An introduction to the geometry and topology of point cloud data

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Motivation: Given a set of points that “looks like a circle”

We would like to be able to say so mathematically.
2. The Rips complex
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The Rips complex

$X_1$ $r_1$ $X_2$ $r_2$ $X_3$ $r_3$ $X_4$ $r_4$
The Rips complex

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The Rips complex

A diagram showing points labeled $X_1$, $X_2$, $X_3$, and $X_4$ connected by arcs with radii $r_1$, $r_2$, $r_3$, and $r_4$. The radii $r_1 + r_3\pi$ and $r_2 + r_3$ are also indicated.
The Rips complex

\[ X_4 \]

\[ X_1 \]

\[ X_3 \]

\[ r_4 \]

\[ r_2 \]

\[ r_3 \]

\[ r_1 \]

\[ 0 \]

\[ r_1 \]

\[ r_2 \]

\[ r_3 \]

\[ r_4 \]

\[ r_1 + r_3 \pi \]

\[ r_2 + r_3 \]
The Rips complex

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The Rips complex

\[ X_1 \quad X_2 \quad X_3 \quad X_4 \]

\[ r_1 + r_3 \pi \]

\[ r_2 + r_3 \]

\[ r_1 \quad r_2 \quad r_3 \quad r_4 \]
The Rips complex

\[ X_4 \rightarrow X_1 \rightarrow X_3 \rightarrow X_2 \rightarrow X_4 \]

\[ r_1 + r_3 \pi \]
The Rips complex

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The Rips complex

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3. Homology of Simplices

Consider the following simplicial complex $\Delta$:

It consists of

- four 0-simplices: $X_1, X_2, X_3, X_4$,
- five 1-simplices: $\{X_1, X_2\}, \{X_1, X_3\}, \{X_2, X_3\}, \{X_1, X_4\}, \{X_3, X_4\}$,
- and one 2-simplex: $\{X_1, X_2, X_3\}$. 
Homology of Simplicies

Sums of $n$-simplicies are called $n$-chains.

- the boundary of $\{X_1, X_2, X_3\}$ is $\{X_1, X_2\} + \{X_2, X_3\} + \{X_3, X_1\}$
- the boundary of $\{X_i, X_j\}$ is $X_j - X_i$.
- the boundary of $\{X_i\}$ is $0$.
By linearity this defines the boundary on all $n$-chains.

**Cycles** are $n$-chains with boundary equal to zero.

For example \( \{X_1, X_2\} + \{X_2, X_3\} + \{X_3, X_1\} \) is a cycle and so is \( X_1 \).

One can check that boundaries are always cycles.
Homology of Simplicies

The homology $H_n(\Delta)$ is the quotient of the cycles modulo the boundaries.

The Betti number $B_n(\Delta)$ is the dimension of $H_n(\Delta)$.

$B_0(\Delta)$ is the number of connected components of $\Delta$.
$B_1(\Delta)$ is the number of holes in $\Delta$.
$B_2(\Delta)$ is the number of voids in $\Delta$. 
In our example $B_0(\Delta) = 1$, $B_1(\Delta) = 1$, and all higher Betti numbers are zero.
4. Persistent homology

Assume we have a simplicial complex that changes as we vary some parameter $r$.

The homology that persists as $r$ changes is called persistent homology.

We can record how the Betti numbers change as $r$ changes using Betti barcodes.

We illustrate this using the Rips complex on our earlier example.
The Rips complex

The Rips complex is a construction in geometric topology that allows for the study of the topological structure of point cloud data. It is defined as the nerve of the balls of radius $r$ around each point in the data set. In the diagram, we see a 4-point data set $X = \{X_1, X_2, X_3, X_4\}$ with radii $r_1, r_2, r_3, r_4$.
The Rips complex

\[ \begin{align*}
X_1 & \quad r_2 \\
X_2 & \quad r_4 \\
X_3 & \quad r_1 \\
X_4 & \quad r_3
\end{align*} \]

\[ r_2 + r_3 \]

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The Rips complex
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The Rips complex

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The Rips complex
The Rips complex

$X_1 \quad X_2 \quad X_3 \quad X_4$

$r_1 \quad r_2 \quad r_3 \quad r_4$

$r_2 + r_3$

$x \quad r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_1 + r_3 \pi$
The Rips complex

\[ X_4 \]

\[ X_3 \]

\[ X_2 \]

\[ X_1 \]

\[ r_1 \]

\[ r_2 \]

\[ r_3 \]

\[ r_4 \]

\[ r_2 + r_3 \]

\[ r_1 + r_3 \pi \]
Betti 0-barcode

\[ 0 \quad r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_1 + r_3 \]
Betti 1-barcode