

# Directed topology and concurrent computing

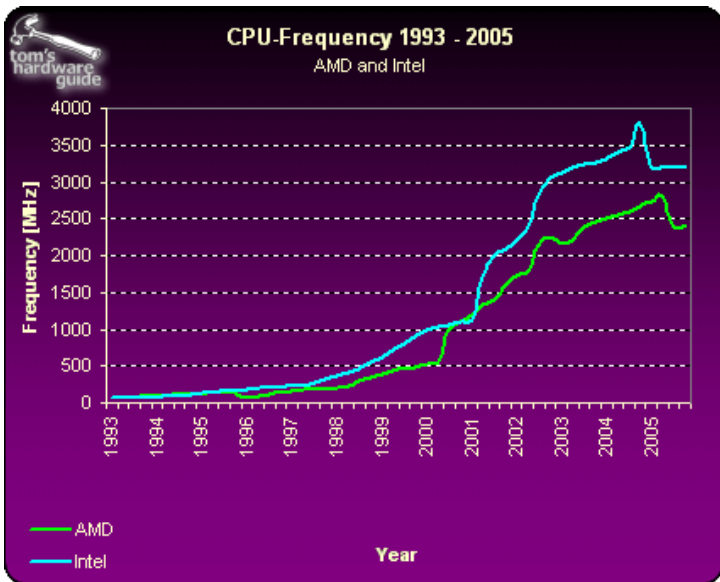
Peter Bubenik

Cleveland State University  
[http://academic.csuohio.edu/bubenik\\_p/](http://academic.csuohio.edu/bubenik_p/)

University of Akron  
April 24, 2008

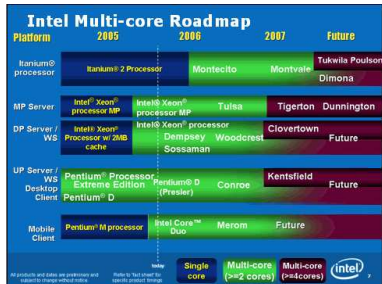
- 1 Motivation
- 2 Directed topology
- 3 The fundamental category
- 4 A Van Kampen Theorem

# The end of Moore's Law?

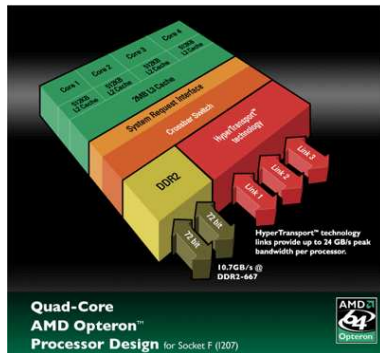


# Example: Multi-core processors

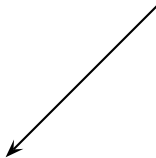
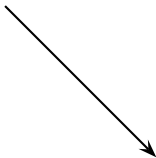
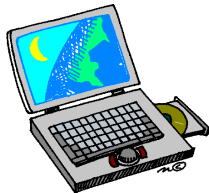
Intel



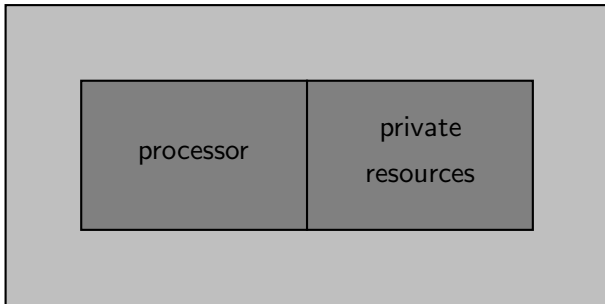
AMD



# Example: Internet database

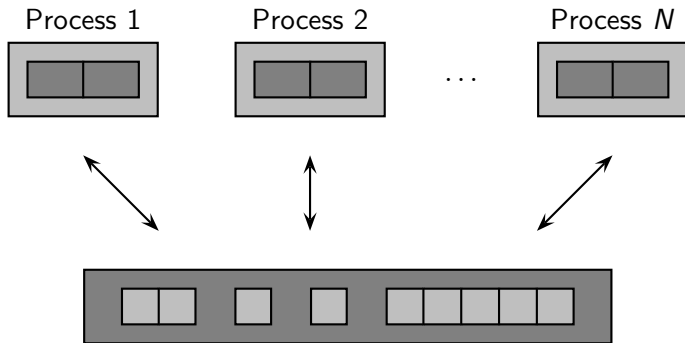


Process



A process with its own private resources

# Concurrent parallel computing



Several processes with shared resources

## Definition

- A **partial order** denoted  $\leq$  on a set  $S$  is a reflexive, transitive, anti-symmetric relation  $R = \{(x, y) \in S \times S \mid x \leq y\}$ .
- A **po-space** is a topological space  $U$  with a partial order  $\leq$  which is a closed subset of  $U \times U$ .

## Remark

*Subspaces and products of po-spaces inherit a po-space structure.*

## Definition

A **directed map (dimap)** is a continuous map  $f : U_1 \rightarrow U_2$  between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

## Example

$\mathbb{R}$  is a po-space with the usual ordering:  $x \leq y \iff y - x$  is non-negative.

## Example

$\mathbb{R}^n$  is a po-space with the product order:

$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \iff x_i \leq y_i$  for  $1 \leq i \leq n$ .

## Notation

Let  $\vec{I} = [0, 1]$  and  $\vec{I}^n = [0, 1]^n$  denote the po-spaces whose orderings are induced from  $(\mathbb{R}, \leq)$  and  $(\mathbb{R}^n, \leq)$ .

# Directed homeomorphisms

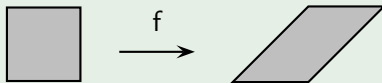
## Definition

A **directed homeomorphism** is a directed map  $f : X \rightarrow Y$  such that there exists a directed map  $g : Y \rightarrow X$  where  $g \circ f$  is the identity on  $X$  and  $f \circ g$  is the identity on  $Y$ .

## Example

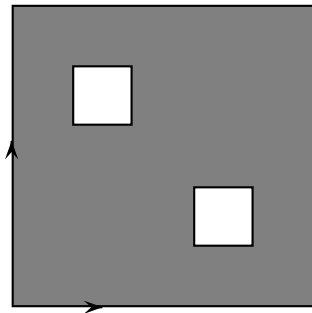
A directed map which is a homeomorphism need not be a directed homeomorphism.

Consider the shear map  $f : \vec{I}^2 \xrightarrow{\cong} Y \subset \mathbb{R}^2$  given by  $(x, y) \mapsto (x + y, y)$ .



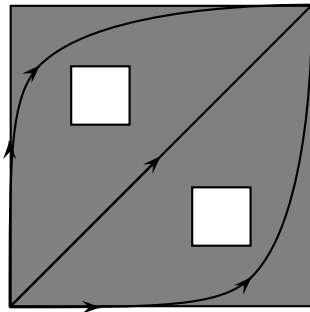
## Definition

A **directed path** in a pospace  $X$  is a directed map  $\gamma : \vec{I} \rightarrow X$ .



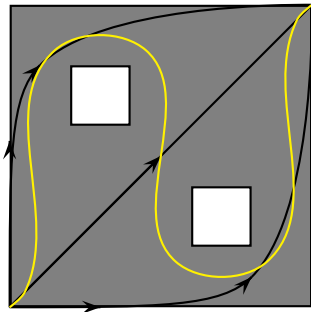
## Definition

A **directed path** in a pospace  $X$  is a directed map  $\gamma : \vec{I} \rightarrow X$ .



## Definition

A **directed path** in a pospace  $X$  is a directed map  $\gamma : \vec{I} \rightarrow X$ .



There are paths which are not homotopic to directed paths.

# A concurrent system

## Example

2 processes using 2 shared resources  $a$  and  $b$  which can only be used by one process at a time

## Notation

$P_x$  - a process locks resource  $x$

$V_x$  - a process releases resource  $x$

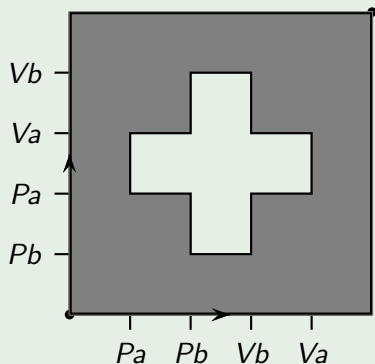
## Program

The first process:       $P_a$     $P_b$     $V_b$     $V_a$

The second process:    $P_b$     $P_a$     $V_a$     $V_b$

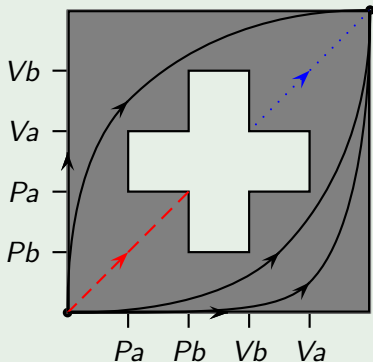
# The Swiss flag

## Example



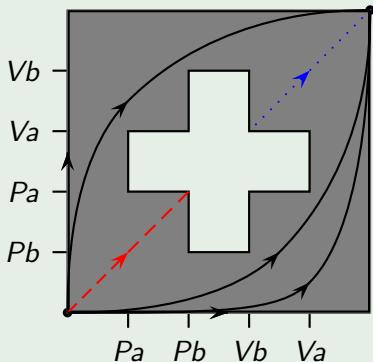
# The Swiss flag

## Example



# The Swiss flag

## Example



**Problem:** Uncountably many states and execution paths.

We would like to reduce the size of the state space and the set of execution paths.

We have seen that directed homeomorphisms are too rigid.

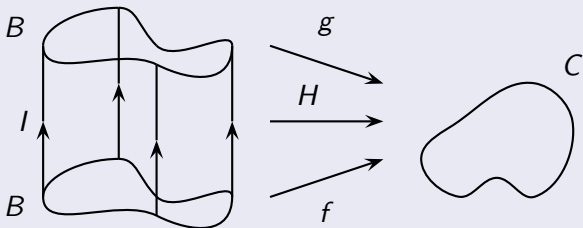
## Idea

*We will use directed homotopies.*

# Directed homotopies

## Definition

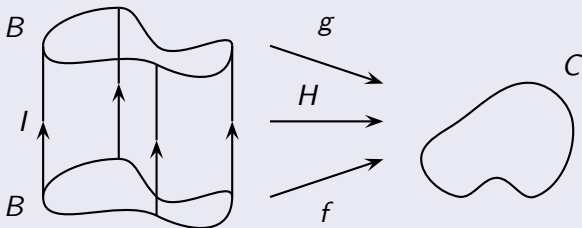
A **homotopy** between directed maps  $f, g : B \rightarrow C$  is a directed map  $H : B \times \vec{I} \rightarrow C$  restricting to  $f$  and  $g$ . Write  $H : f \xrightarrow{\sim} g$ .



# Directed homotopies

## Definition

A **homotopy** between directed maps  $f, g : B \rightarrow C$  is a directed map  $H : B \times \vec{I} \rightarrow C$  restricting to  $f$  and  $g$ . Write  $H : f \xrightarrow{\cong} g$ .

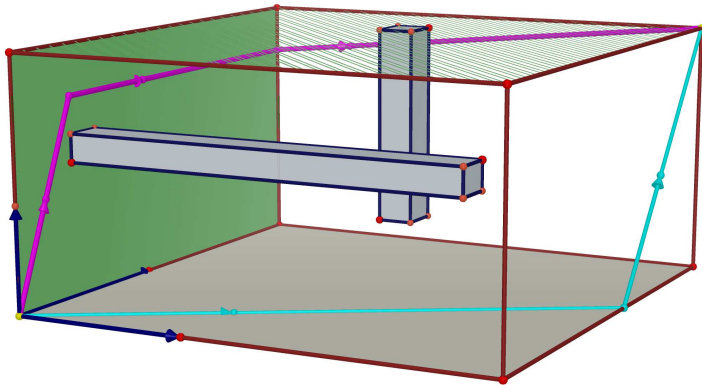


## Definition

Directed maps  $f, g$  are homotopic if there is a chain of homotopies

$$f \xrightarrow{\cong} f_1 \xleftarrow{\cong} f_2 \xrightarrow{\cong} \dots \xleftarrow{\cong} f_n \xrightarrow{\cong} g.$$

# A room with two barriers



Two directed paths which are homotopic as paths, but not as directed paths.

## Definition

Let  $x, y \in$  the po-space  $B$ .

- A **directed path** is a directed map  $\vec{I} \rightarrow B$ .
- Directed paths are **homotopy equivalent** if they are so relative to their endpoints.
- Let  $\vec{\pi}_1(B)(x, y)$  be the set of homotopy equivalence classes of directed paths from  $x$  to  $y$ .

## Definition

Let  $x, y \in$  the po-space  $B$ .

- A **directed path** is a directed map  $\vec{I} \rightarrow B$ .
- Directed paths are **homotopy equivalent** if they are so relative to their endpoints.
- Let  $\vec{\pi}_1(B)(x, y)$  be the set of homotopy equivalence classes of directed paths from  $x$  to  $y$ .
- As  $x$  and  $y$  vary over  $B$ , these sets assemble to give the **fundamental category**  $\vec{\pi}_1(B)$ , which we will describe next.

## Definition

- For  $x \in X$ , the **fundamental group**  $\pi_1(X, x)$  is the set of homotopy classes of paths beginning and ending at  $x$ .

# The fundamental group and groupoid

## Definition

- For  $x \in X$ , the **fundamental group**  $\pi_1(X, x)$  is the set of homotopy classes of paths beginning and ending at  $x$ .
- The **fundamental groupoid**  $\pi_1(X)$ , is a category whose objects are the points in  $X$ , and whose morphisms  $\pi_1(X)(x, y)$  are the homotopy classes of paths from  $x$  to  $y$ . This category is a **groupoid** since every morphism is invertible.

## Remark

*The existence of composition with associativity and identity is built into the definition of category.*

# The fundamental category

## Definition

The **fundamental category**  $\vec{\pi}_1(X)$  has objects the points in  $X$  and morphisms  $\vec{\pi}_1(B)(x, y)$  the homotopy classes of directed paths from  $x$  to  $y$ . Since these morphisms are not invertible, this category is not a groupoid.

# Full subcategories of the fundamental category

## Problem

*The fundamental category is enormous.*

## Plan

*We would like to find a “small” subcategory of the fundamental category which still contains useful information.*

# Full subcategories of the fundamental category

## Problem

*The fundamental category is enormous.*

## Plan

*We would like to find a “small” subcategory of the fundamental category which still contains useful information.*

## Definition

Given  $A \subseteq X$ , let  $\vec{\pi}_1^X(A)$  denote the subcategory whose objects are points in  $A$  and whose morphisms are homotopy classes of paths in  $X$ .

# The fundamental bipartite graph

Assume  $X$  is a compact pospace.

## Definition

Let  $\text{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$ .

Let  $\text{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}$ .

# The fundamental bipartite graph

Assume  $X$  is a compact pospace.

## Definition

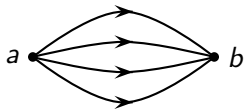
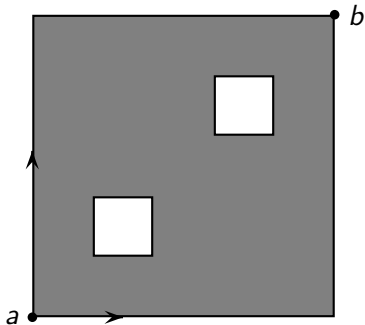
Let  $\text{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$ .

Let  $\text{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}$ .

## Definition

The **fundamental bipartite graph** of  $X$  is  $\bar{\pi}_1^X(\text{Min } X \cup \text{Max } X)$ .

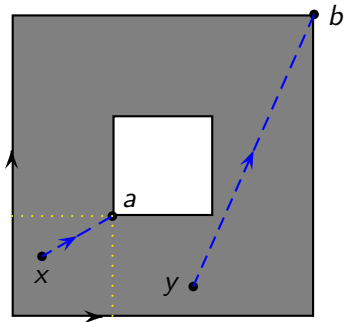
# Example of the fundamental bipartite graph



# Future retracts

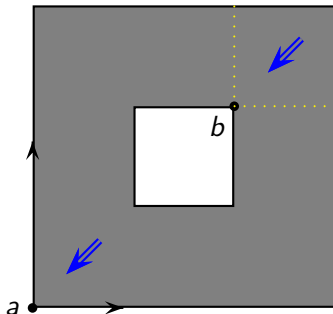
Definition (Grandis, 2005)

A **future retract** of  $\vec{\pi}_1(X)$  moves each  $x \in X$  along a directed path in  $X$  to a point  $x^+$  which “has the same future”.



## Definition (Grandis, 2005)

A **past retract** of  $\vec{\pi}_1(X)$  moves each  $x \in X$  backwards along a directed path in  $X$  to a point  $x^-$  which “has the same past”.



## Definition

An **extremal model** is given by a sequence of future retracts and past retracts each of which gives a bijection on minimal and maximal points.

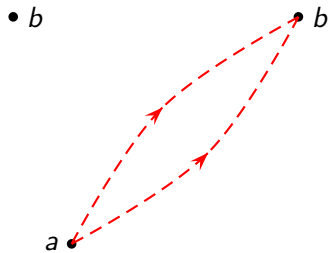
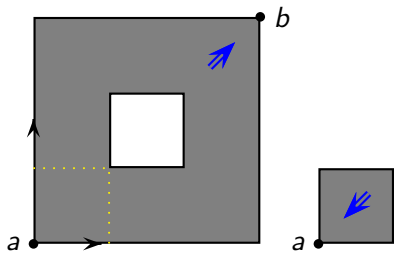
## Definition

An **extremal model** is given by a sequence of future retracts and past retracts each of which gives a bijection on minimal and maximal points.

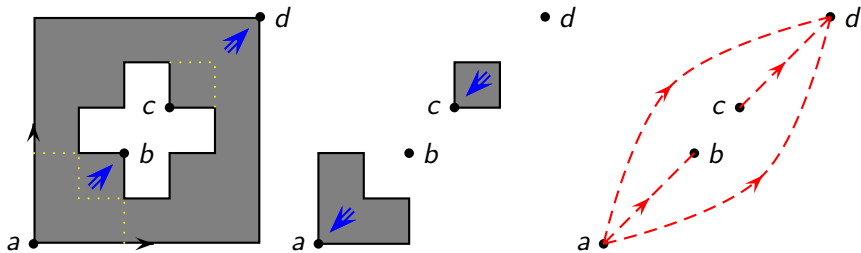
## Theorem (B,2008)

*An extremal model preserves the fundamental bipartite graph.*

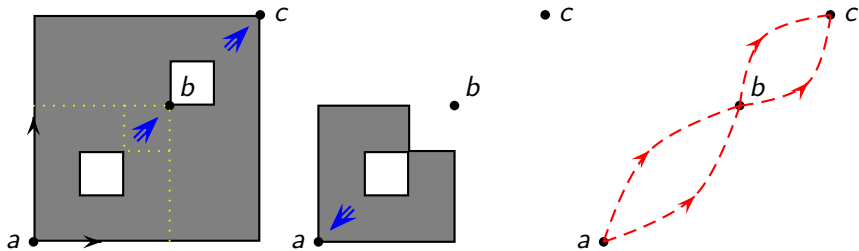
# Examples of extremal models (1)



## Examples of extremal models (2)



# Examples of extremal models (3)



# Piece-by-piece simplification

Simple examples, such as the Swiss flag, can easily be analyzed and simplified by hand.

We would like to have algorithms that do the same for much larger, and higher-dimensional examples.

To do so, we would like a way to simplify small pieces of the fundamental category, and then patch these simplifications together.

# Van Kampen Theorem for the fundamental category

Theorem (Grandis 2003, Goubault 2003)

Assume  $X = \text{int}(X_1) \cup \text{int}(X_2)$  and let  $X_0 = X_1 \cap X_2$ . Then the pushout of pospaces:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \dashrightarrow & X \end{array}$$

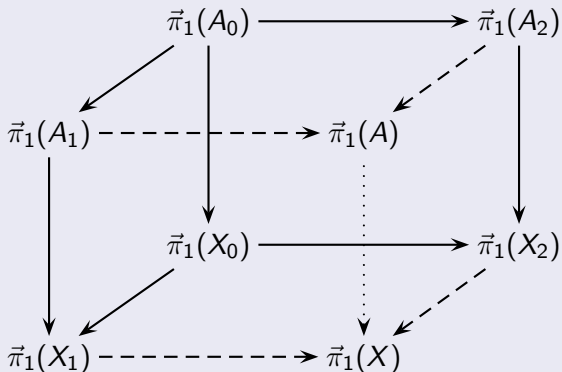
induces a pushout of fundamental categories:

$$\begin{array}{ccc} \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_2) & \dashrightarrow & \vec{\pi}_1(X) \end{array}$$

# Van Kampen Theorem for extremal models

Theorem (B,2008)

*Given compatible future retracts (solid arrows)*



*there is an induced retraction on the pushouts (dotted arrow).  
Furthermore, this can be extended to extremal models.*

- Po-spaces provide a good mathematical model for concurrent parallel computing.
- Using directed homotopies one can hope to cope with the “state space explosion”.
- The fundamental bipartite graph captures the essential schedules.
- Future and past retracts of the fundamental category which preserve minimal and maximal points can be combined to form an extremal model.
- The extremal model preserves the fundamental bipartite graph.
- This model is amenable to a piece-by-piece analysis.

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.
- E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.

- How can higher directed homotopy be used to dramatically simplify the state space?
- The fundamental bipartite graph detects deadlock states and captures the essential schedules. Can this be turned into a homology theory?