An introduction to the geometry and topology of point cloud data

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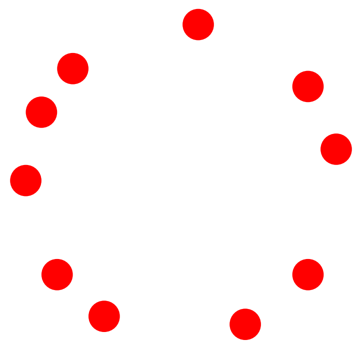
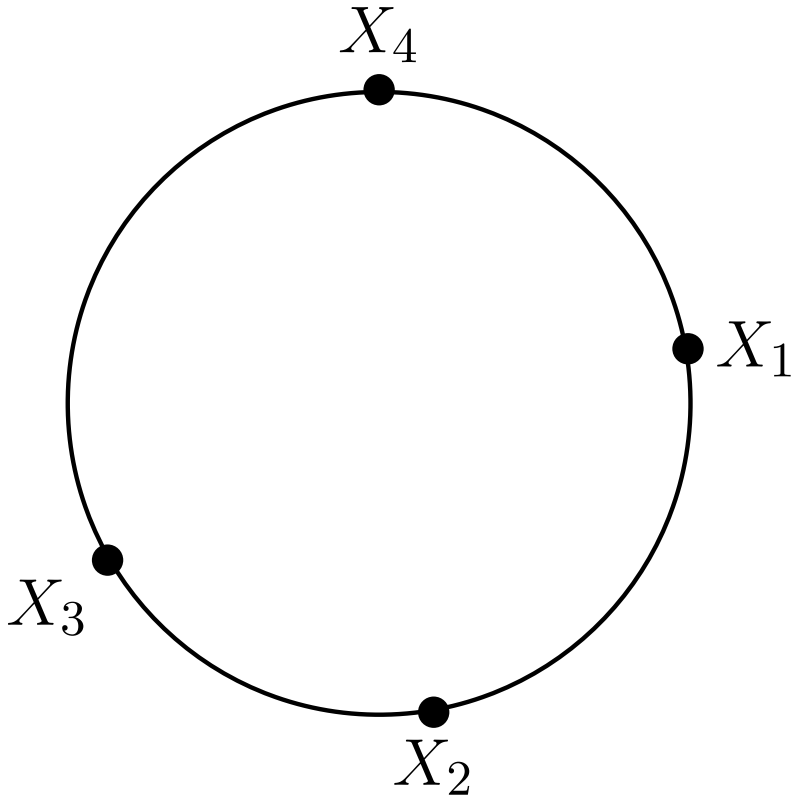
1. Point Cloud Data

Motivation: Given a set of points that “looks like a circle”

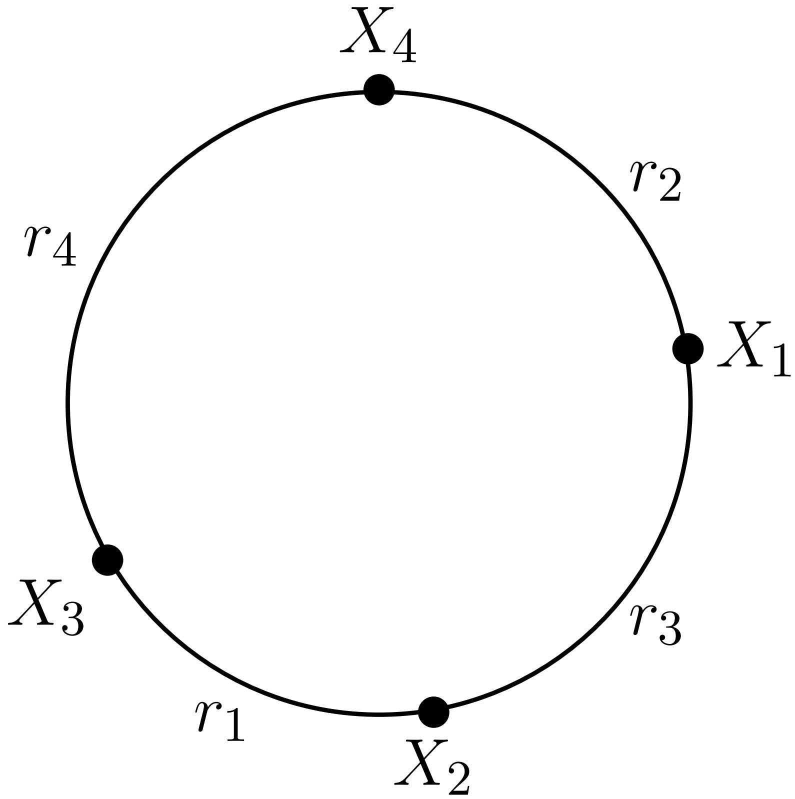


We would like to be able to say so mathematically.

2. The Rips complex



2. The Rips complex



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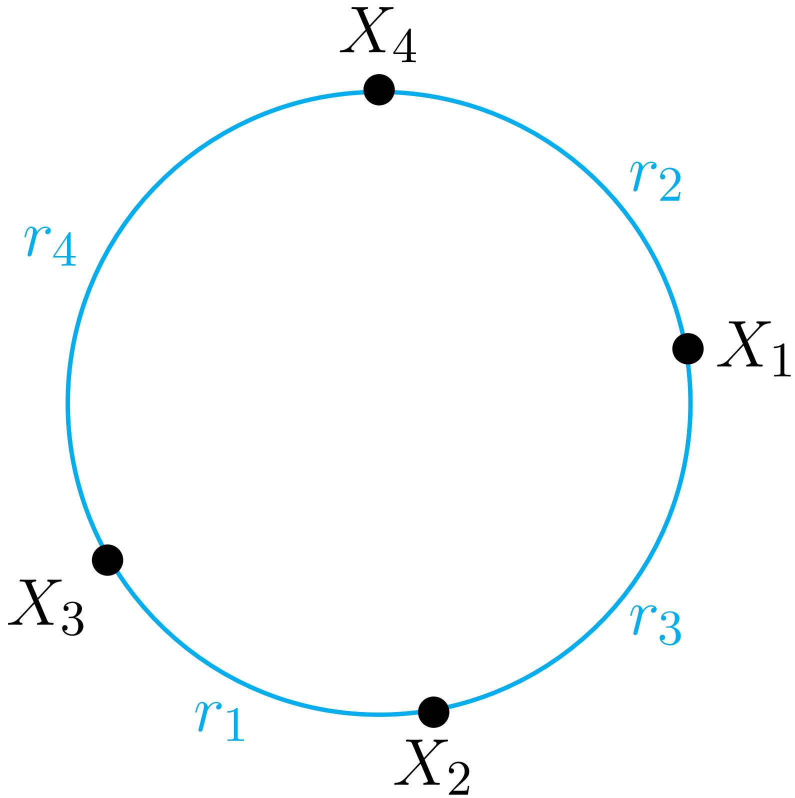

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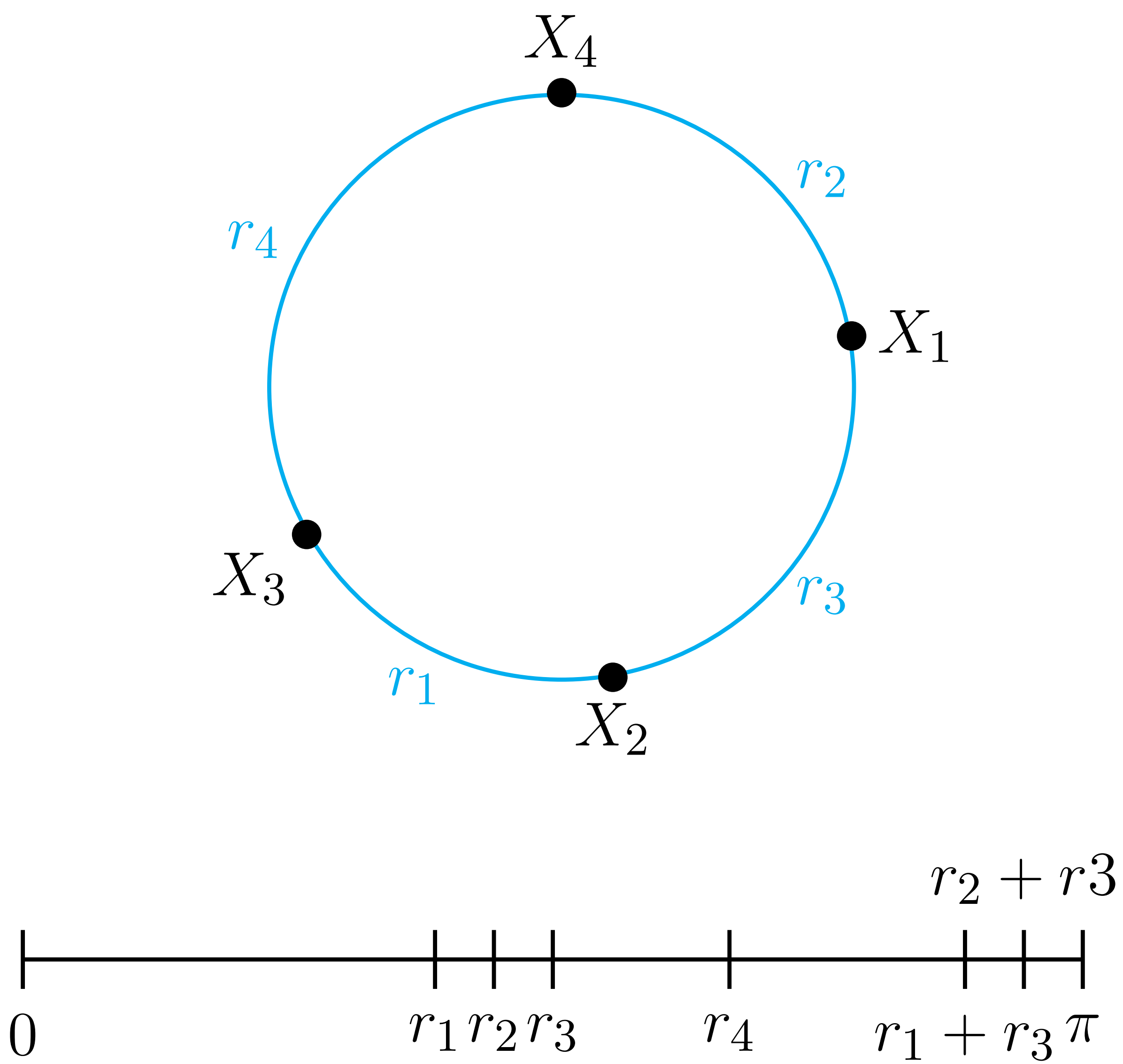

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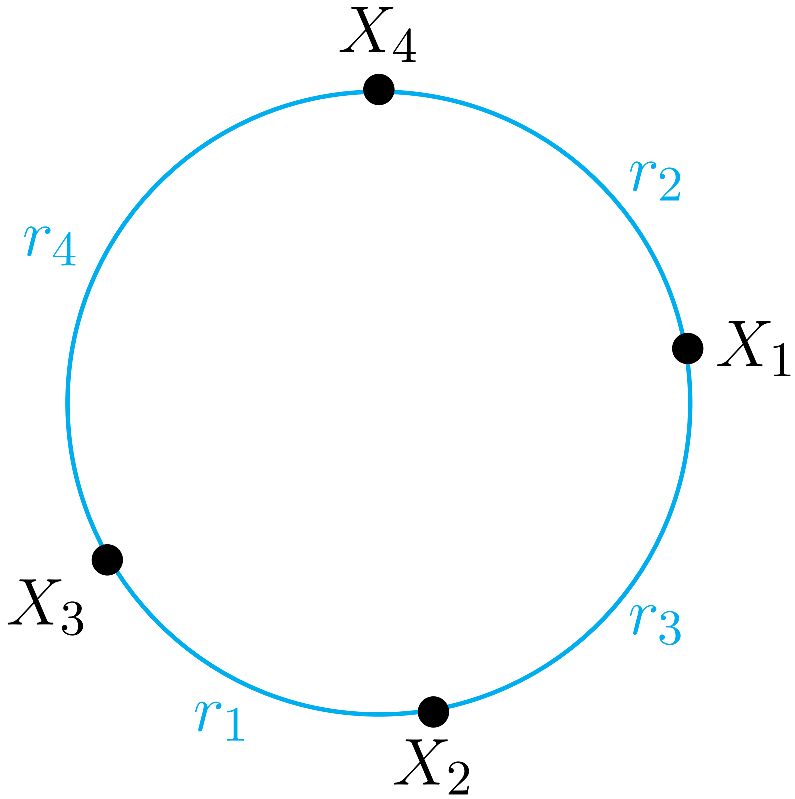

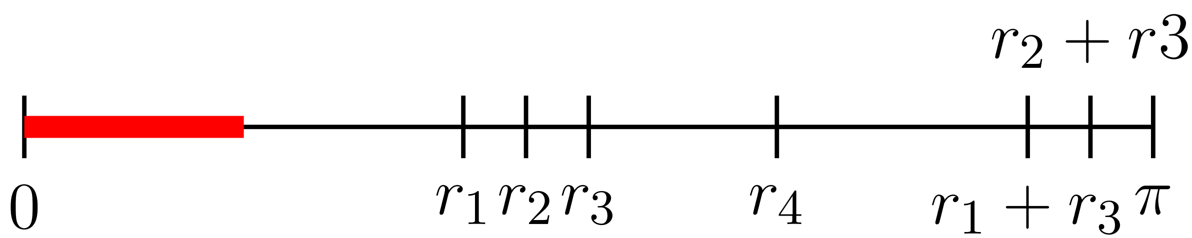

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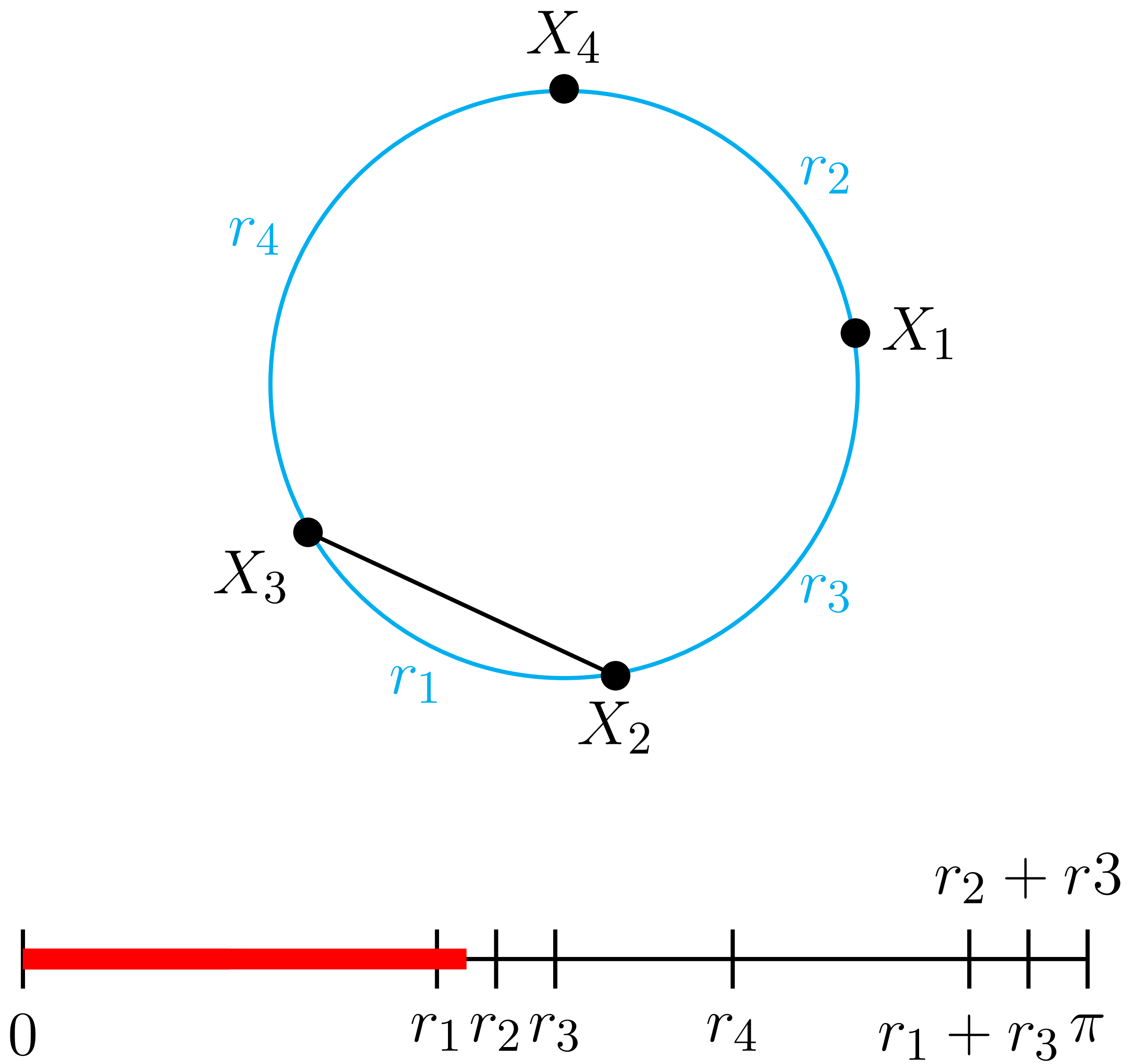

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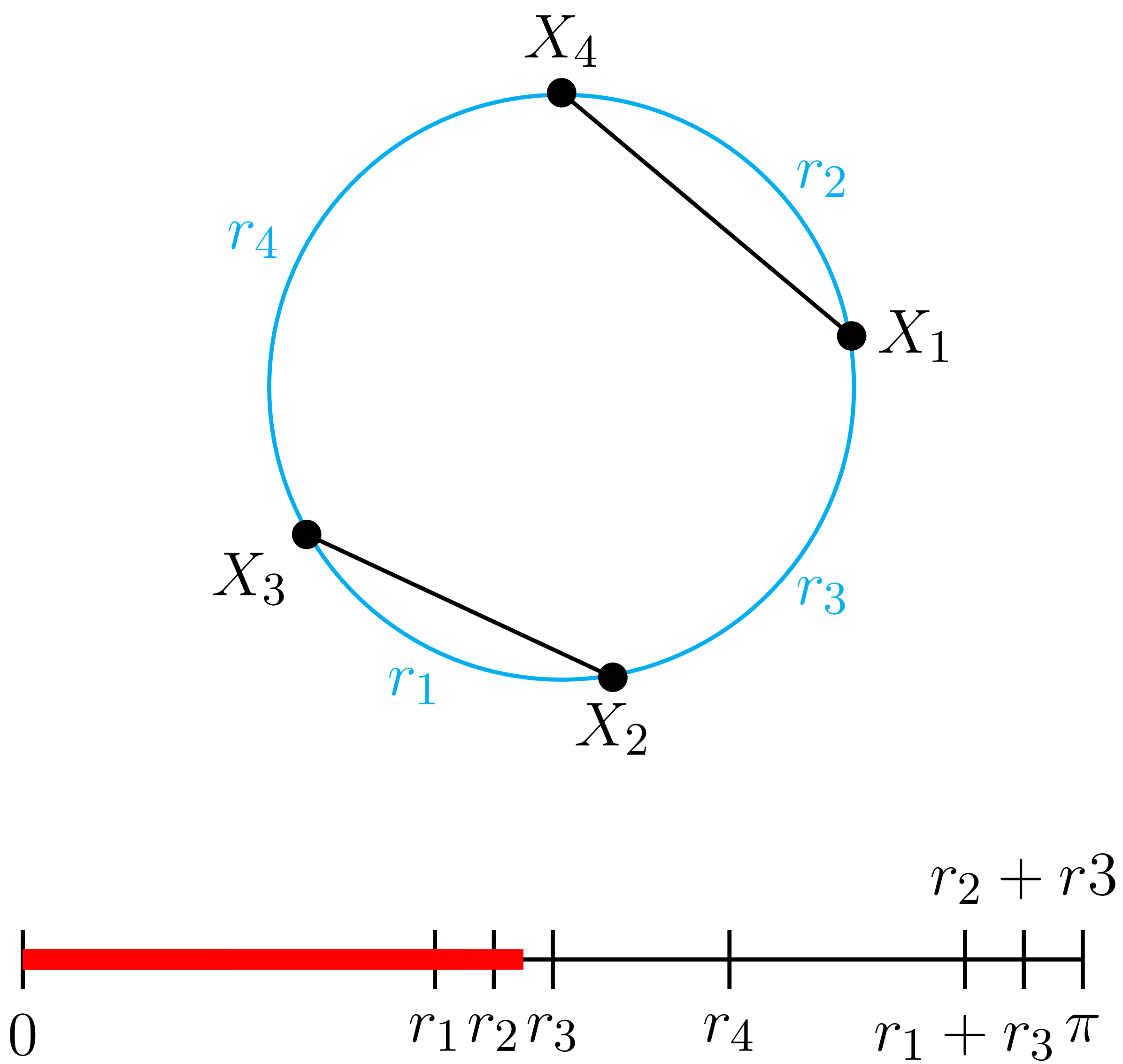

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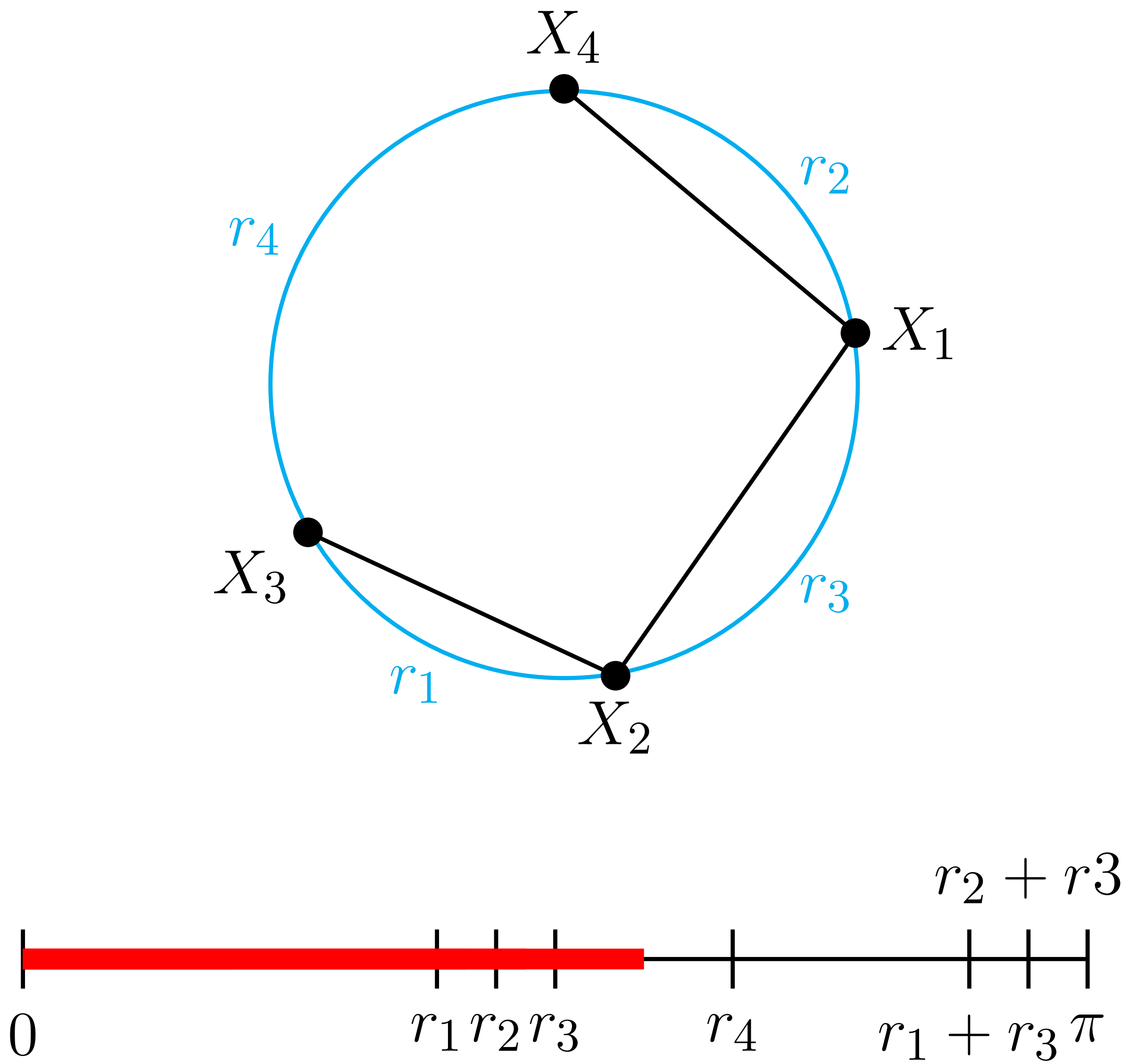

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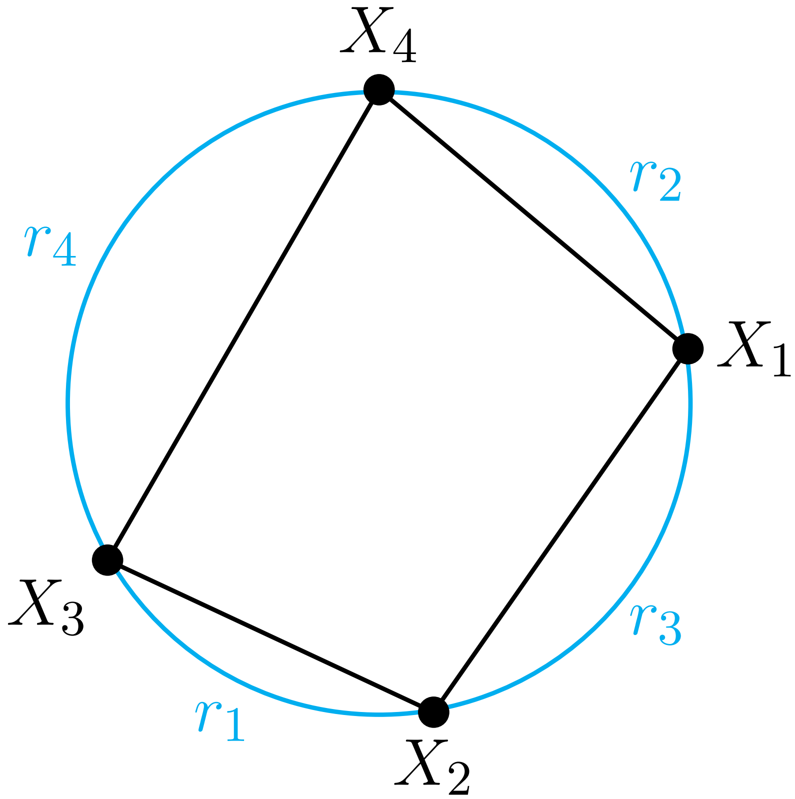

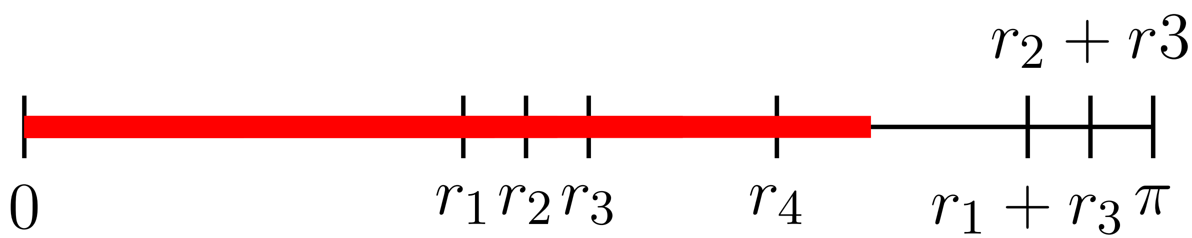

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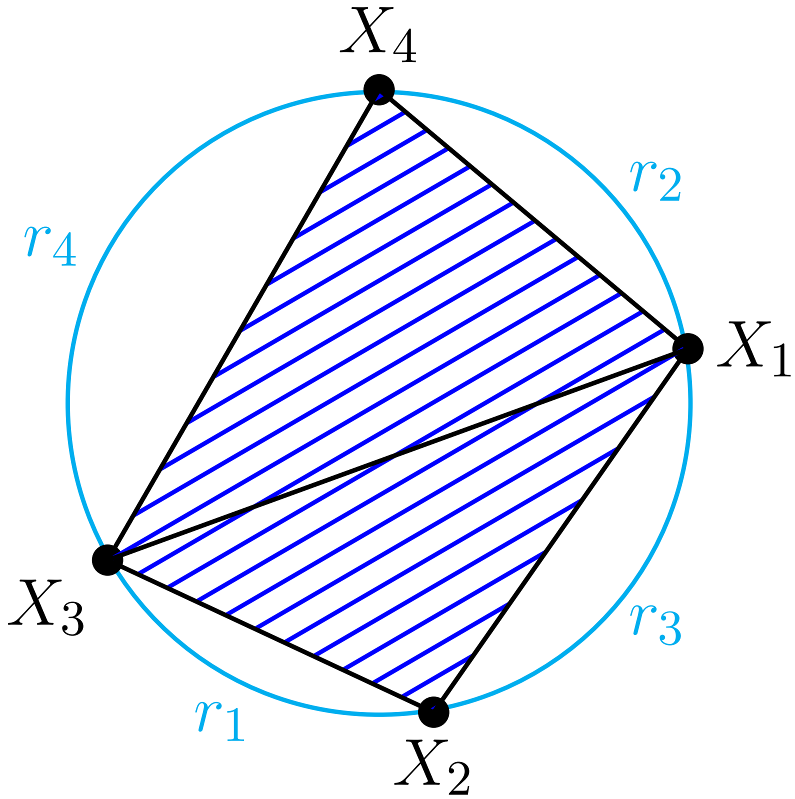

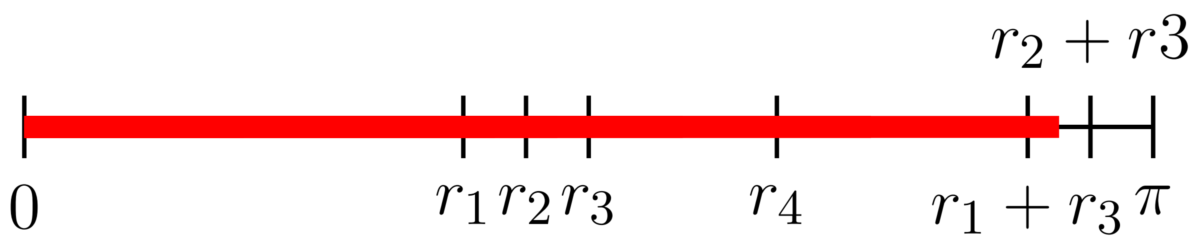

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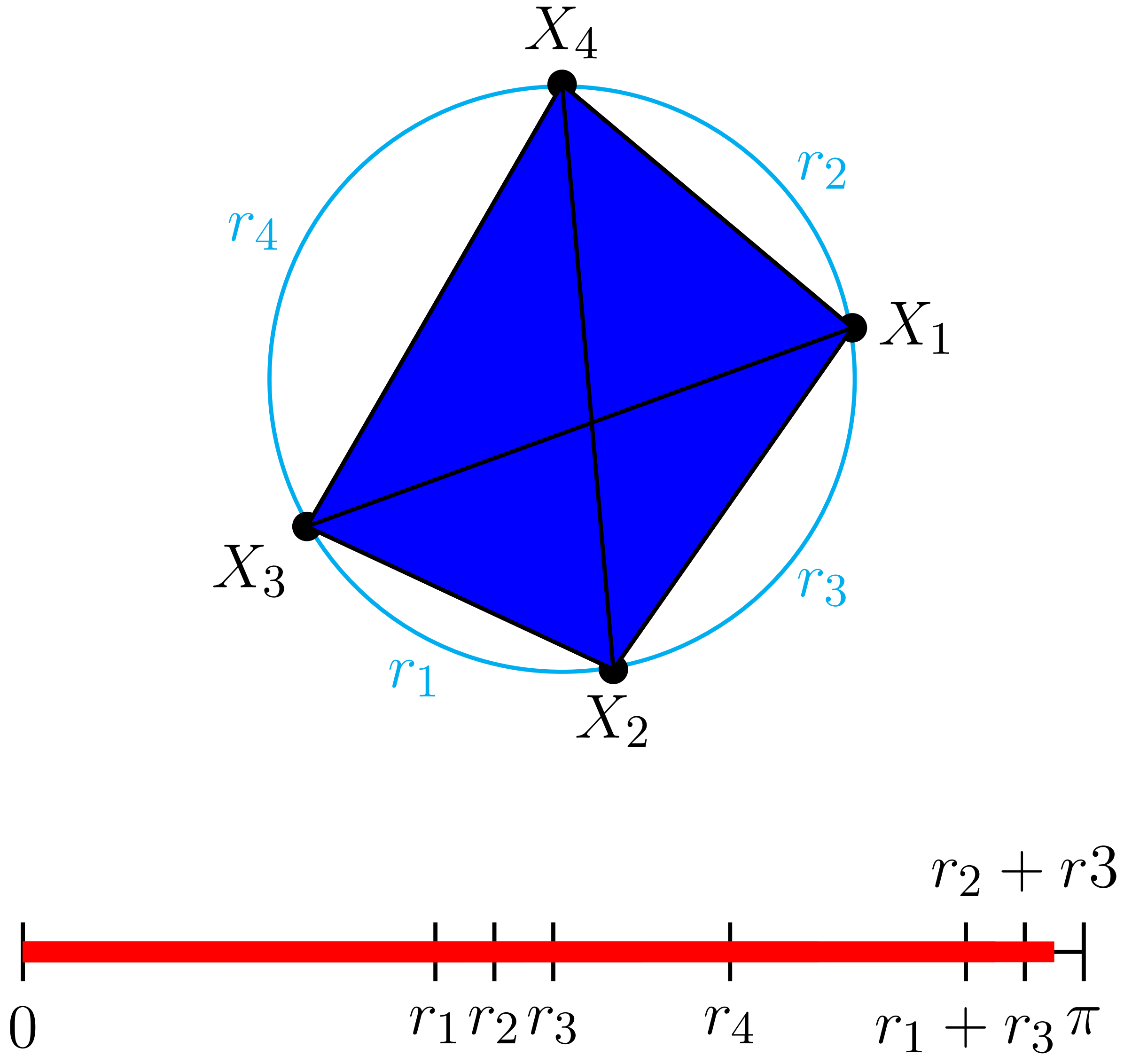
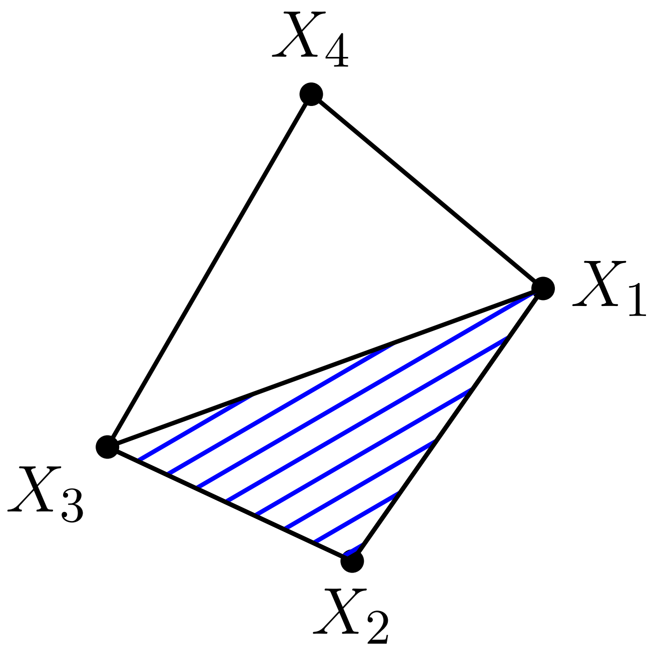


3. Homology of Simplices

Consider the following simplicial complex Δ:

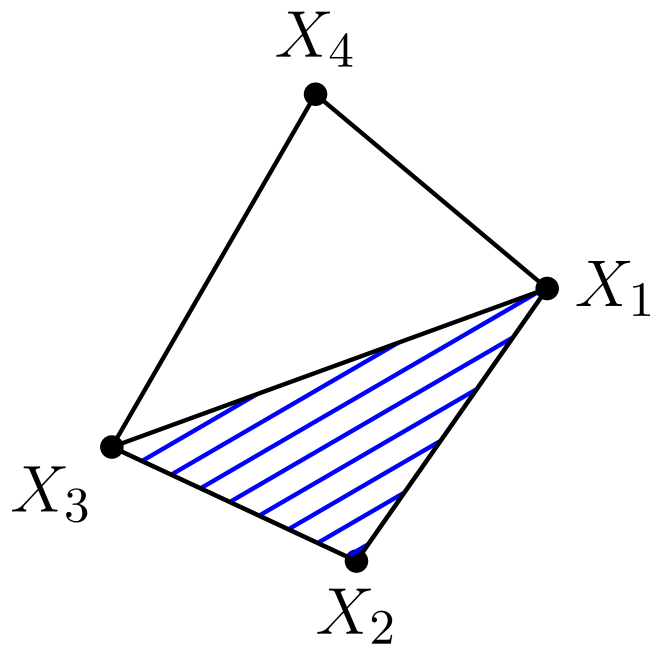


It consists of

- four 0-simplices: X_1, X_2, X_3, X_4,
- five 1-simplicies: $\{X_1, X_2\}$, $\{X_1, X_3\}$, $\{X_2, X_3\}$, $\{X_1, X_4\}$, $\{X_3, X_4\}$,
- and one 2-simplex: $\{X_1, X_2, X_3\}$.

Homology of Simplicies

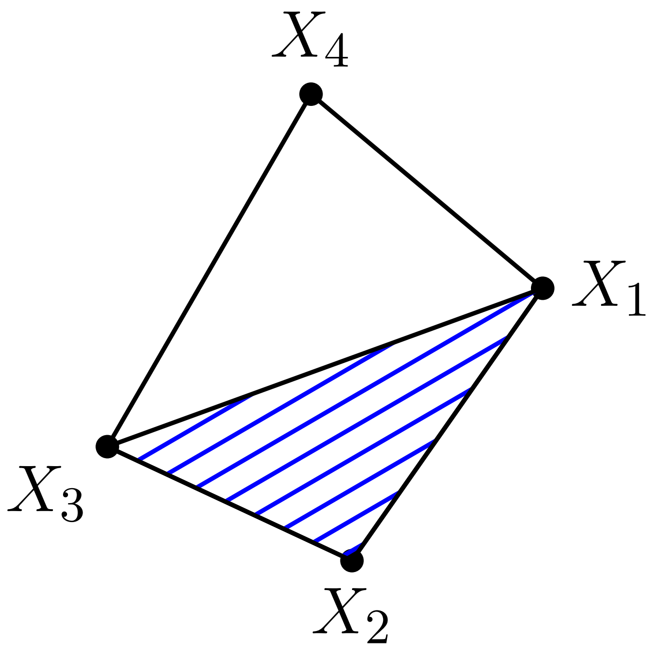
Sums of n-simplicies are called n-chains.



- the boundary of $\{X_1, X_2, X_3\}$ is $\{X_1, X_2\} + \{X_2, X_3\} + \{X_3, X_1\}$
- the boundary of $\{X_i, X_j\}$ is $X_j - X_i$,
- the boundary of $\{X_i\}$ is 0

Homology of Simplicies

By linearity this defines the boundary on all n-chains.



Cycles are n-chains with boundary equal to zero.

For example $\{X_1, X_2\} + \{X_2, X_3\} + \{X_3, X_1\}$ is a cycle and so is X_1.

One can check that boundaries are always cycles.

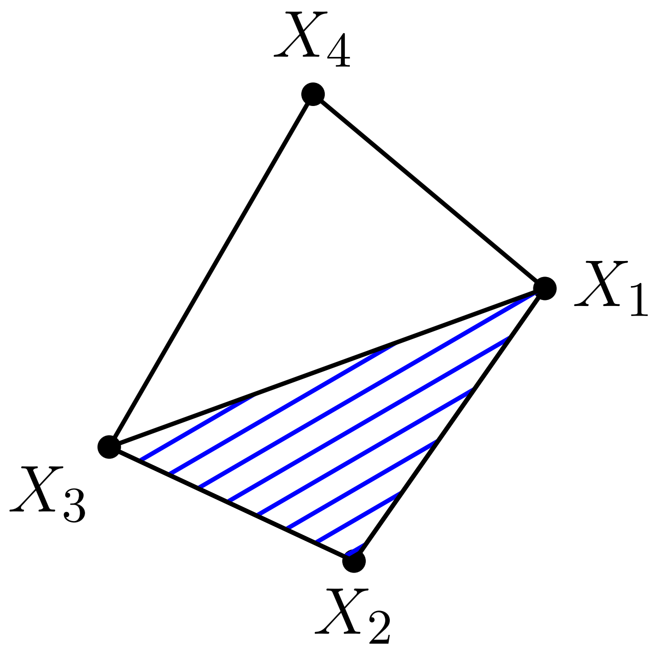
Homology of Simplicies

The homology $H_n(\Delta)$ is the quotient of the cycles modulo the boundaries.

The Betti number $B_n(\Delta)$ is the dimension of $H_n(\Delta)$.

$B_0(\Delta)$ is the number of connected components of Δ.
$B_1(\Delta)$ is the number of holes in Δ.
$B_2(\Delta)$ is the number of voids in Δ.

Homology of Simplicies



In our example $B_0(\Delta) = 1$, $B_1(\Delta) = 1$, and all higher Betti numbers are zero.

4. Persistent homology

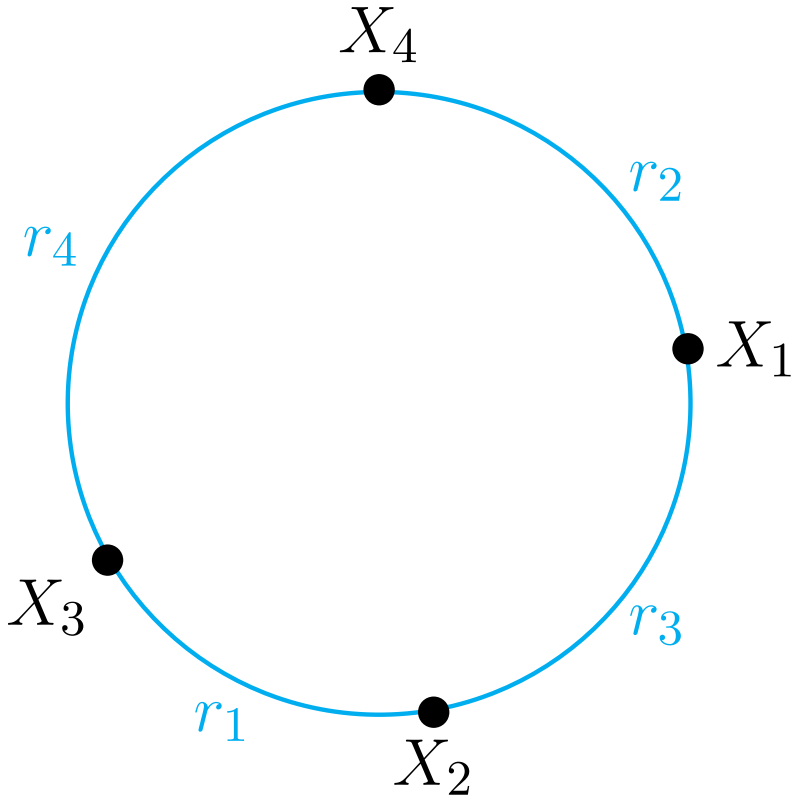
Assume we have a simplicial complex that changes as we vary some parameter r.

The homology that persists as r changes is called persistent homology.

We can record how the Betti numbers change as r changes using Betti barcodes.

We illustrate this using the Rips complex on our earlier example.

The Rips complex



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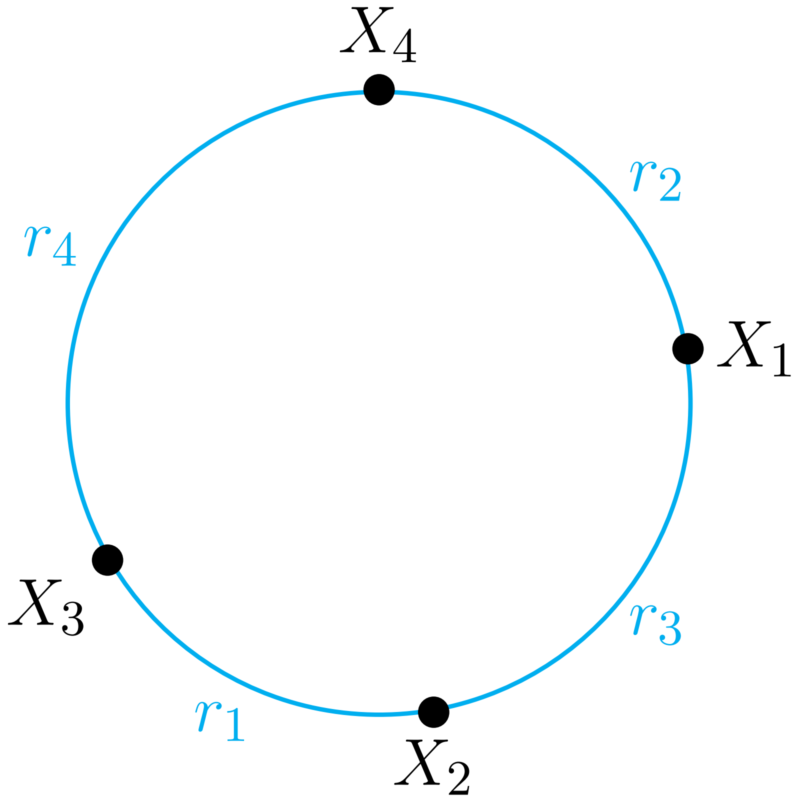

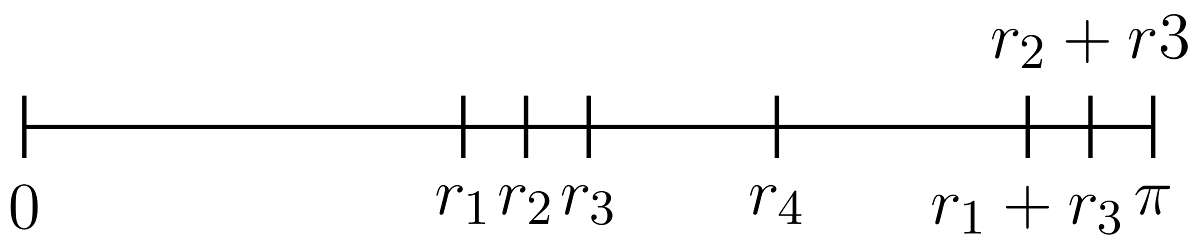

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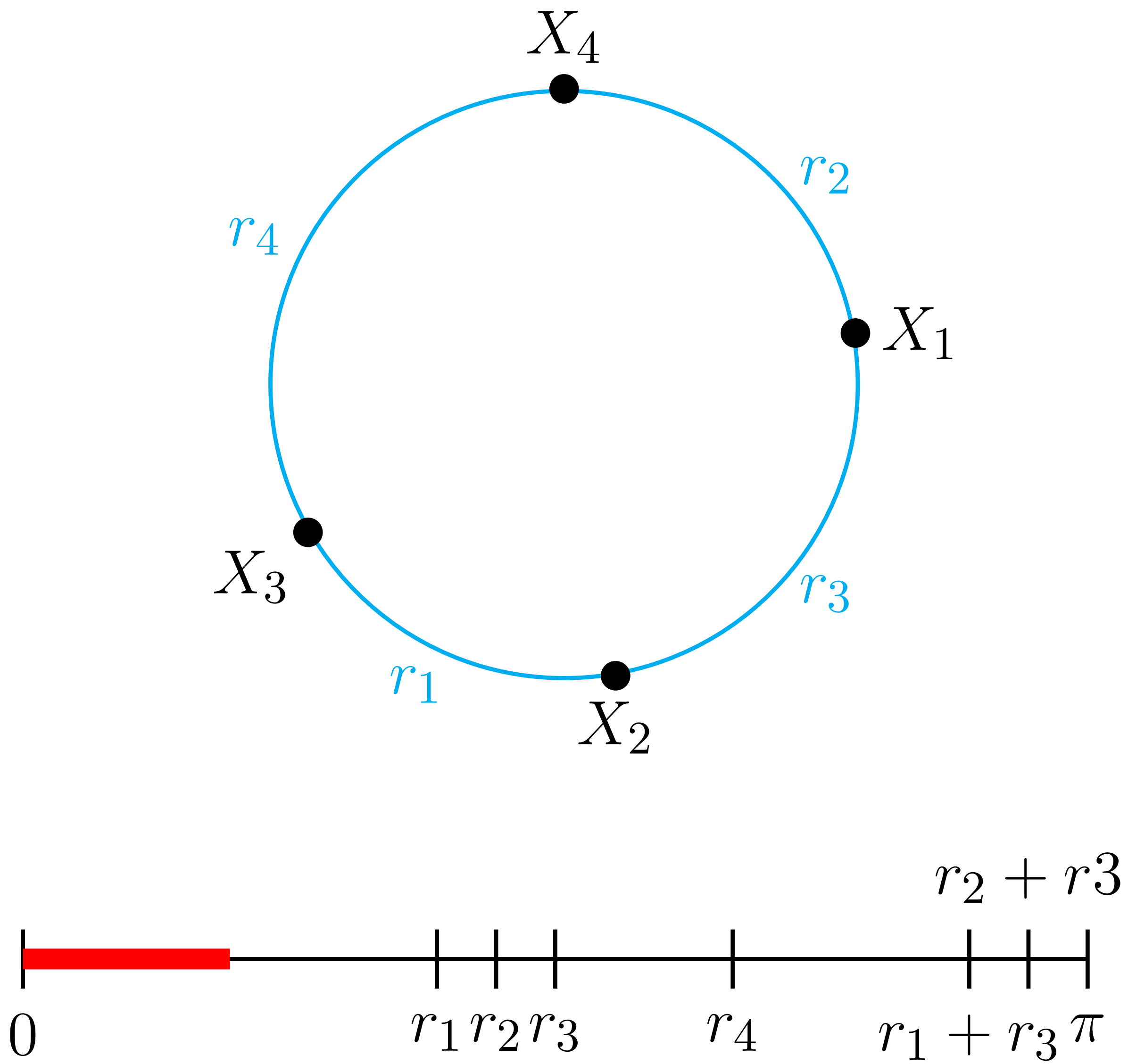

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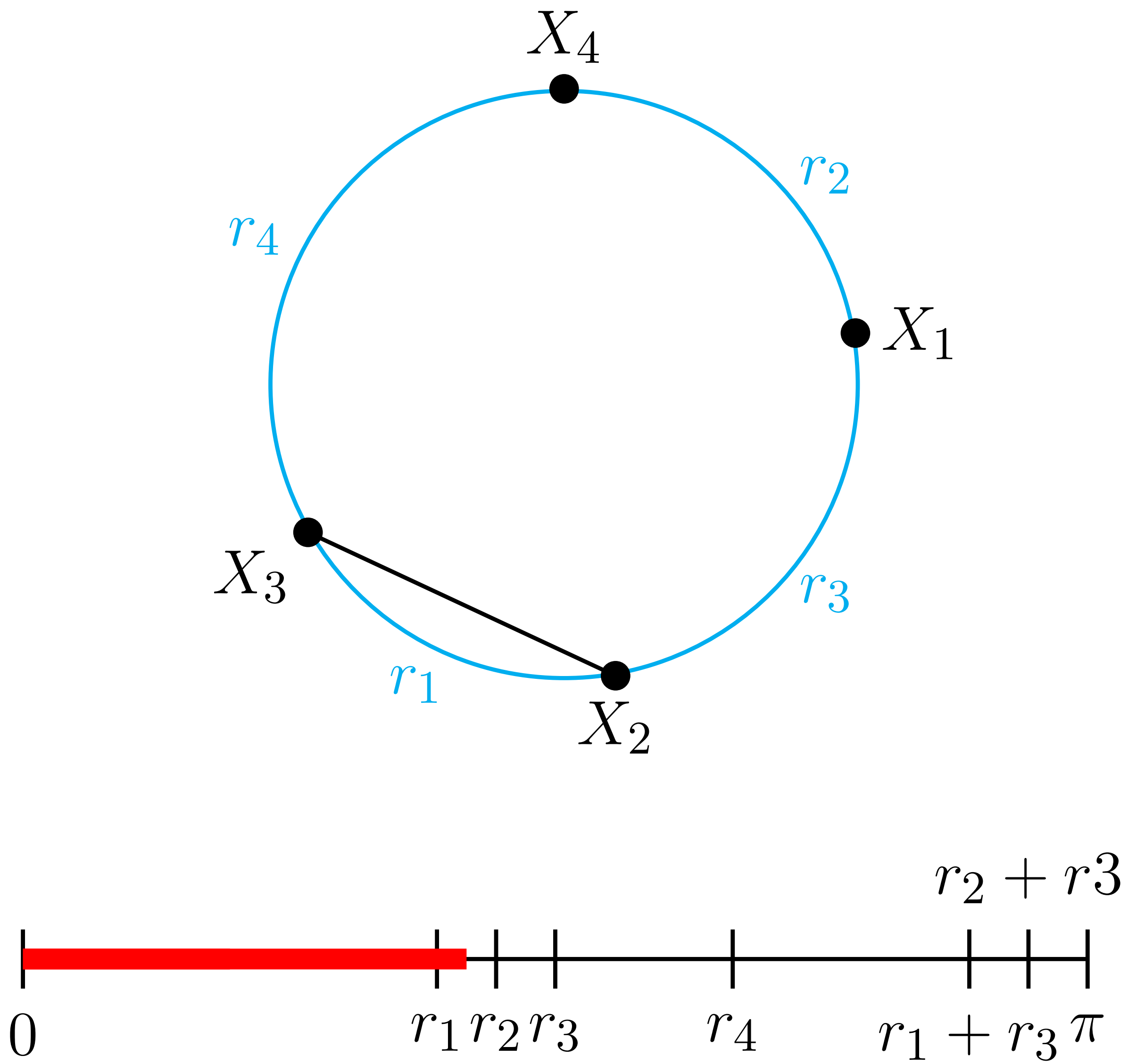

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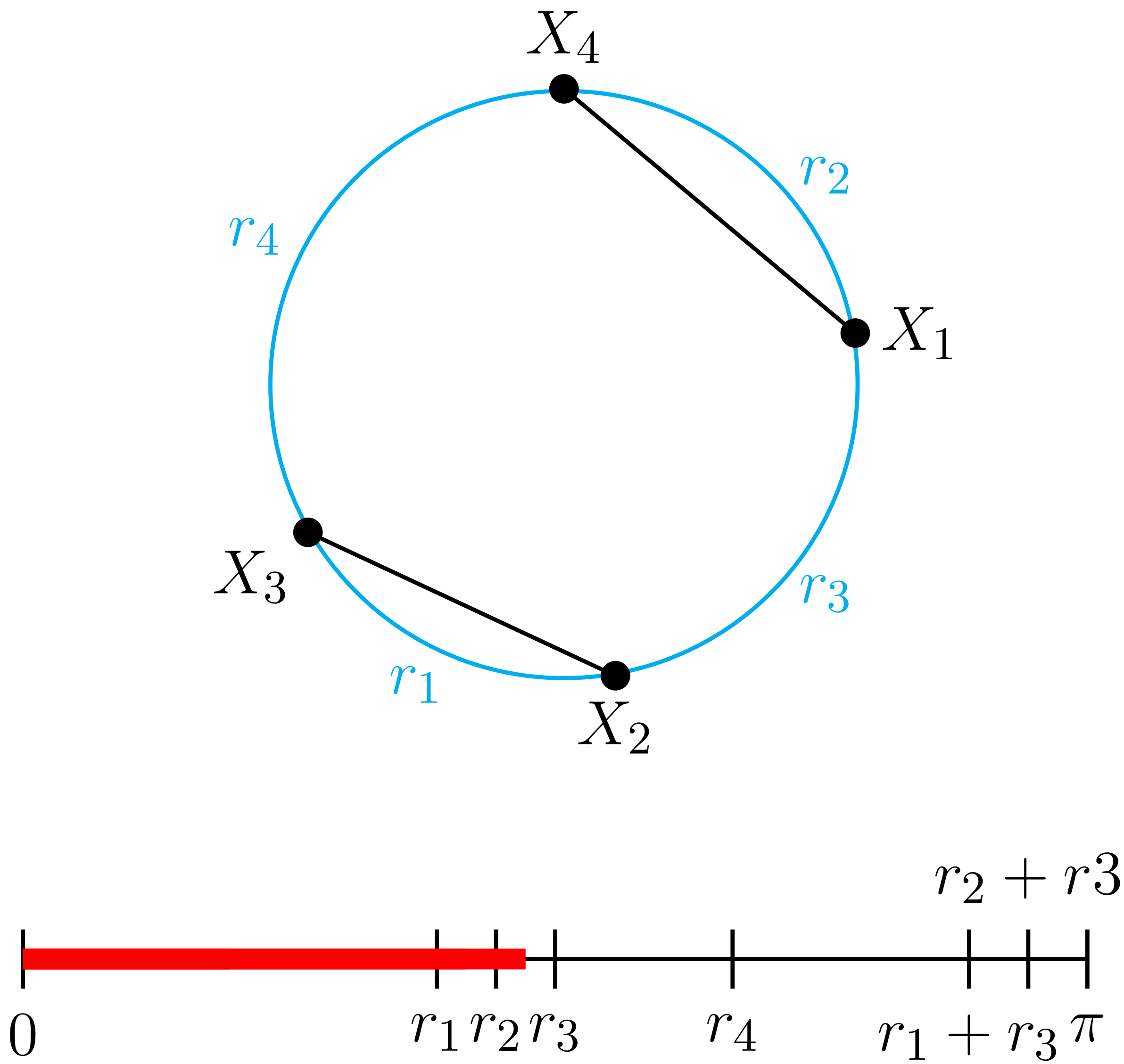

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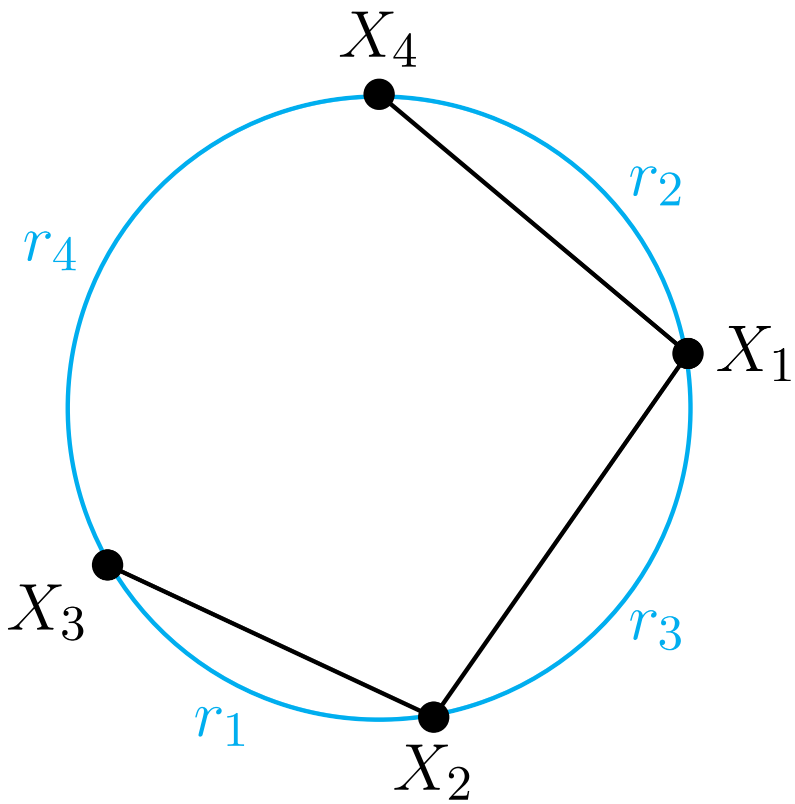

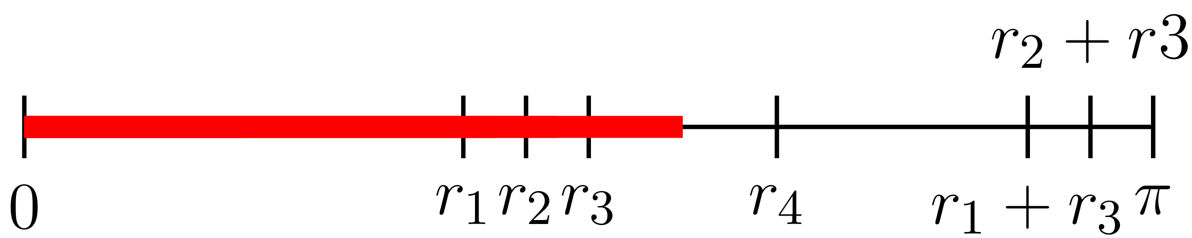

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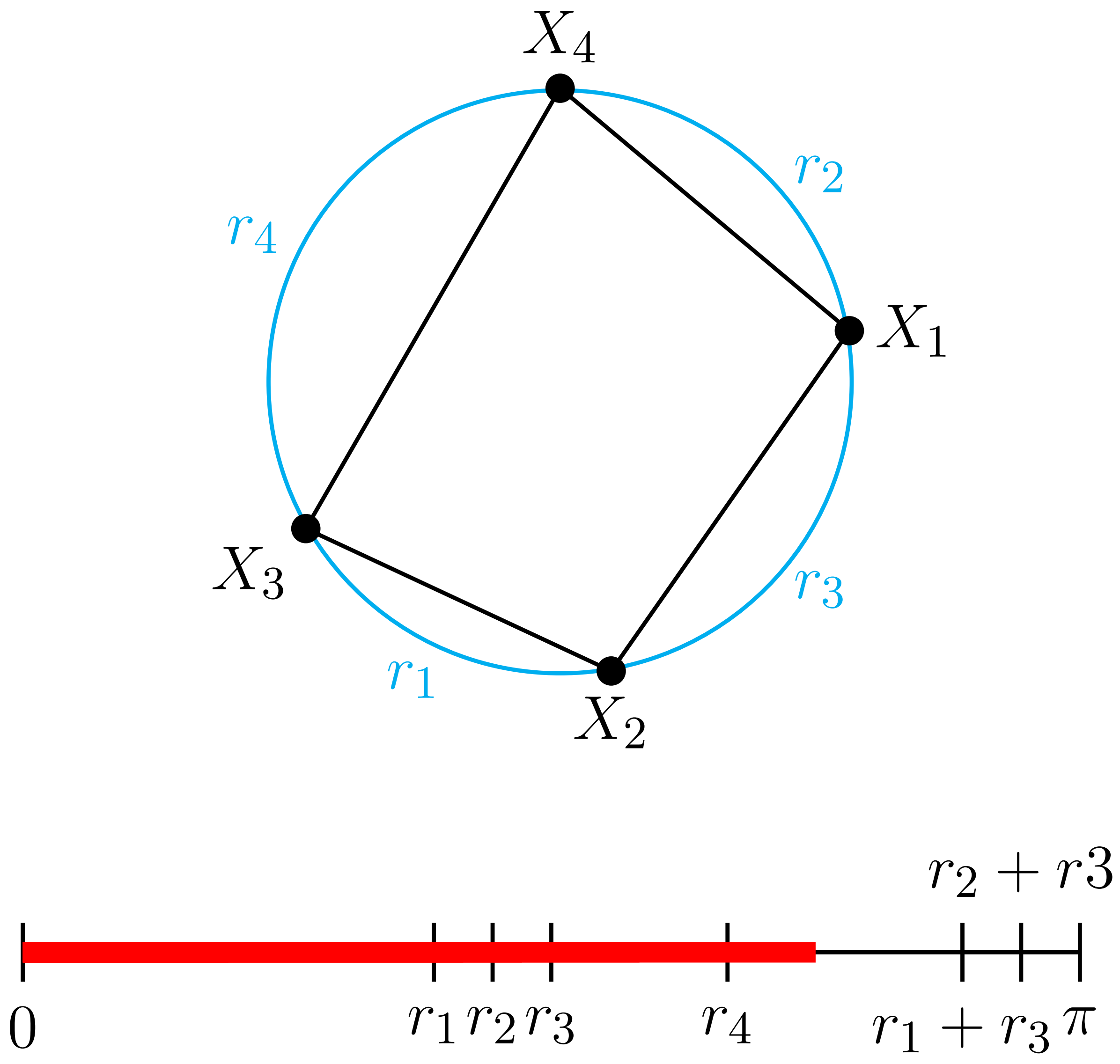

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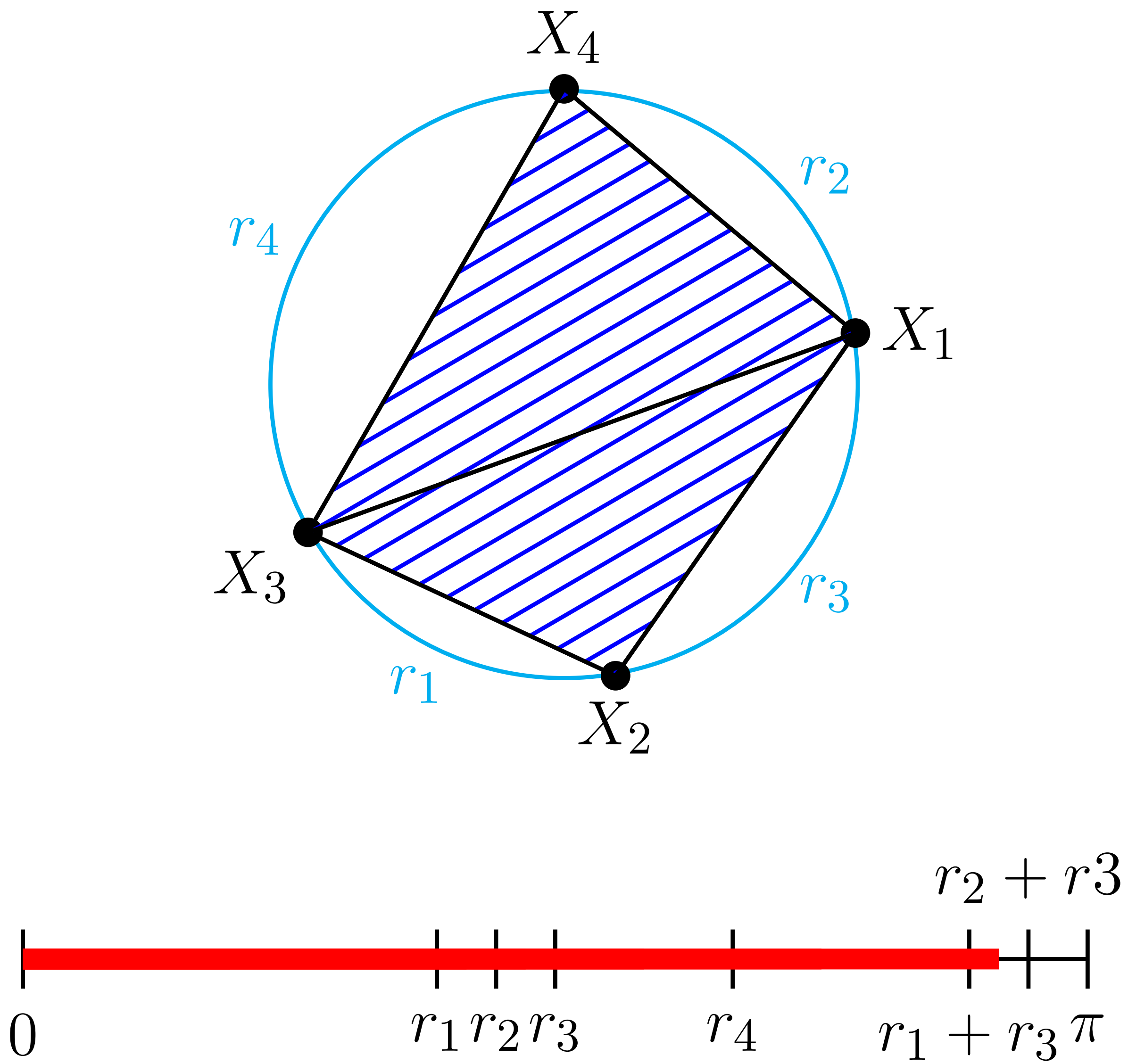

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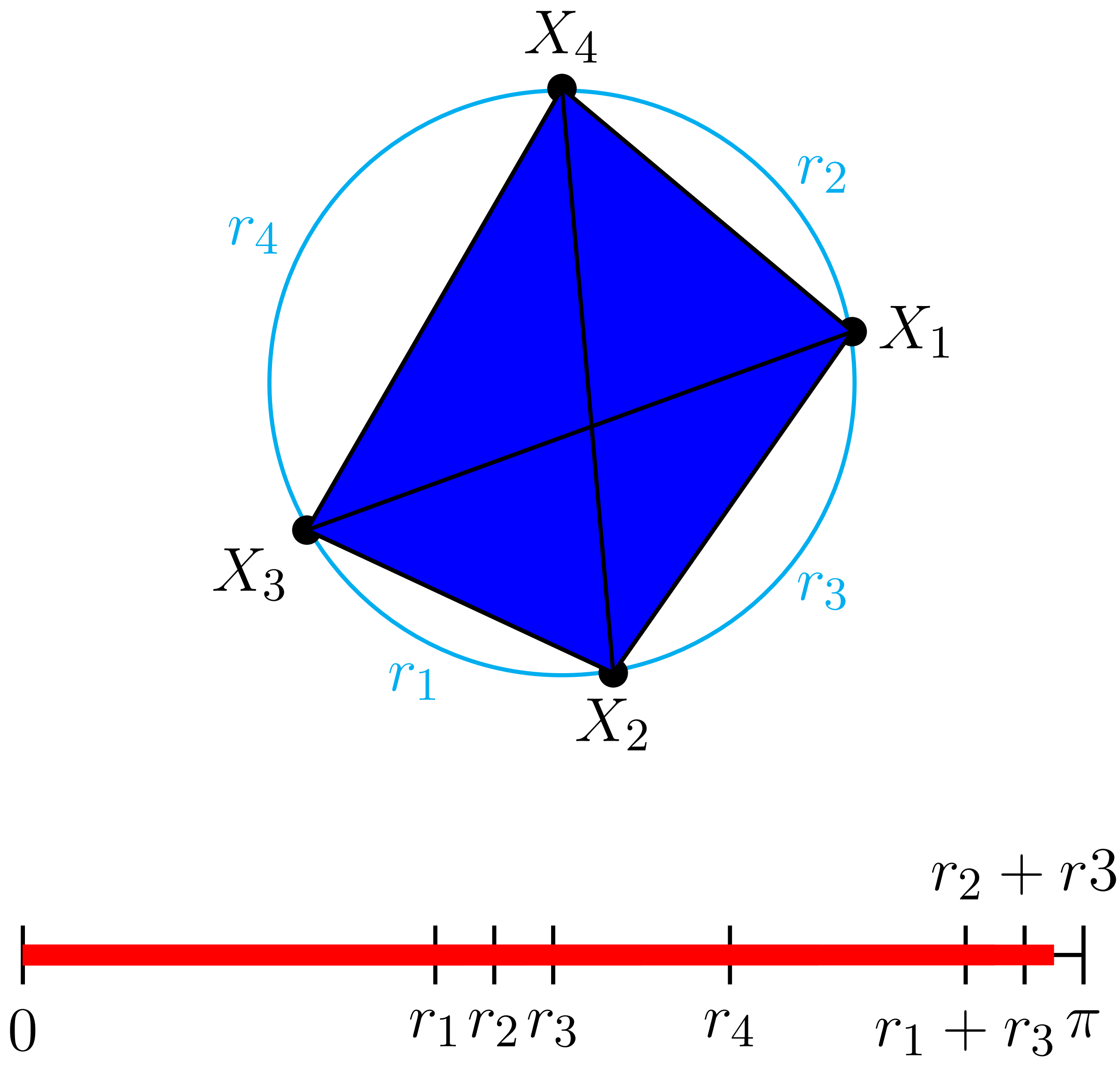
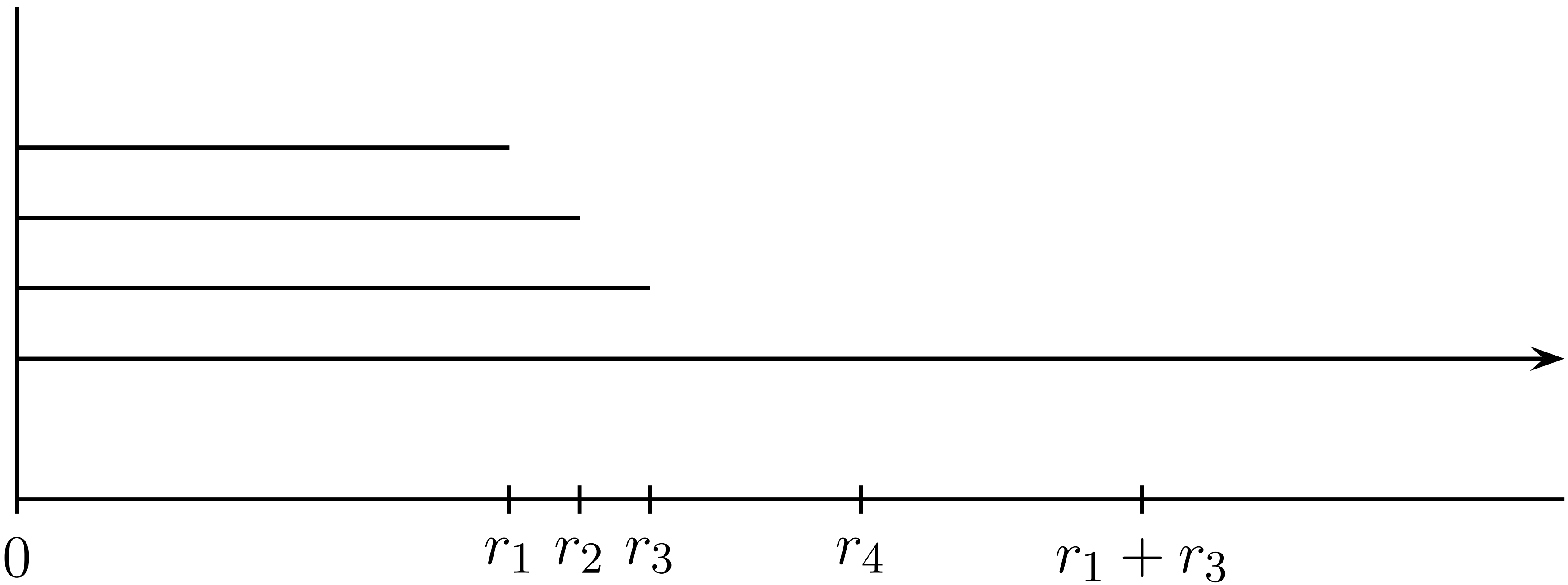


Betti 0-barcode



Betti 1-barcode