An introduction to directed homotopy theory

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Directed spaces

Definition

A directed space is a topological space X together with a set dX of continuous maps $[0,1] \to X$ called directed paths satisfying the following:

1. all constant paths are directed paths;
2. directed paths are closed under concatenation; and
3. if γ is a directed path and $f : [0,1] \to [0,1]$ is a non-decreasing continuous map then $\gamma \circ f$ is a directed path.

A directed map $f : (X, dX) \to (Y, dY)$ is a continuous map $f : X \to Y$ such that $dX \subseteq dY$.

Examples of directed spaces

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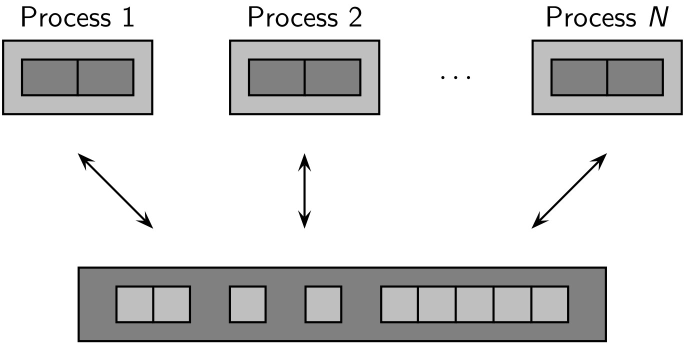
- Given two directed spaces X and Y, then $X \times Y$ is a directed space with $d(X \times Y) = dX \times dY$ where $(f,g)(t) = (f(t), g(t))$.
- If X is a directed space and $A \subseteq X$, then A is a directed space with dA equal to the subset of paths in dX whose image is in A.

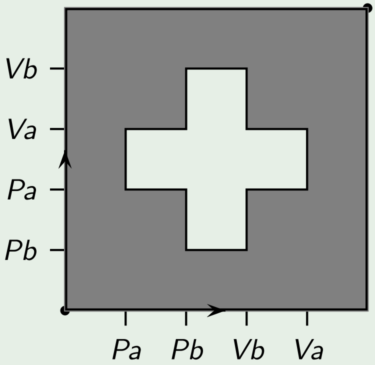
Concurrent parallel computing



Several processes with shared resources

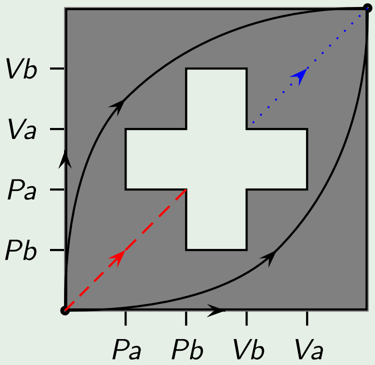
The Swiss flag

Example



The Swiss flag

Example



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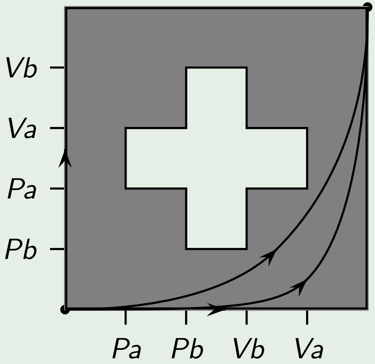
Example

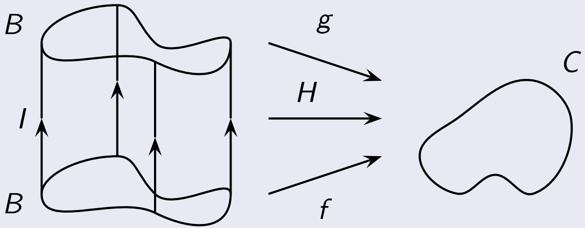


Problem: Uncountably many states and execution paths.

Directed homotopies

Definition

A homotopy between directed maps $f, g : B \to C$ is a directed map $H : B \times \vec{I} \to C$ restricting to f and g. Write $H : f \xrightarrow{\simeq} g$.



Definition

Directed maps f, g are homotopic if there is a chain of homotopies

$$f \xrightarrow{\simeq} f_1 \xleftarrow{\simeq} f_2 \xrightarrow{\simeq} \dots \xleftarrow{\simeq} f_n \xrightarrow{\simeq} g.$$

Equivalence classes of directed paths

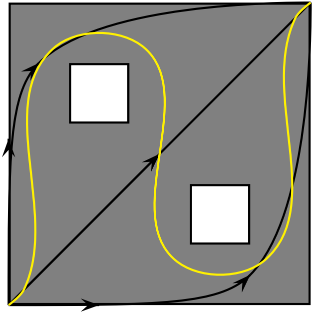
Definition

Directed paths are homotopy equivalent if they are so relative to their endpoints.

Remark

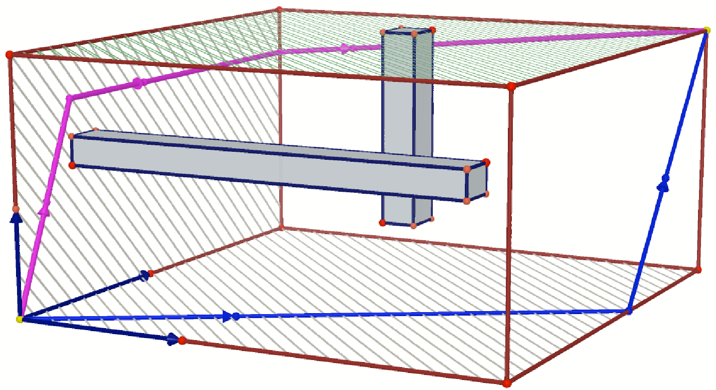
Directed paths up to homotopy are significantly different from paths up to homotopy!

Directed paths



There are paths which are not homotopic to directed paths.

A room with two barriers



Two directed paths which are homotopic as paths, but not as directed paths.

The fundamental category

Definition

The fundamental category $\vec{\pi}_1(X)$ has

- objects: the points in X
- morphisms: homotopy classes of directed paths

Remark

The existence of composition with associativity and identity is built into the definition of a category.

Full subcategories of the fundamental category

Problem

The fundamental category is enormous.

Plan

We would like to derive a "small" category from the fundamental category that still contains useful information.

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Definition

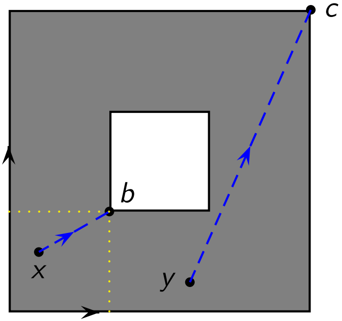
Given $A \subseteq X$, let $\vec{\pi}_1(X, A)$ have

- objects: points in A
- morphisms: homotopy classes of paths in X

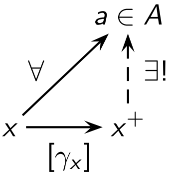
Future retracts

Definition

A future retract of $\vec{\pi}_1(X)$ moves each $x \in X$ along a directed path in X to a point x^+ which “has the same future”.



$$P^+ : \vec{\pi}_1(X) \to \vec{\pi}_1(X, A)$$



Retracts for triples

For $A \subseteq B \subseteq X$, one can similarly define future retracts

$$P^+ : \vec{\pi}_1(X, B) \to \vec{\pi}_1(X, A).$$

This functor is a left adjoint to the inclusion functor.

Dually, one has past retracts.

Extremal models

Definition (B)

An extremal model is a chain of future retracts and past retracts

$$\vec{\pi}_1(X) \xrightarrow{P_1^+} \vec{\pi}_1(X, X_1) \xrightarrow{P_2^-} \vec{\pi}_1(X, X_2) \xrightarrow{P_3^+} \dots \xrightarrow{P_n^\pm} \vec{\pi}_1(X, A),$$

such that $\mathrm{Min}(X) \cup \mathrm{Max}(X) \subseteq A$.

Extremal models and the fundamental group

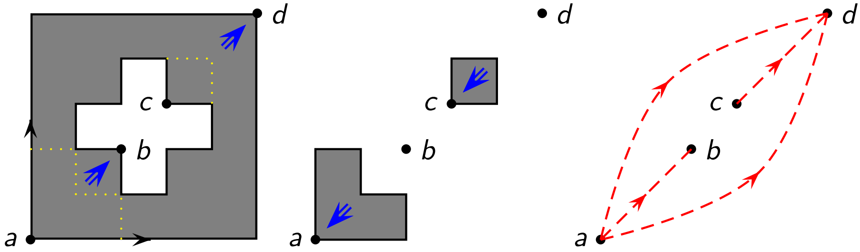
Let X be a path–connected space with dX all paths.
Choose $x \in X$.

There is a unique functor $\vec{\pi}_1(X) \rightarrow \vec{\pi}_1(X, x)$.

This functor is a future retract, a past retract, and a minimal extremal model.

It coincides with the functor from the fundamental groupoid to the fundamental group.

An extremal model for the Swiss flag



An extremal model of $\vec{S}^1$

Let $x \in \vec{S}^1$.

There is a future retract

$$P^+ : \vec{\pi}_1(\vec{S}^1) \to \vec{\pi}_1(\vec{S}^1, x) \cong (\mathbb{N}, +).$$

It is a minimal extremal model.

Van Kampen Theorem for the fundamental category

Theorem (Grandis 2003, Goubault 2003)

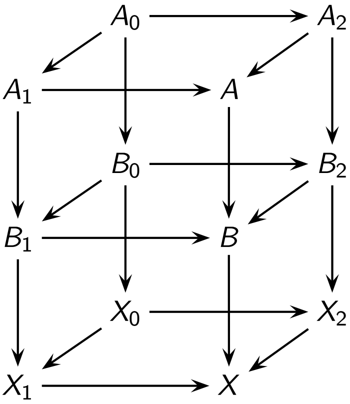
Assume $X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$ *and let* $X_0 = X_1 \cap X_2$. *Then the pushout of directed spaces :*

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \dashrightarrow & X \end{array}$$

induces a pushout of fundamental categories:

$$\begin{array}{ccc} \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_2) & \dashrightarrow & \vec{\pi}_1(X) \end{array}$$

Compatible triples



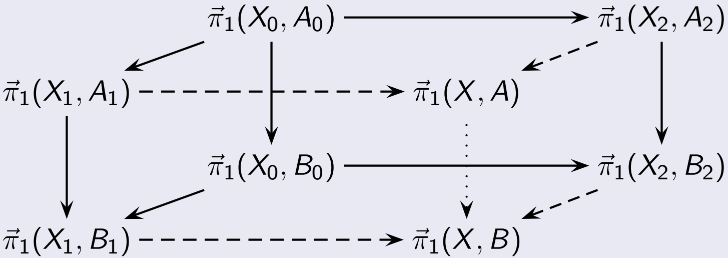
Compatible retracts

$$
\begin{array}{ccccc}
 & & \vec{\pi}_1(X_0, A_0) & \longrightarrow & \vec{\pi}_1(X_2, A_2) \\
 & \swarrow & \uparrow P_0^+ & & \uparrow P_2^+ \\
\vec{\pi}_1(X_1, A_1) & & \vec{\pi}_1(X_0, B_0) & \longrightarrow & \vec{\pi}_1(X_2, B_2) \\
\uparrow P_1^+ & \swarrow & & & \\
\vec{\pi}_1(X_1, B_1) & & & &
\end{array}
$$

A van Kampen theorem for full subcategories

Theorem (B)

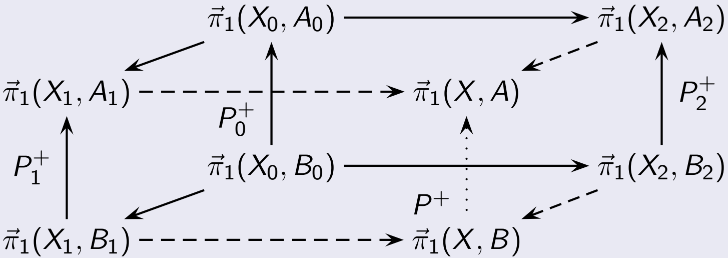
The inclusions above induce the following pushout in the arrow category on **Cat**.



A van Kampen theorem for future (past) retracts

Theorem (B)

There is an induced retraction P^+, which is a pushout.

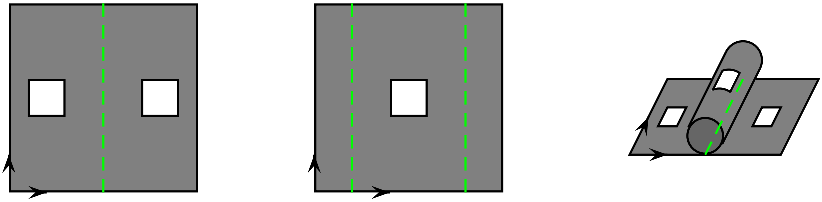


A van Kampen theorem for extremal models

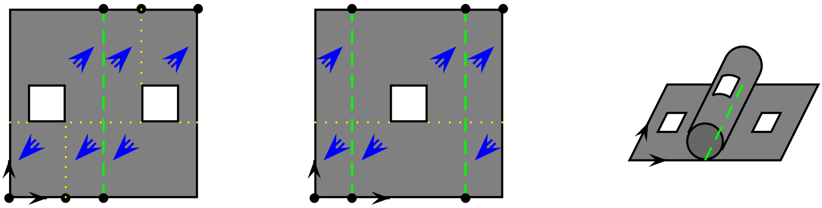
Theorem (B)

The pushout of compatible extremal models is an extremal model.

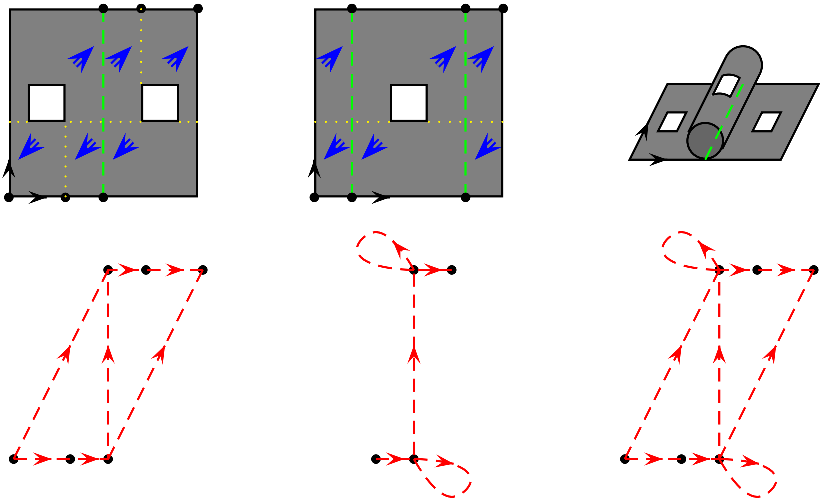
Van Kampen for extremal models example



Van Kampen for extremal models example



Van Kampen for extremal models example



Summary

- Directed spaces provide a good mathematical model for concurrent parallel computing.
- Directed paths up to homotopy are different from paths up to homotopy.
- The homotopy classes of directed paths assemble into the fundamental category.
- Minimal extremal models provide a way to generalize the fundamental group to directed spaces.
- There is a van Kampen theorem for extremal models.

Applications

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.
- E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.

Open problems

- The fundamental bipartite graphs detects deadlocks and captures the essential schedules. Is this part of a homology theory?
- What can we do with higher directed homotopy?