dimaps



$$(t, t) \leftarrow t$$

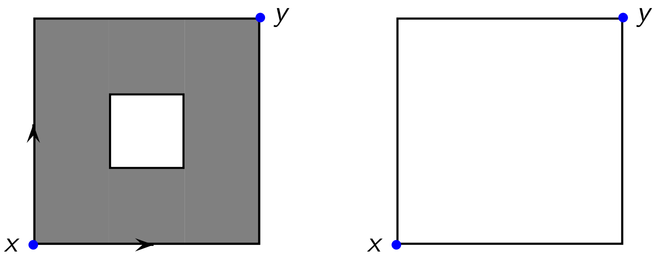
$$(x, y) \mapsto \max(x, y)$$



give a dihomotopy equivalence.

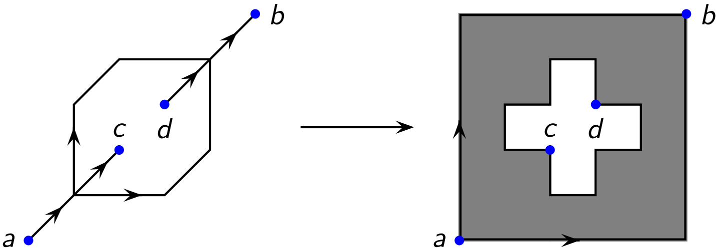
Another equivalence

$\vec{I} \times \vec{I}$ with a square removed and two marked points



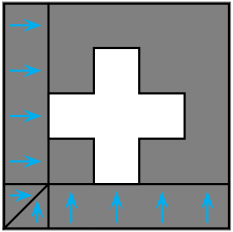
is dihomotopy equivalent to its boundary.

Let $A = \{a, b, c, d\}$. Then the inclusion

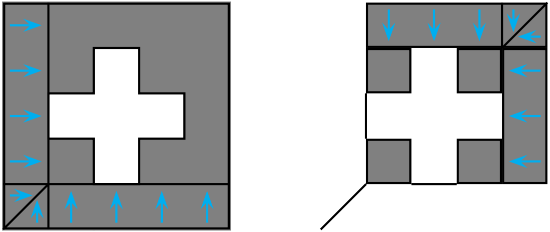


is a dihomotopy equivalence in the context of the four marked points.

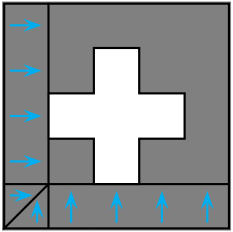
Sketch of the proof

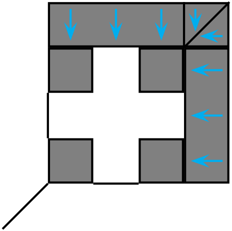


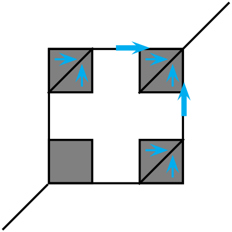
Sketch of the proof



Sketch of the proof







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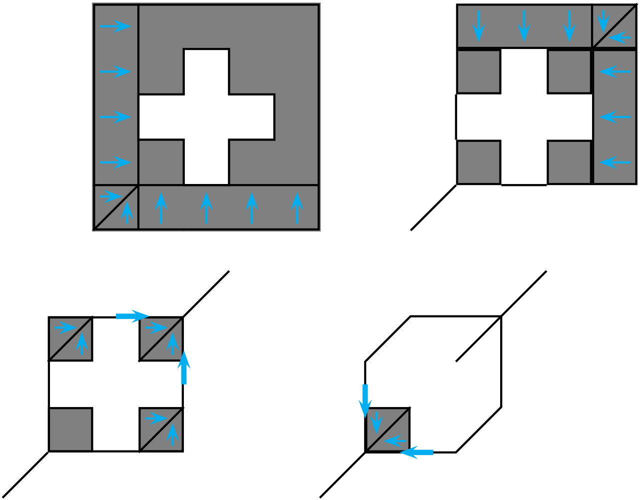


Directed paths (execution paths)

Definition

Let $x, y \in$ the po-space B.

- A dipath is a dimap $\vec{I} \to B$.
- Dipaths are dihomotopy equivalent if they are so in the context of their endpoints.
- Let $\vec{\pi}_1(B)(x, y)$ be the set of dihomotopy equivalence classes of dipaths from x to y.

Dipaths in equivalent spaces

Proposition (B, 2004)

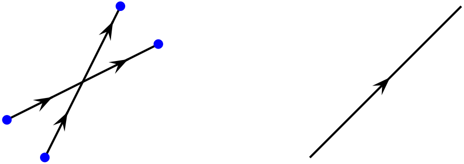
Given a dimap $f : B \to C$ respecting the context and $x, y \in A$ there is an induced map

$$\vec{\pi}_1(f)(x, y) : \vec{\pi}_1(B)(x_B, y_B) \to \vec{\pi}_1(C)(x_C, y_C).$$

If f is a dihomotopy equivalence then it is an isomorphism.

Example

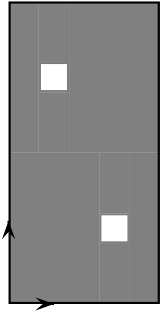
In the context of its four endpoints

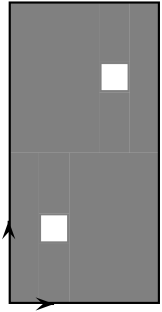


the left hand po-space is not dihomotopy equivalent to $\vec{I}$.

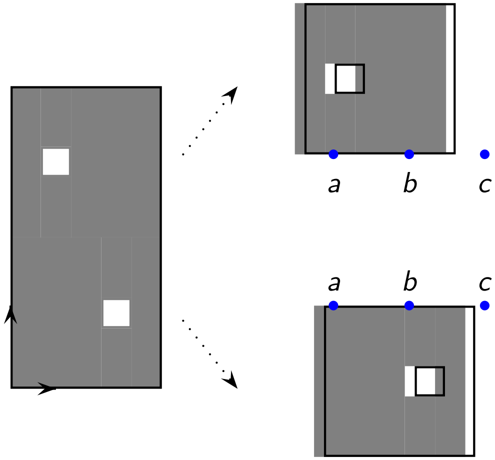
Compound examples

We would like to find equivalent po-spaces to the following examples by analyzing them piece-by-piece.

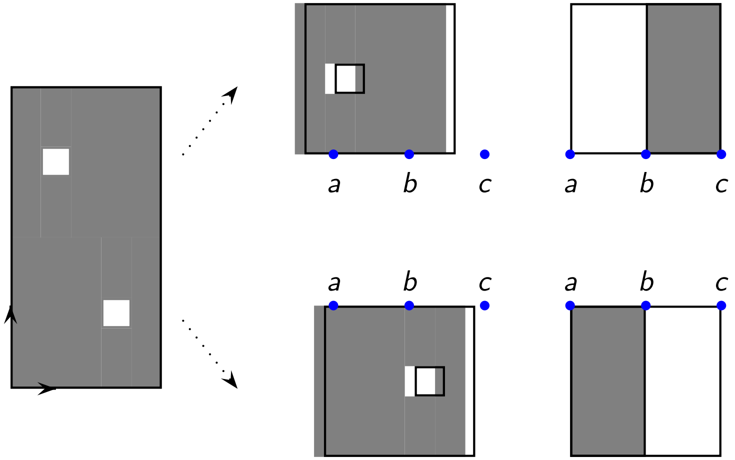




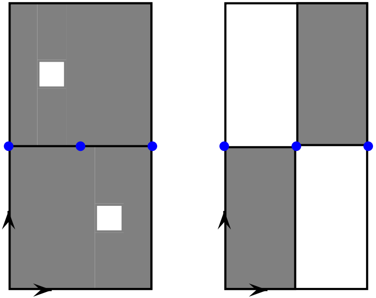
Equivalences of pieces



Equivalences of pieces



Patching the pieces together



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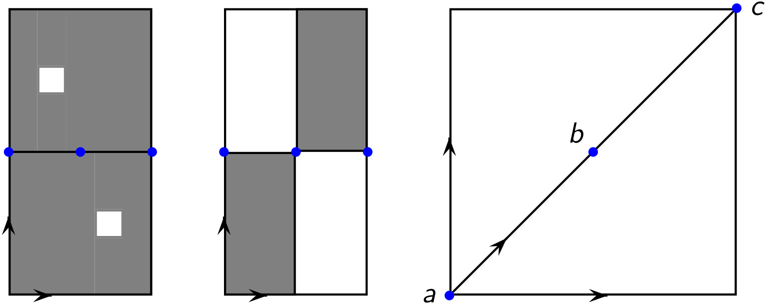
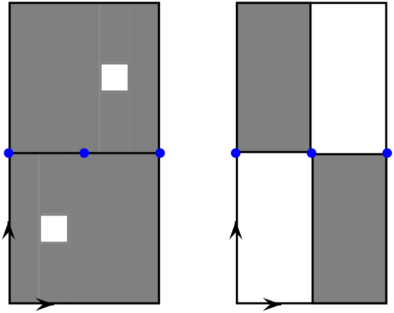
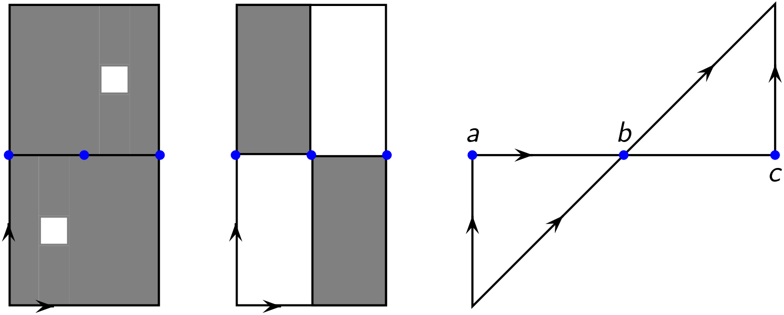


Second example

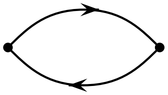


Second example



A more general model

We would like to model execution loops such as



These cannot be modeled by po-spaces.

However they can be modeled by local po-spaces.

Local po-spaces

Definition

- An order atlas is a open cover of po-spaces with compatible partial orders.
- A local po-space is a topological space together with an order atlas.
- A morphism of local po-spaces is a continuous map which respect the orders.

Equivalences of local po-spaces

Just as with po-spaces, we can define local po-spaces under some context A, and we consider morphisms which respect the context.

We can also define dihomotopy equivalences using context exactly the same way as with po-spaces.

Enter some machinery

A powerful framework for studying equivalences is given by model categories.

Definition

A model category is a category and with three distinguished classes of morphisms: weak equivalences, cofibrations, and fibrations satisfying five simple axioms.

The structure of a model category allows one to apply the machinery of homotopy theory.

A model for concurrent systems

Theorem (B-Worytkiewicz, 2006)

The category of local po-spaces under a context A embeds into a model category such that

- *the weak equivalences are the dihomotopy equivalences, and*
- *the cofibrations are the monomorphisms*
- *pushouts of weak equivalence with cofibrations are weak equivalences*

Summary

- Po-spaces and local po-spaces provide a good mathematical model for concurrent parallel computing.
- Using directed homotopies one can hope to cope with the "state space explosion" and find the essential schedules.
- Relative directed homotopies allow a piece-by-piece analysis of the model.
- All of the above can be studied in the framework of model categories.

Related Work

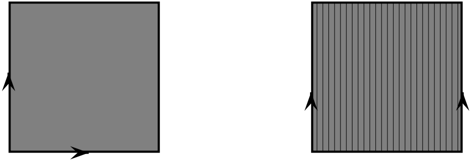
- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss the flag, in any dimension.
- E. Goubault and E. Haucourt (2005) gathered the dihomotopy classes into “components” to develop a static analyzer of concurrent parallel programs.

Open problems

- How can the new applications of directed homotopy theory to parallel computing be used to give new algorithms?
- What are the connections between between context and components of the fundamental category?
- Can the idea of context be used to develop a static analyzer?
- How can higher directed homotopy be used to dramatically simplify the state space?
- Is there a directed homology theory that detects deadlock states or counts essential schedules?

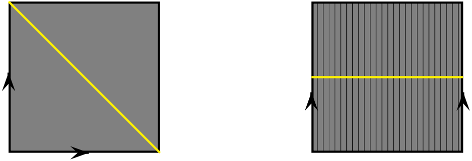
Non-discrete context

Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$



and glue them together along the yellow lines.

Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$



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