# Using context and model categories to define directed homotopies

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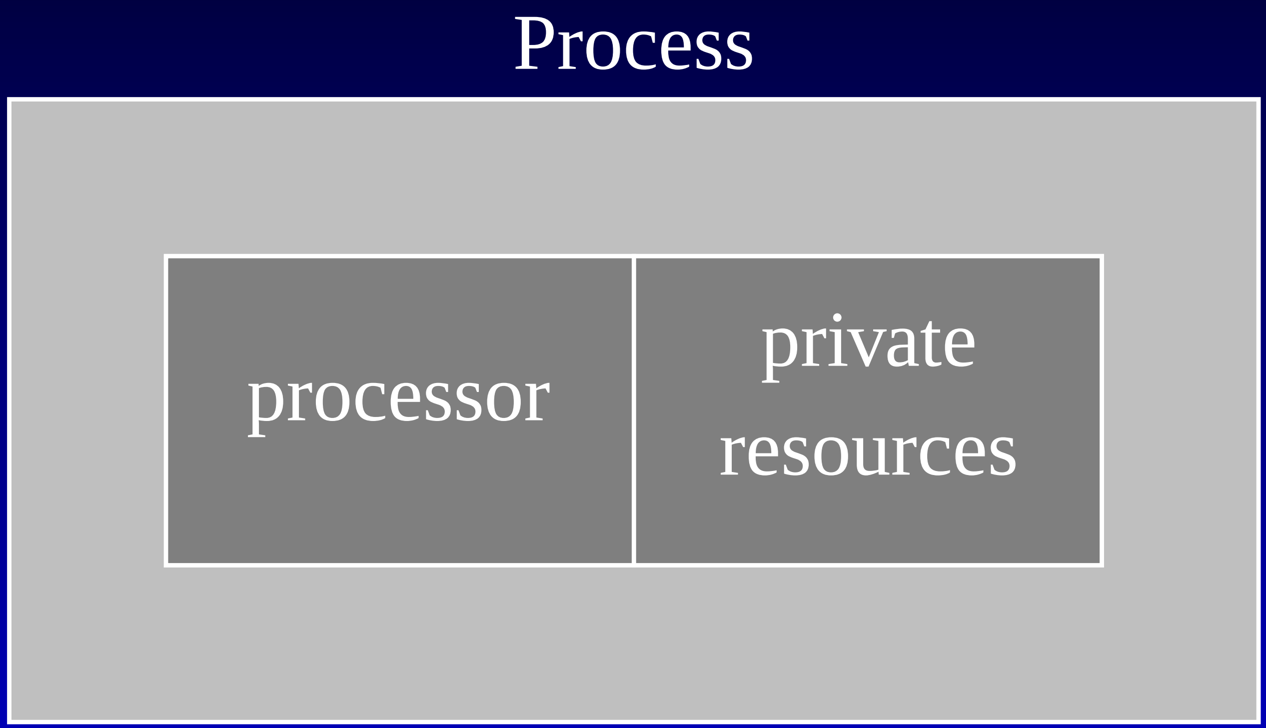
# 1. Introduction

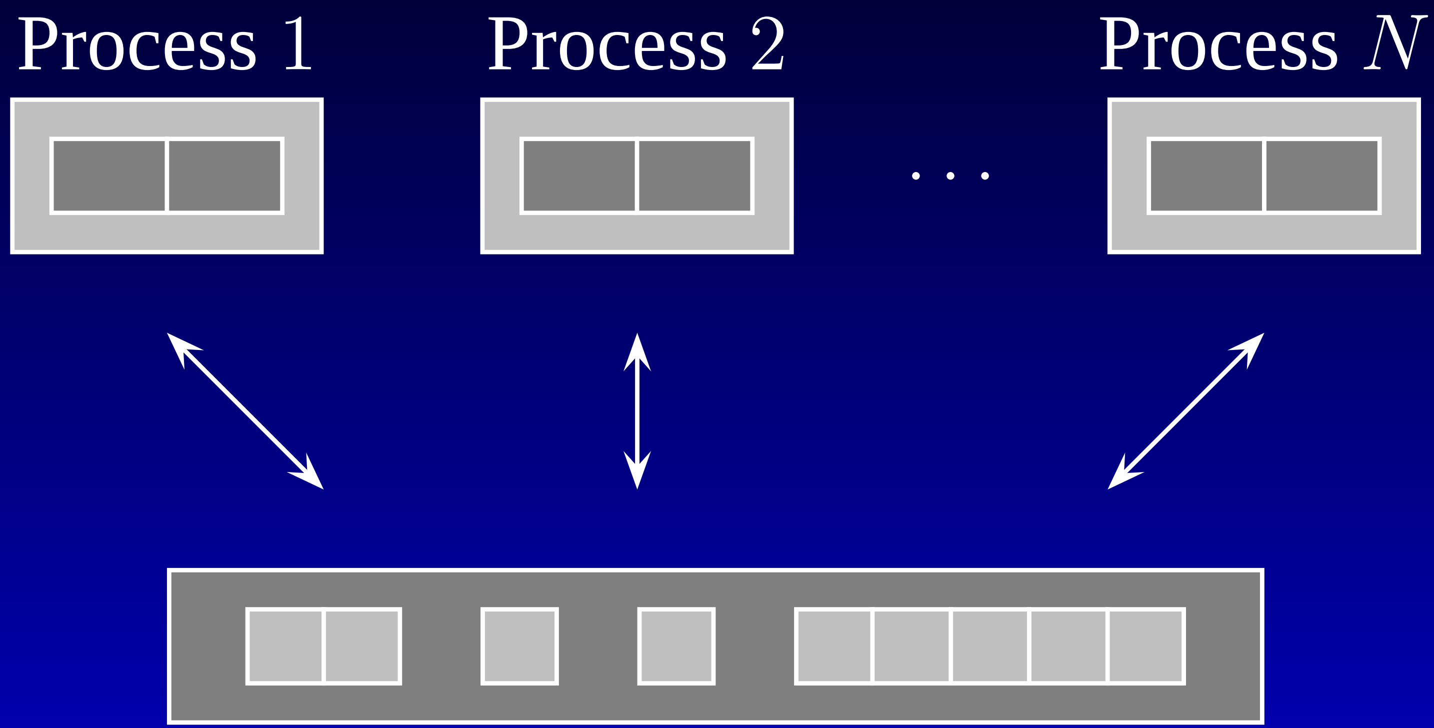
- A concurrent system
- A model for concurrency
- Equivalences - undirected and directed
- The Problem

# A nonconcurrent system



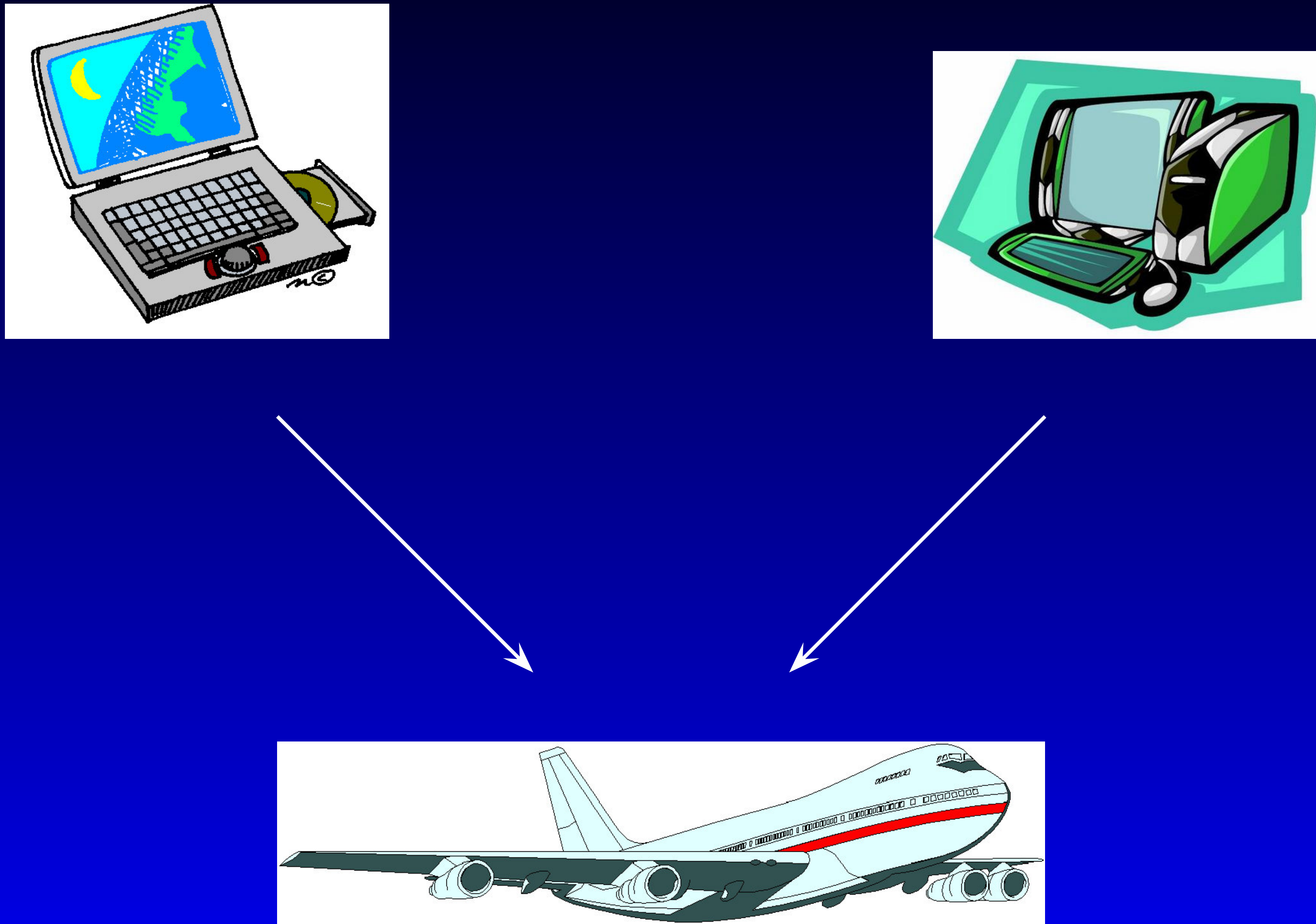

A process with its own private resources

# A concurrent system



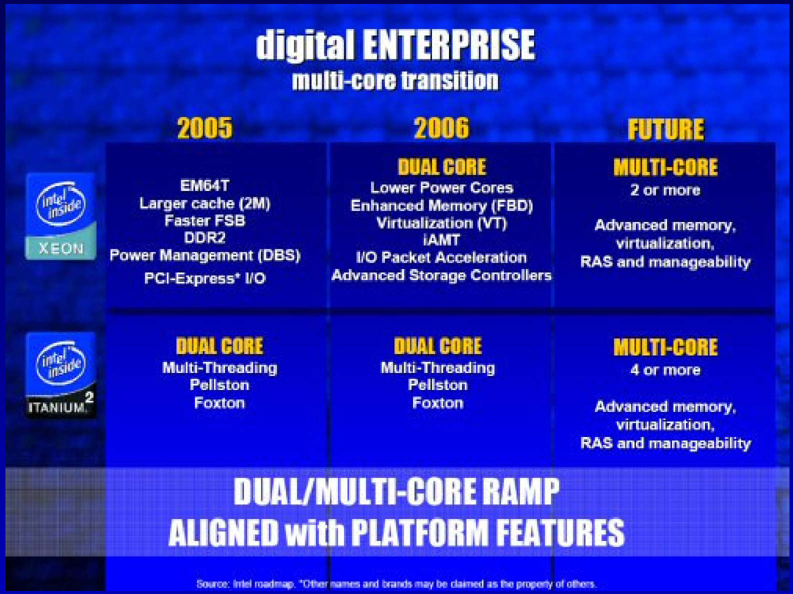

Several processes with shared resources

# Example: Internet database

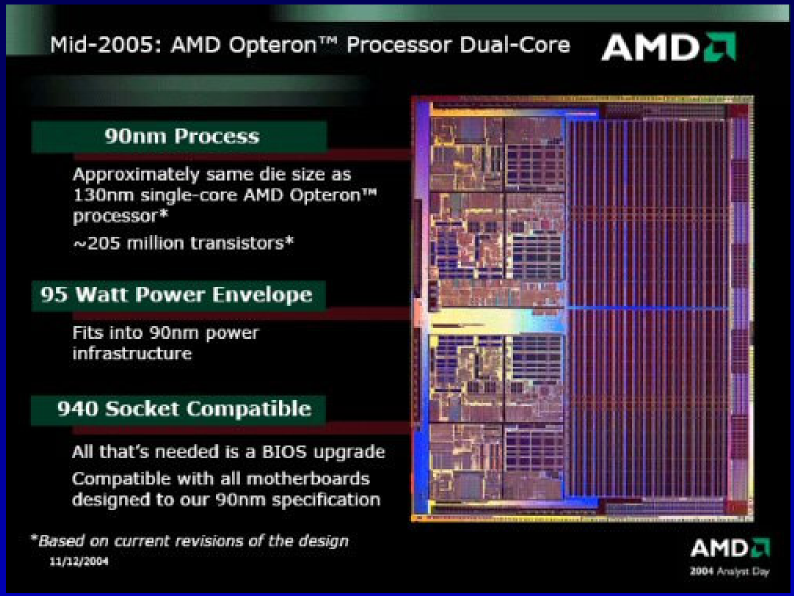


# Example: dual core processors

Intel (Q3 2005)




AMD (Q3 2005)



# A concurrent system

Example:
2 processes using 2 shared resources $a$ and $b$ which can only be used by one process at a time

Notation:
$Px$ - a process locks resource $x$
$Vx$ - a process releases resource $x$

Program of the first process: $Pa \quad Pb \quad Vb \quad Va$

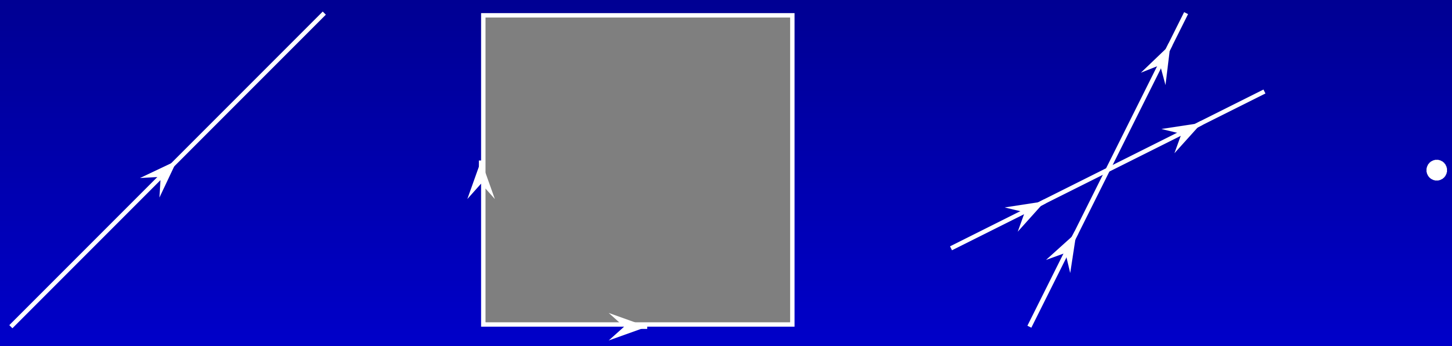
Program of the second process: $Pb \quad Pa \quad Va \quad Vb$

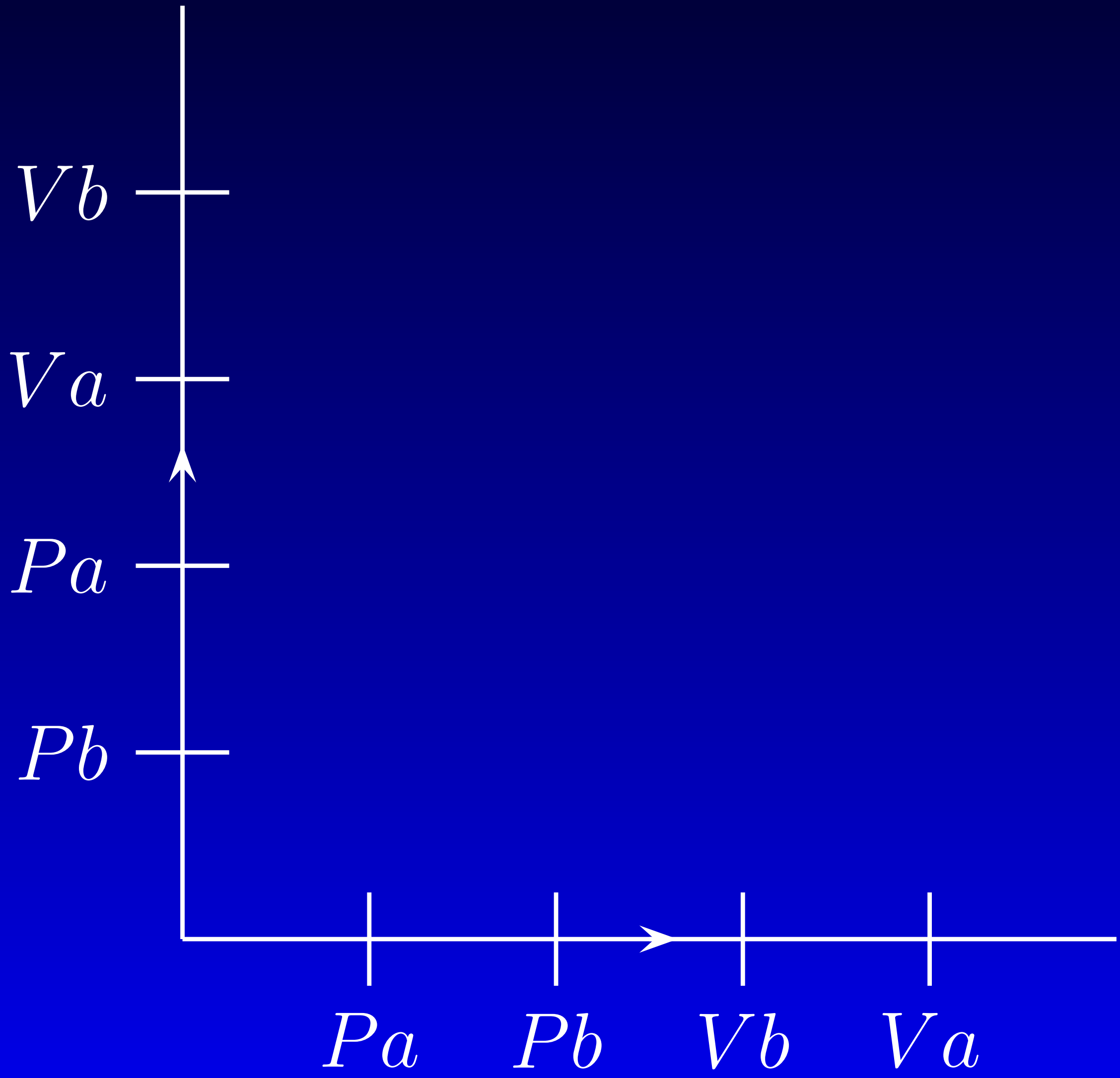
# A model for concurrency

Concurrent systems can be modeled by subspaces of $\mathbb{R}^n$ together with a partial order.

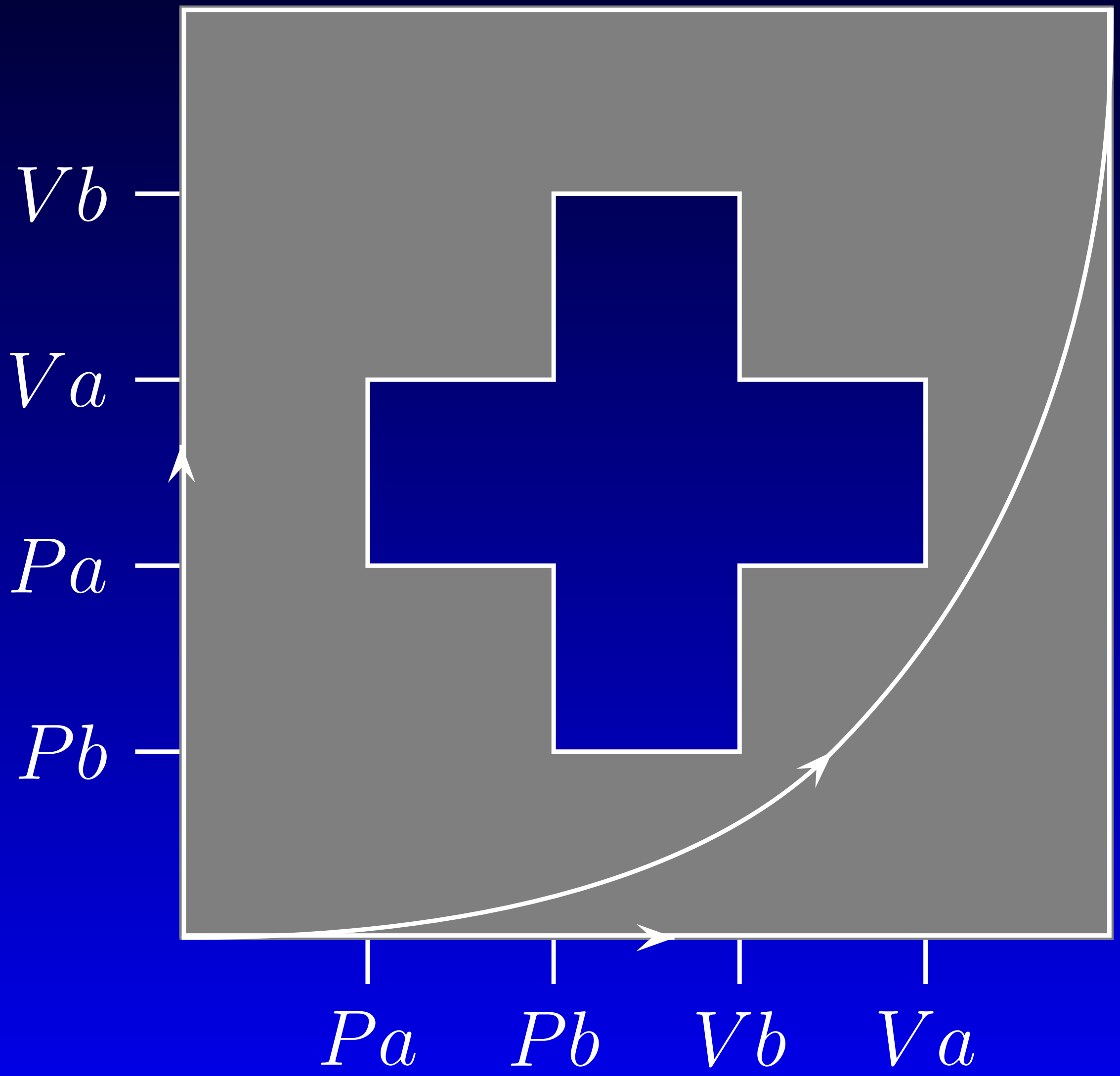
Definition: A po-space is a topological space $U$ with a partial order $\leq$ which is a closed subset of $U \times U$.

Examples:




$Vb$
$Va$
$Pa$
$Pb$
$Pa$
$Pb$
$Vb$
$Va$

# The Swiss flag



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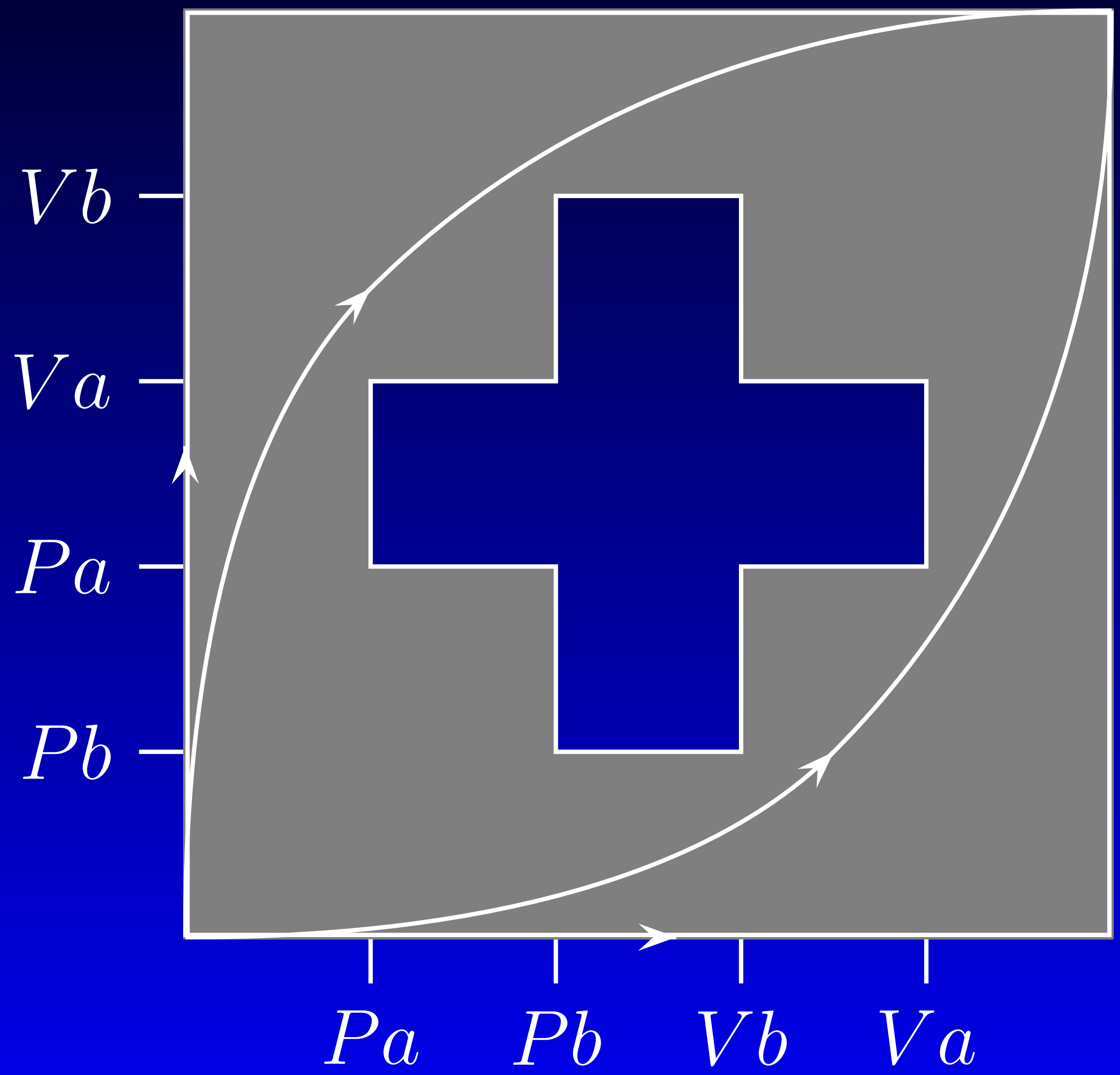

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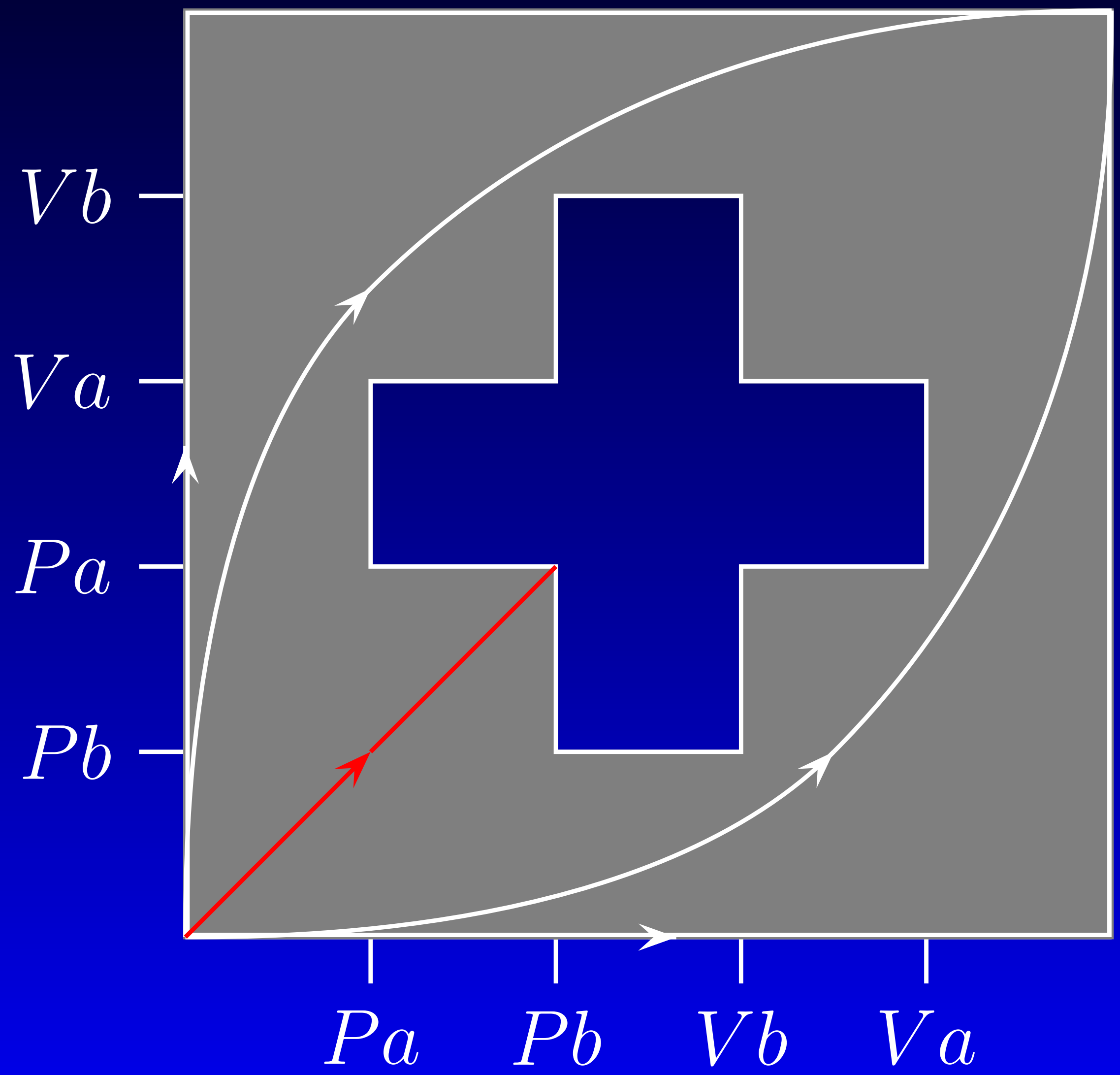

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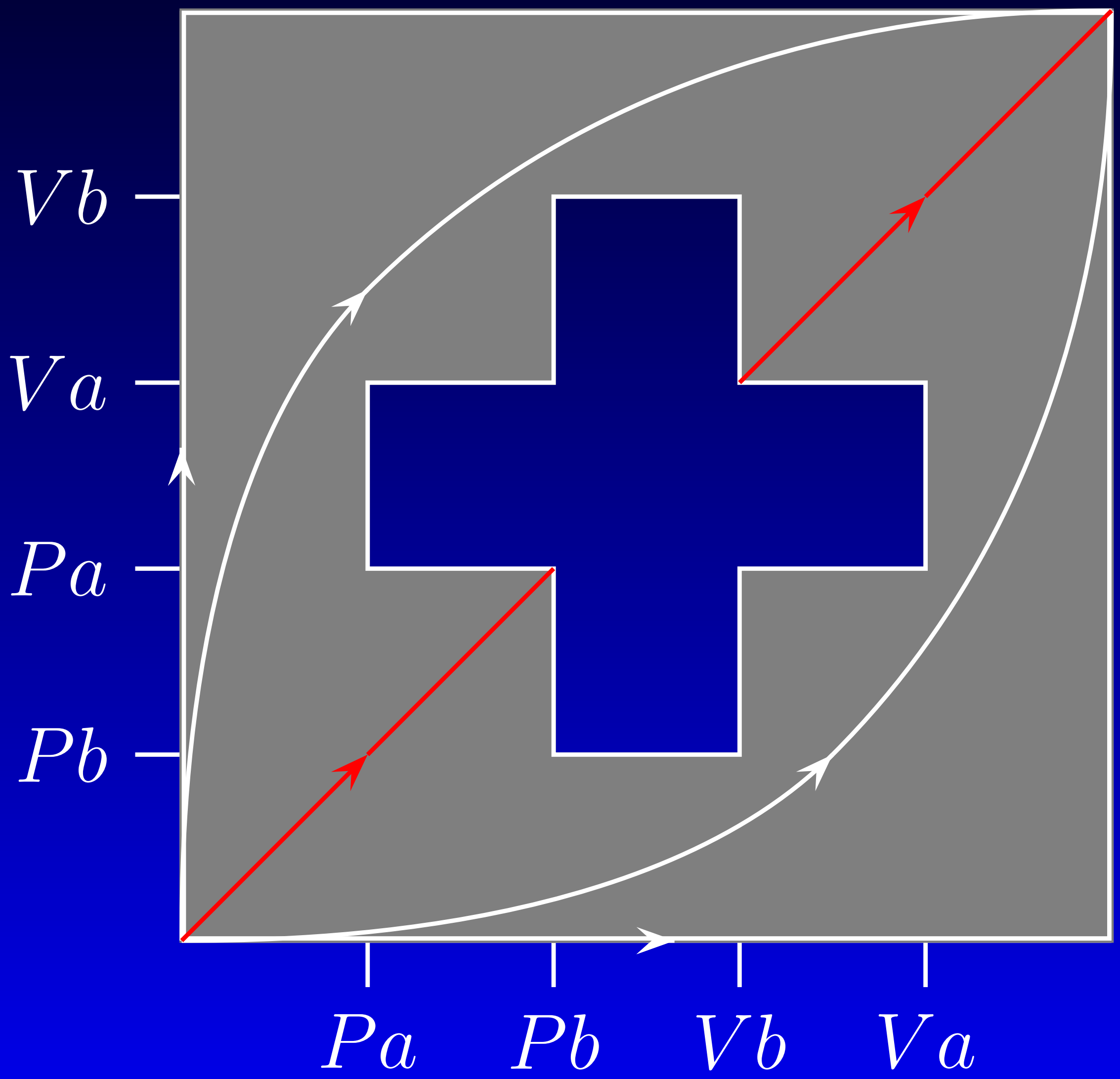

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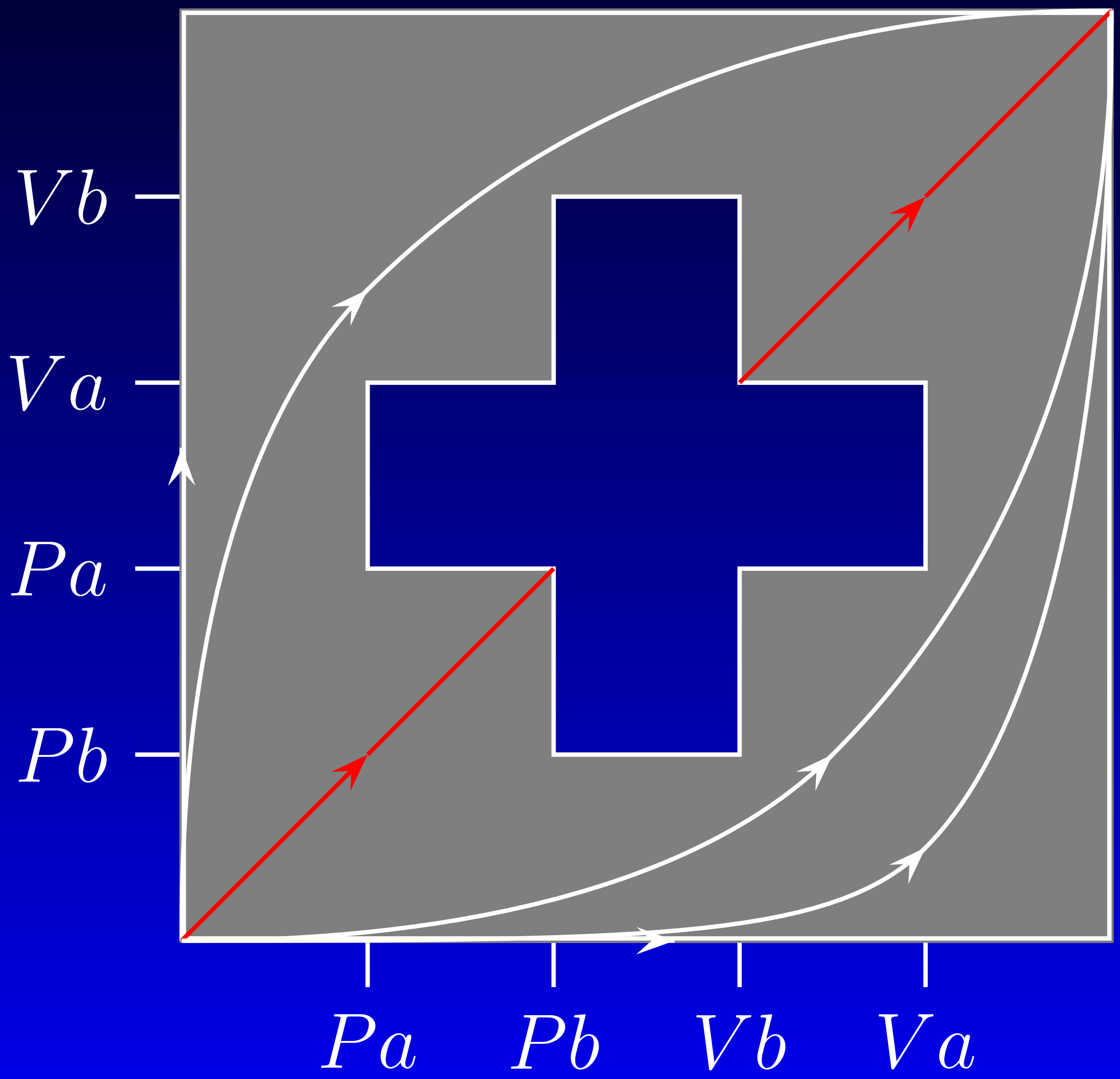
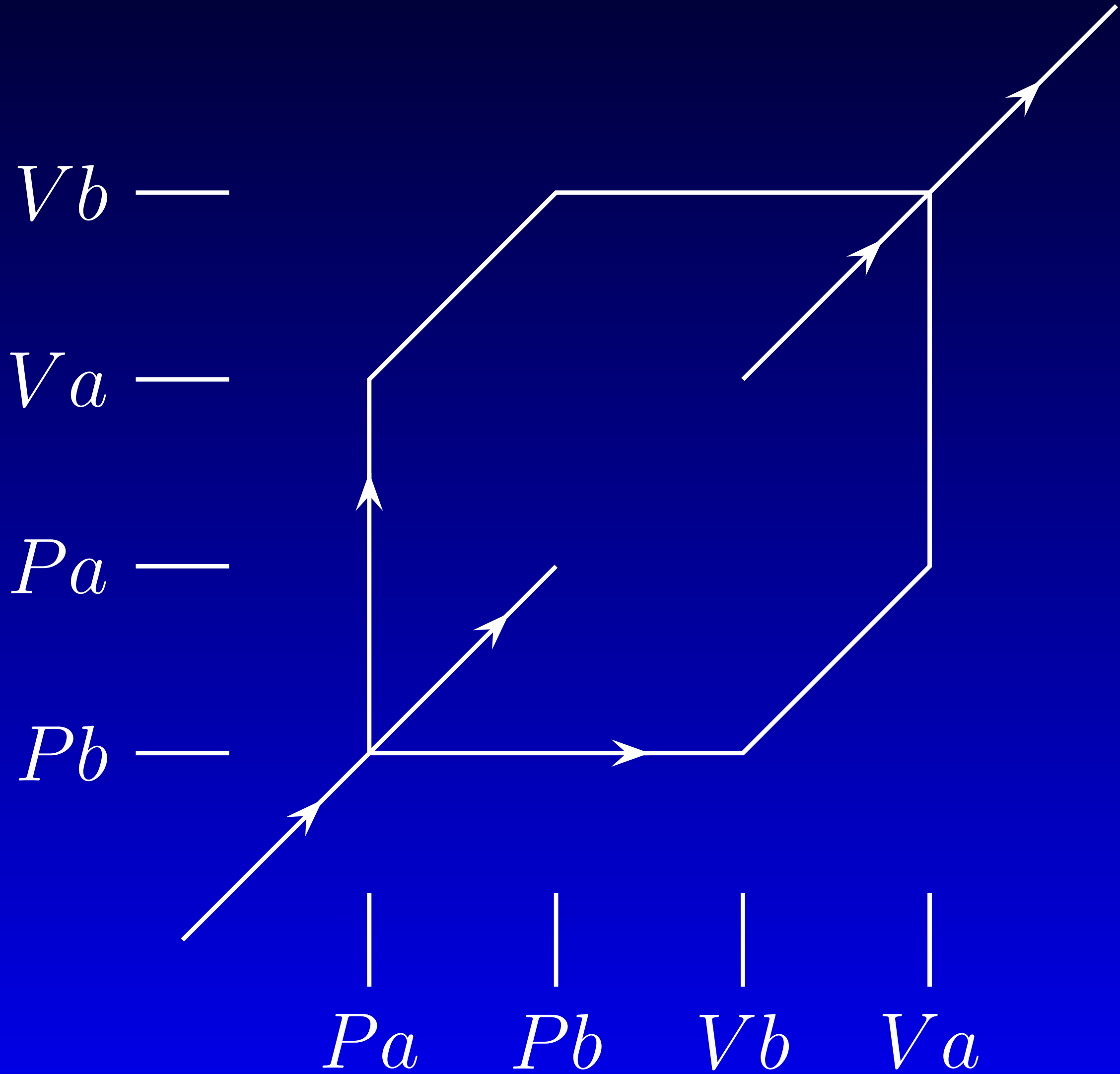


# A sub-po-space of the Swiss flag



# Goal

Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis of po-spaces.

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We would like equivalences that allow a piece-by-piece analysis of po-spaces.

Idea: Use algebraic topology.

# Undirected equivalences

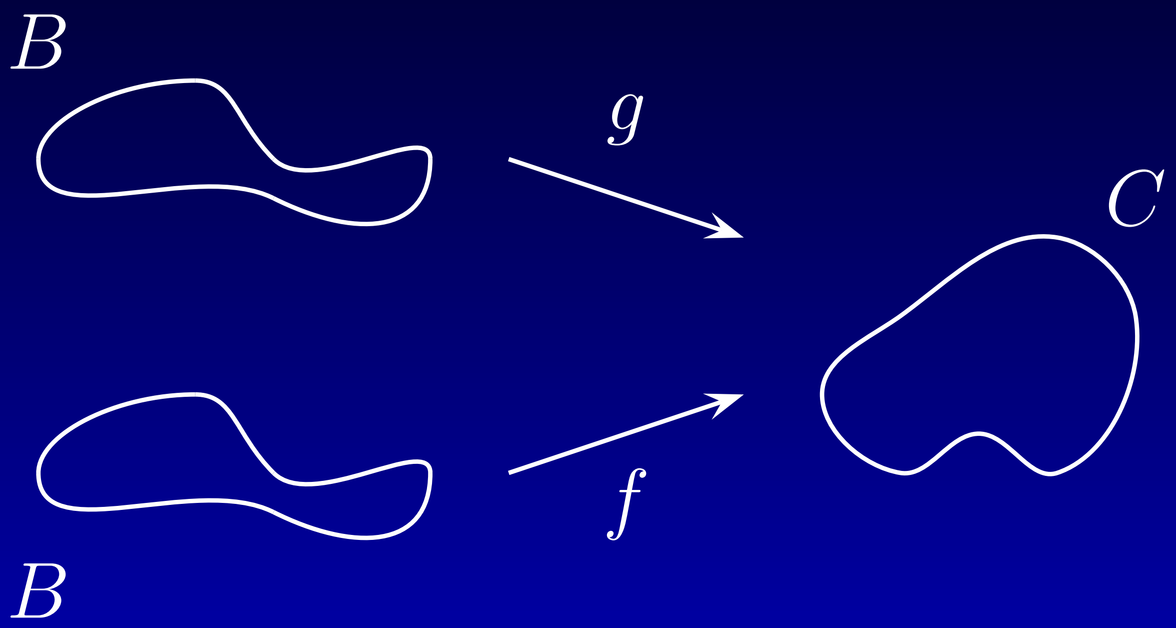
Working with (undirected) spaces and continuous maps, equivalences are defined using homotopies.

Maps are equivalent if there is a homotopy between them.

Spaces are equivalent if there maps between them whose compositions are homotopic to the identity map.

# Undirected equivalences

Definition: Given continuous maps $f, g : B \to C$,



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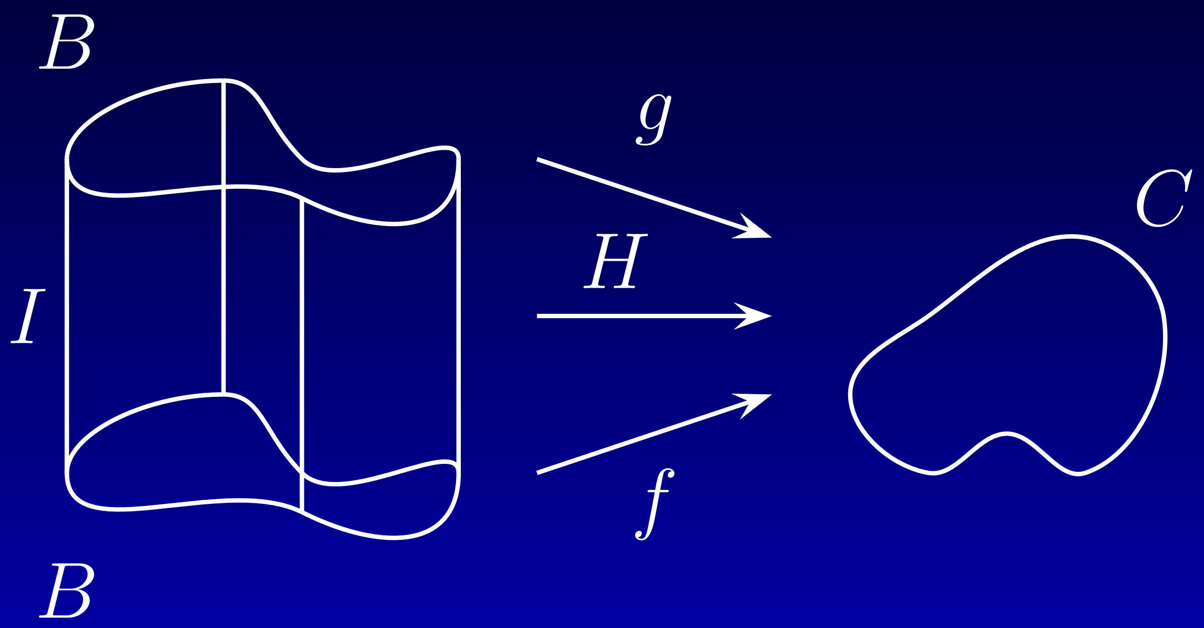
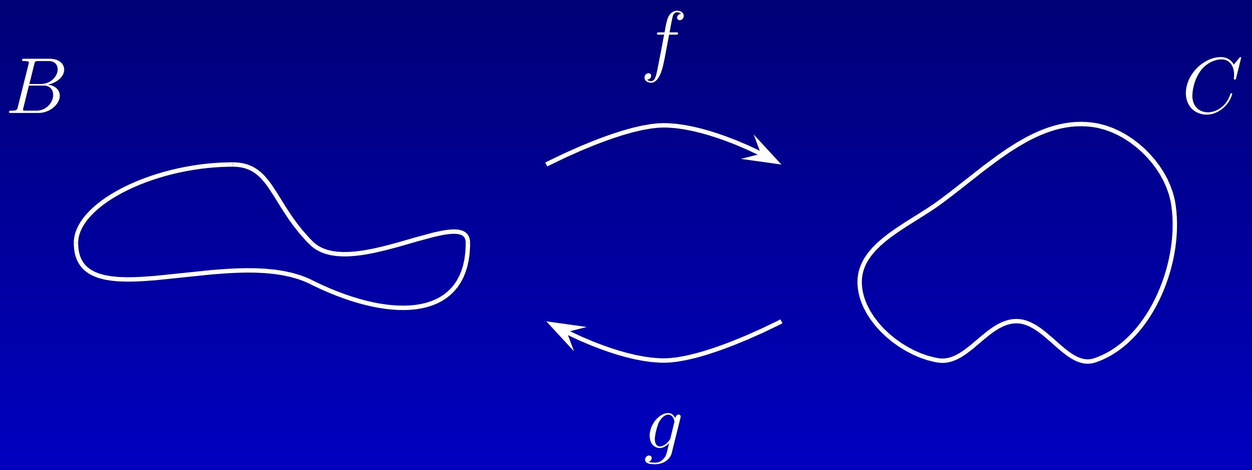



a homotopy between $f$ and $g$ is a continuous map $H : B \times I \to C$ restricting to $f$ and $g$. This is an equivalence relation. Write $H : f \xrightarrow{\simeq} g$.

# Undirected equivalences

Definition: Spaces $B, C$ are homotopy equivalent if there are maps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \text{ and } f \circ g \simeq \mathrm{Id}_C \, .$$



# Directed spaces

Definition:

- A po-space is a topological space $U$ with a partial order $\leq$ which is a closed subset of $U \times U$.
- A directed map (dimap) is a continuous map $f : U_1 \to U_2$ between po-spaces such that

$$x \leq y \implies f(x) \leq f(y).$$

- Let **PoSpc** be the category whose objects are po-spaces and morphisms are dimaps.

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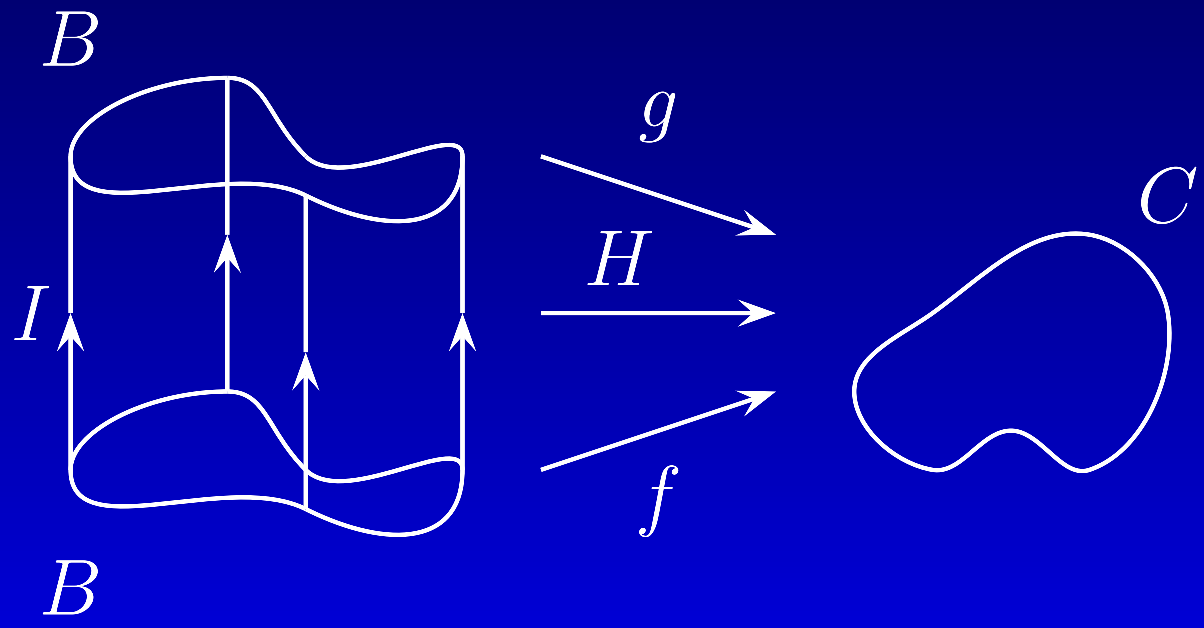
- Let $\mathbf{PoSpc}$ be the category whose objects are po-spaces and morphisms are dimaps.

Remark: Subspaces and products of po-spaces inherit a po-space structure.

# Directed equivalences

Definition:

- A directed homotopy (dihomotopy) between dimaps $f, g : B \to C$ is a dimap $H : B \times \vec{I} \to C$ restricting to $f$ and $g$. Write $H : f \to g$.



# Directed equivalences

Definition:

- Write $f \simeq g$ if there is a chain of dihomotopies

$$f \to f_1 \leftarrow f_2 \to \ldots \leftarrow f_n \to g.$$

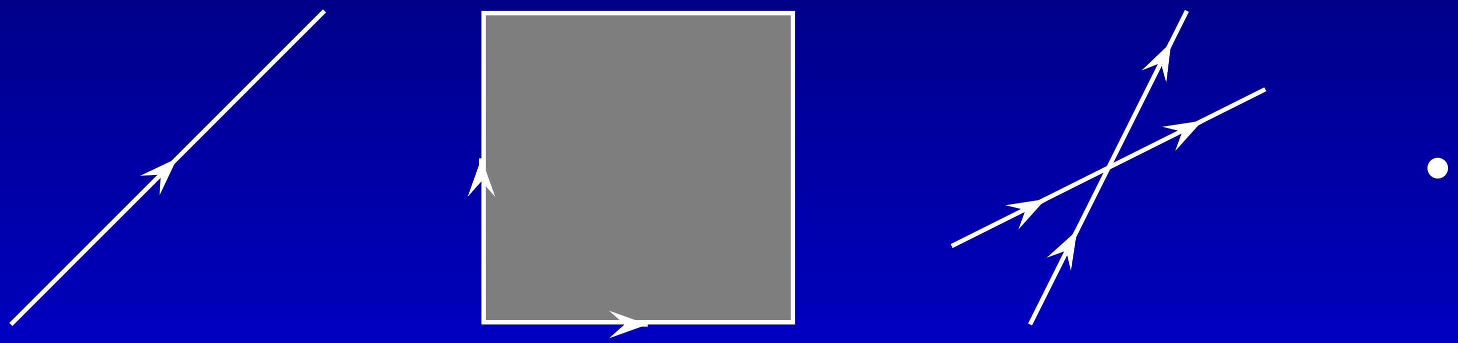
- Po-Spaces $B, C$ are dihomotopy equivalent if there are dimaps $f : B \leftrightarrows C : g$ such that

$$g \circ f \simeq \mathrm{Id}_B \text{ and } f \circ g \simeq \mathrm{Id}_C\,.$$

# The Problem

Recall: We wanted to use dihomotopy equivalences to provide equivalences of concurrent systems.

However all of the following spaces are dihomotopy equivalent.



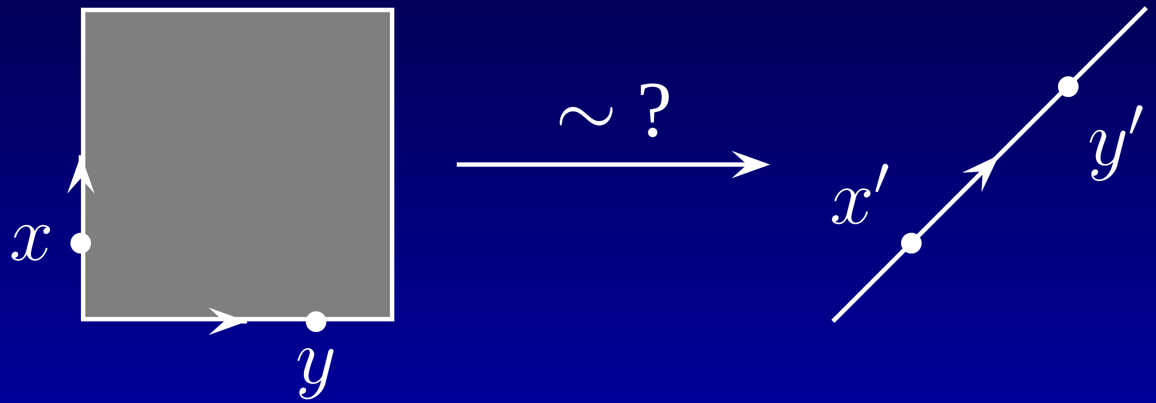
Thus, a stronger notion of equivalence is needed.

# 2. Using context

- A basic example
- A solution
- Equivalences using context
- Non-equivalences using directed paths
- Piece-by-piece example

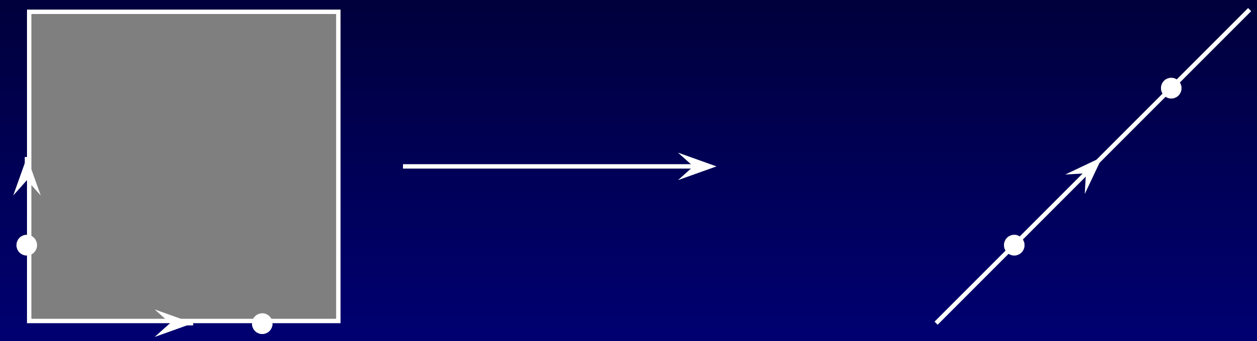
# A basic example

Question: Is there an equivalence between $\vec{I} \times \vec{I}$ and $\vec{I}$?




Our guide: If this map is an equivalence, then so is the map obtained by attaching po-spaces to this example. (Formally, we want the set of equivalences to be closed under pushouts with inclusions.)

# Adding to the basic example



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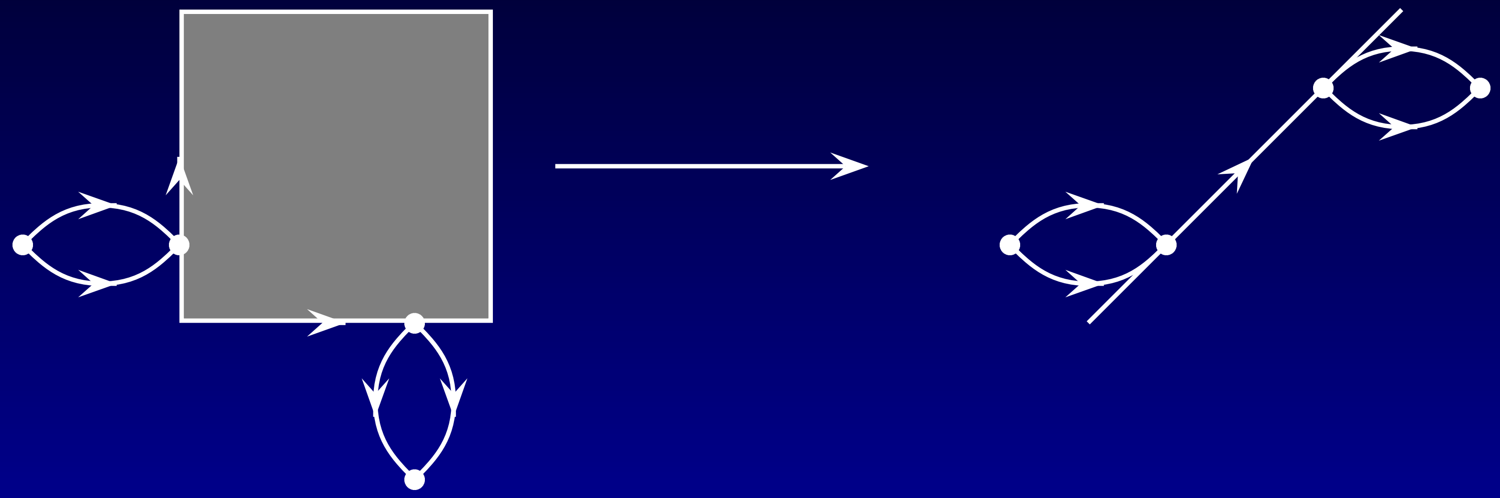

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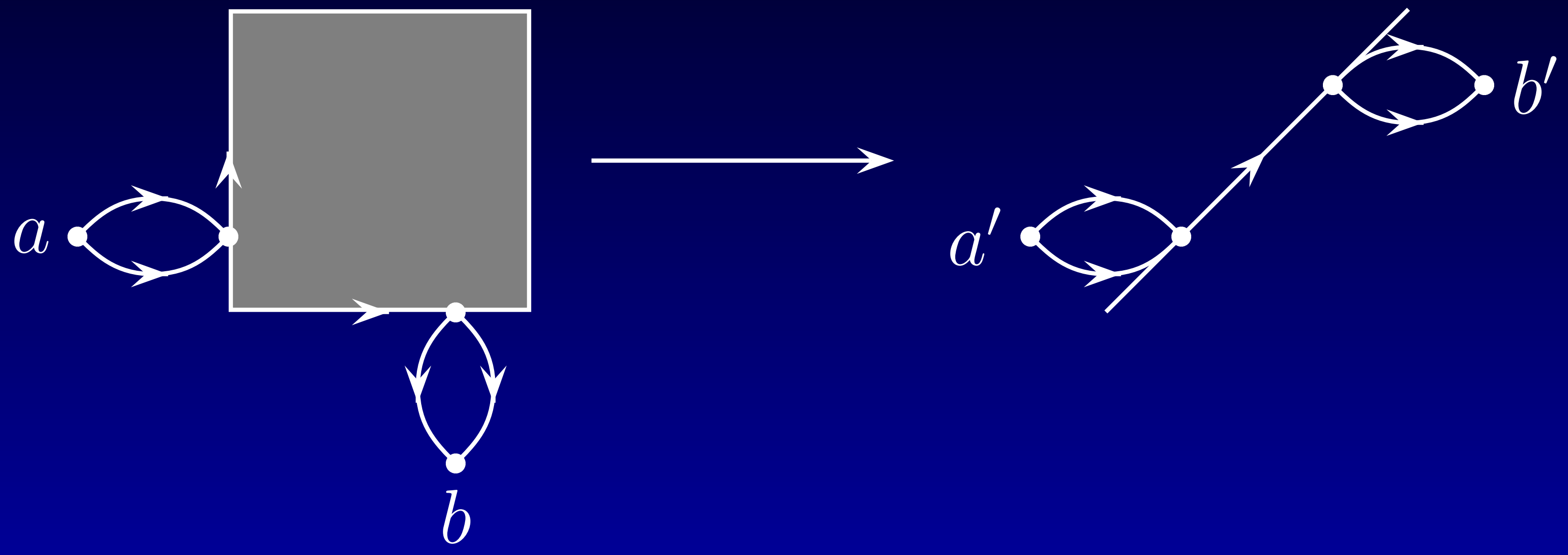


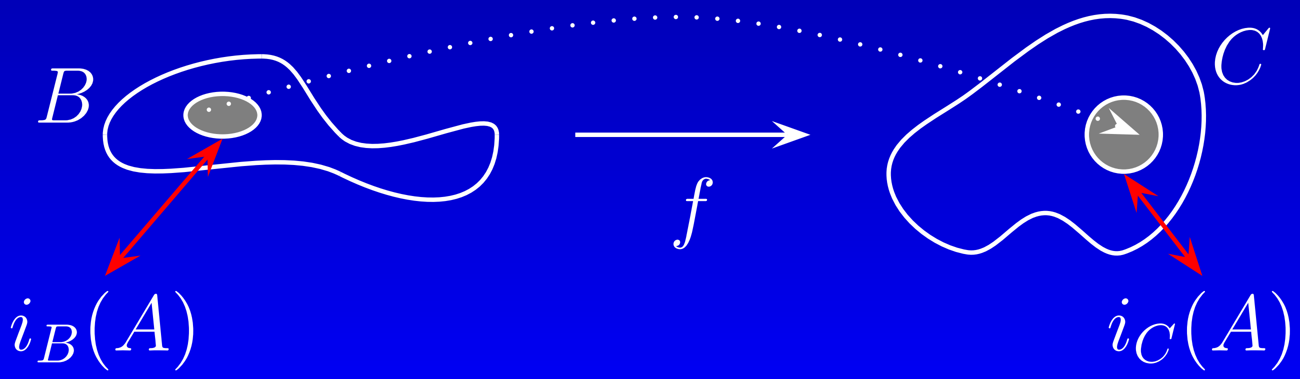

There is no execution path from $a$ to $b$, while there are such paths from $a'$ to $b'$.
Thus, this map should not be an equivalence.

# A Solution

Instead of working with just po-spaces work with po-spaces together with context.

Definition: Choose a po-space $A$ (called the context). This choice will depend on the attachments one wants to consider.
Consider po-spaces $B$ together with a dimap $i_B : A \longrightarrow B$ and consider morphisms which are dimaps such that

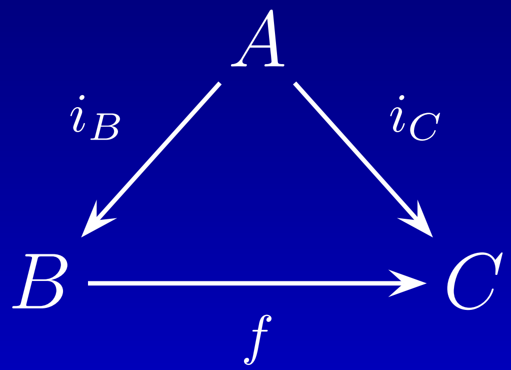
$$f(i_B(a)) = i_C(a) \text{ for all } a \in A$$



# $\mathbf{A} \downarrow \mathbf{PoSpc}$

Definition: Given a pospace $A$, let $\mathbf{A} \downarrow \mathbf{PoSpc}$ be the category whose

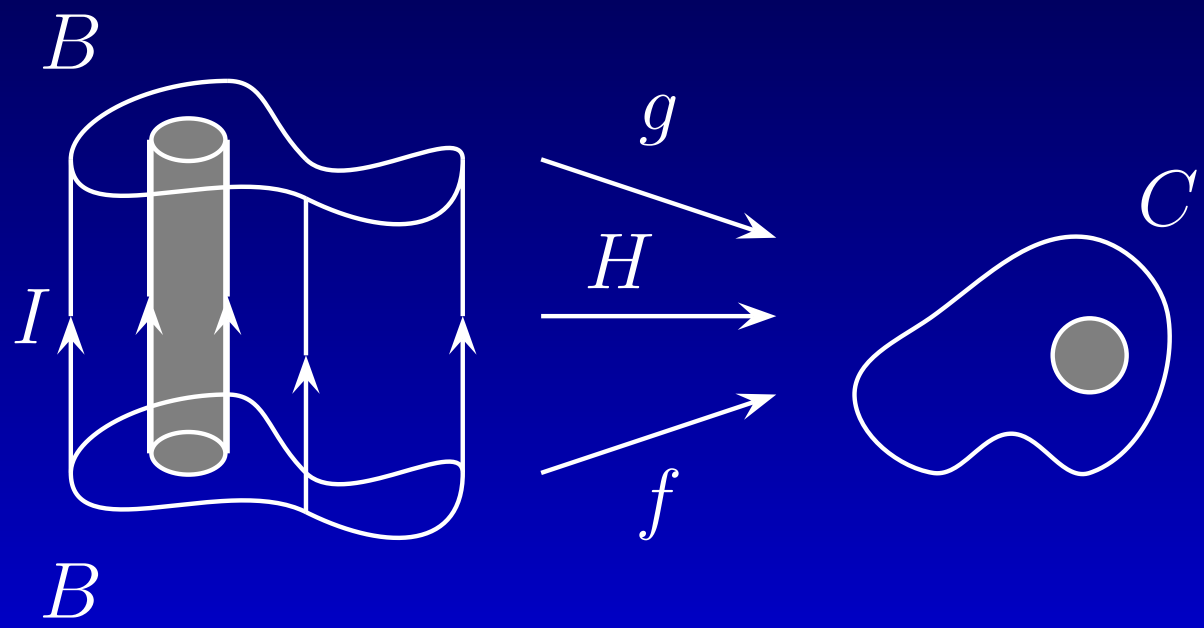
- objects are dimaps $\iota_B : A \to B$ ,
- morphisms are dimaps $f : B \to C$ such that $f \circ \iota_B = \iota_C$



# Equivalences using context

Definition:

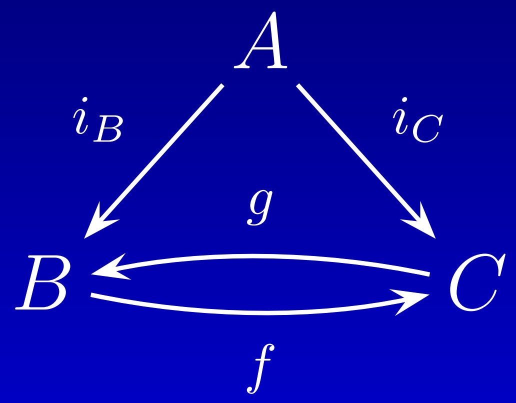
- A dihomotopy between $f, g : B \longrightarrow C$ in the context of $A$ is a dihomotopy $H : f \longrightarrow g$ rel $A$.



# Equivalences using context

Definition:

- Write $f \simeq g$ if there is a chain of dihomotopies $f \to f_1 \leftarrow f_2 \to \ldots \leftarrow f_n \to g$.
- $i_B : A \to B, i_C : A \to C$ are dihomotopy equivalent if there are dimaps



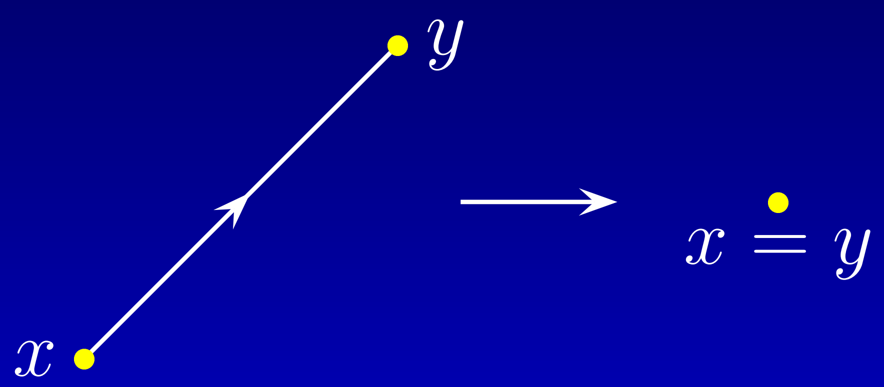

such that $g \circ f \simeq \mathrm{Id}_B$

and $f \circ g \simeq \mathrm{Id}_C$ .

# Example

Let $A = \{x, y\}$ with $x \leq y$.
Then po-spaces under the context $A$ are just po-spaces with two marked points, one of which is after the other.

In this category

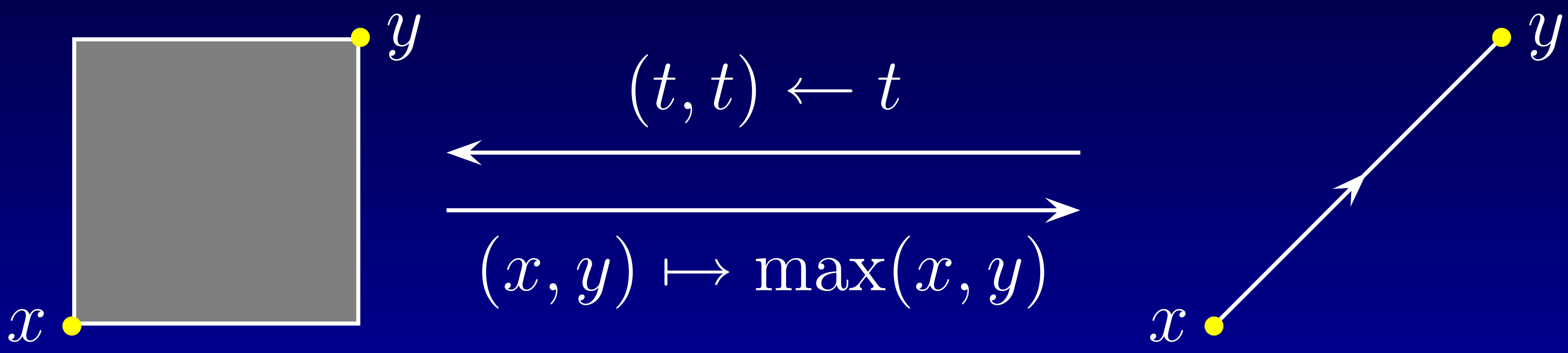



is not

a dihomotopy equivalence since there is no dimap in the reverse direction.

# Example of an equivalence

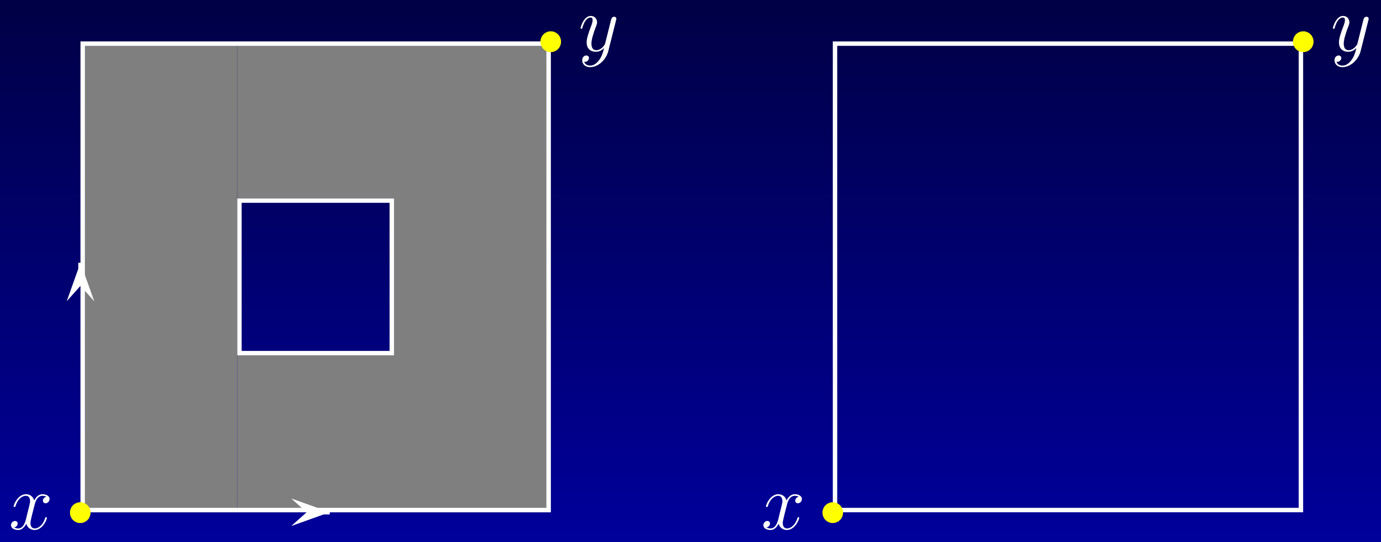
In the same context (of two marked points) the dimaps




give a dihomotopy equivalence.

# Another equivalence

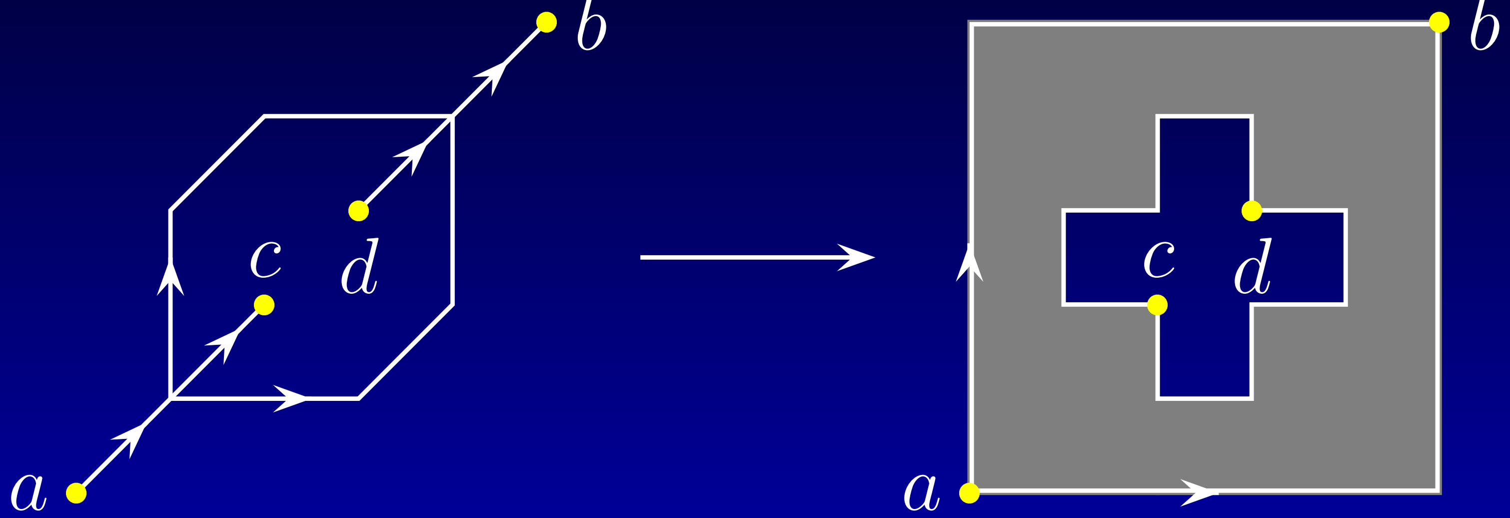
$\vec{I} \times \vec{I}$ with a square removed and two marked points




is dihomotopy equivalent to its boundary.

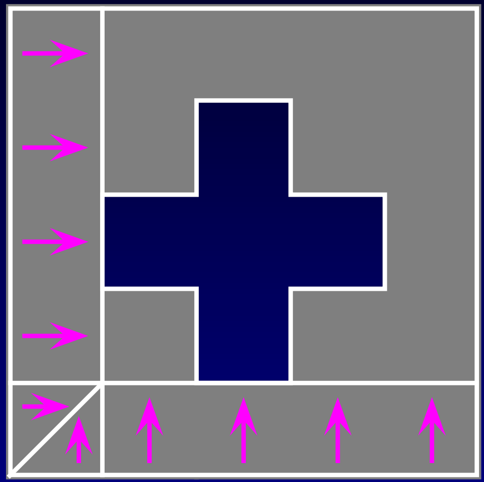
# Context for the Swiss flag

Let $A = \{a, b, c, d\}$. Then the inclusion




is a dihomotopy equivalence in the context of the four marked points.

# Sketch of the proof



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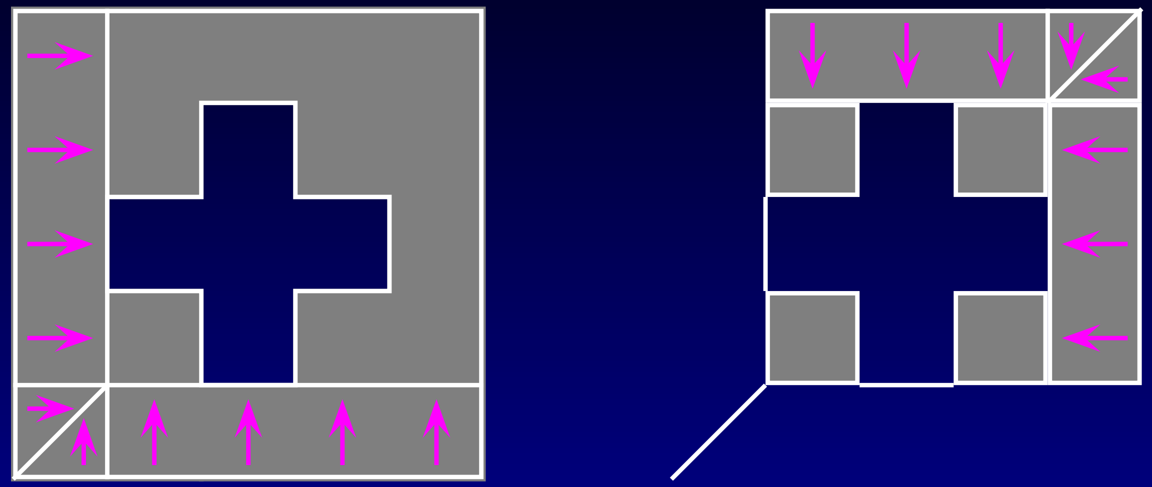

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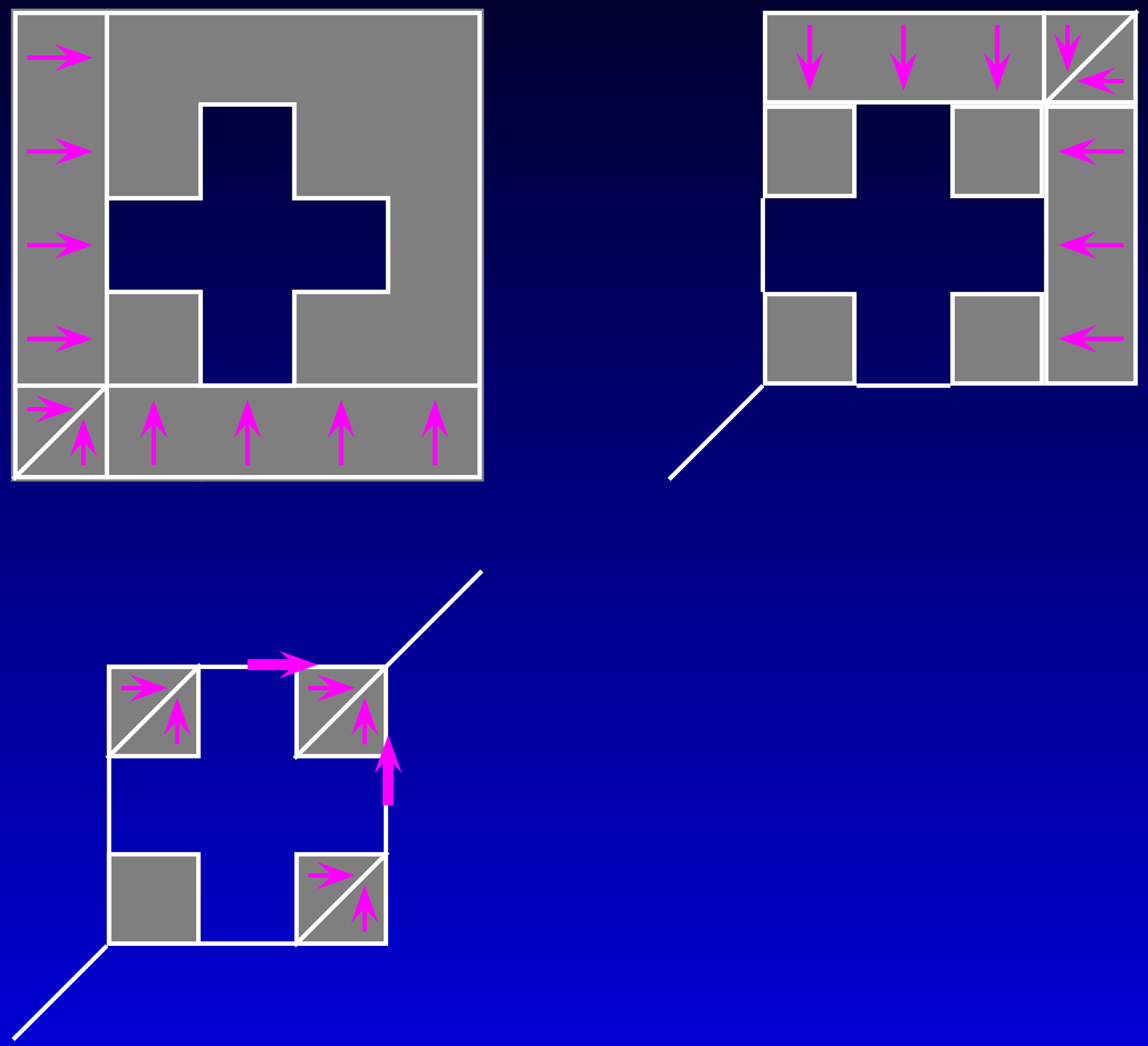

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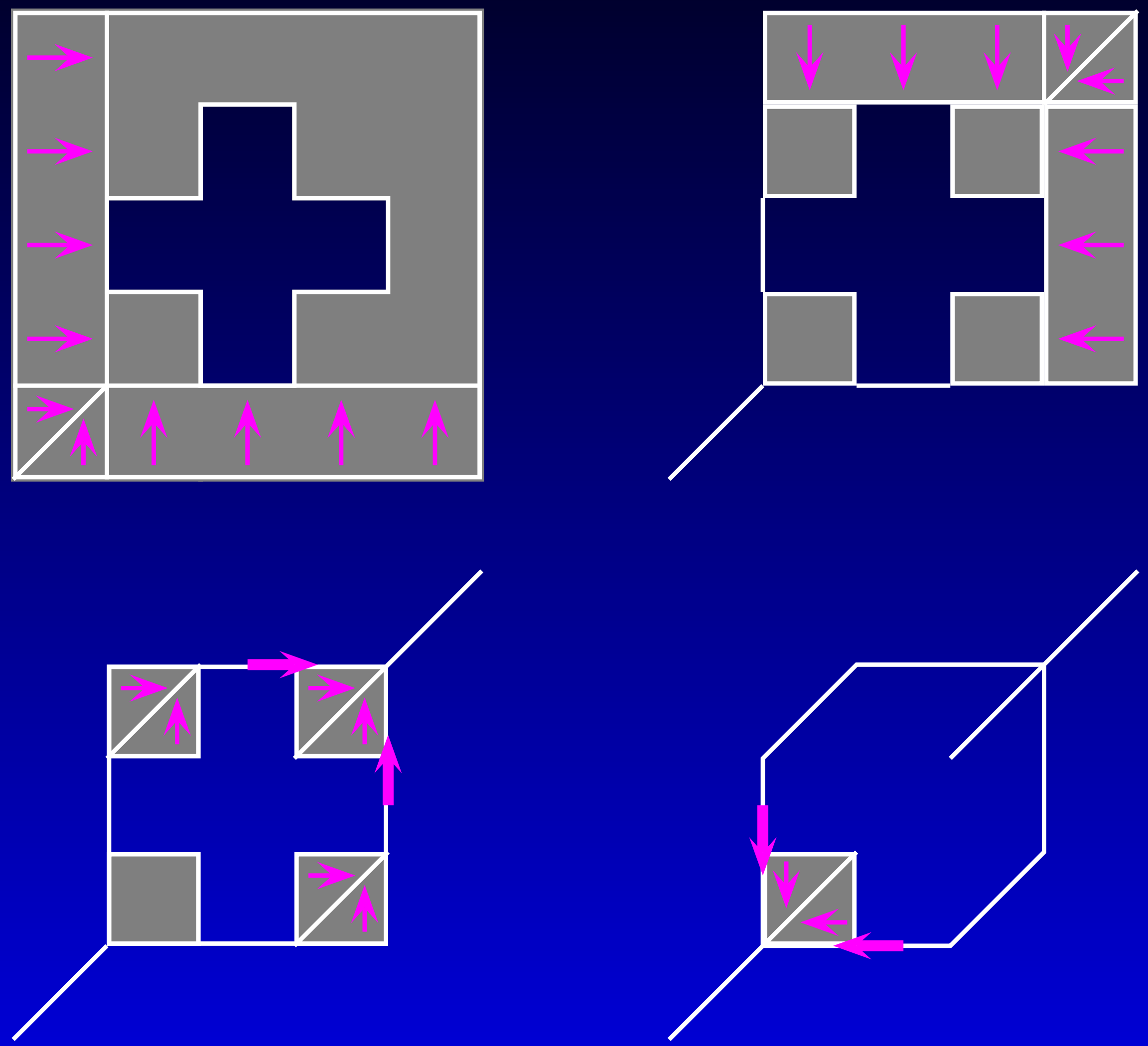


# Directed paths

Definition: Let $x, y \in$ the po-space $B$.

- A dipath is a dimap $\vec{I} \longrightarrow B$.
- Dipaths are dihomotopy equivalent if they are so in the context of their endpoints.
- Let $\vec{\pi}_1(B)(x, y)$ be the set of dihomotopy equivalence classes of dipaths from $x$ to $y$.

# Dipaths in equivalent spaces

Notation: Given a context $A$ and po-spaces $B, C$ together with $i_B : A \longrightarrow B$ and $i_C : A \longrightarrow C$, if $x \in A$ denote $i_B(X)$ by $x_B$ and $i_C(x)$ by $x_C$.

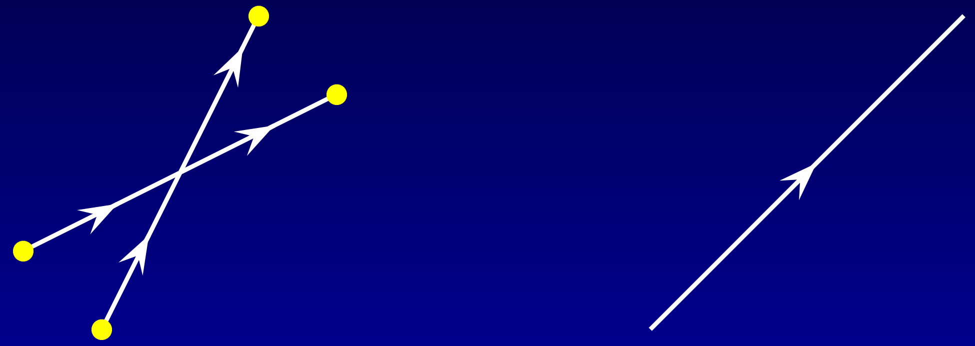
Proposition: Given a dimap $f : B \longrightarrow C$ respecting the context and $x, y \in A$ there is an induced map

$$\vec{\pi}_1(f)(x,y) : \vec{\pi}_1(B)(x_B, y_B) \longrightarrow \vec{\pi}_1(C)(x_C, y_C).$$

If $f$ is a dihomotopy equivalence then it is an isomorphism.

# Example

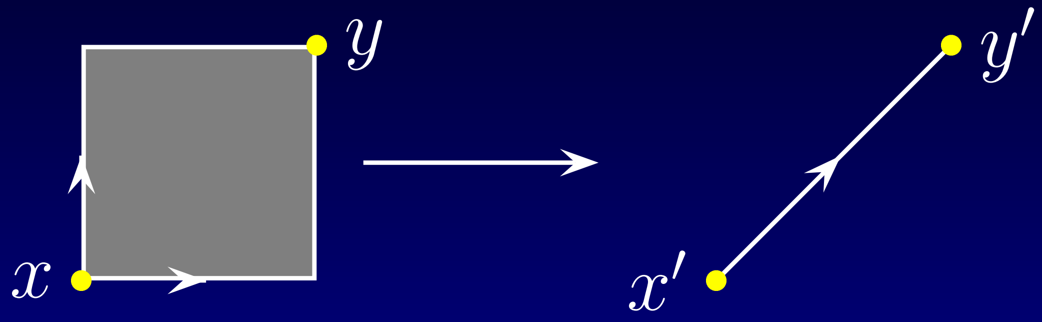
In the context of its four endpoints



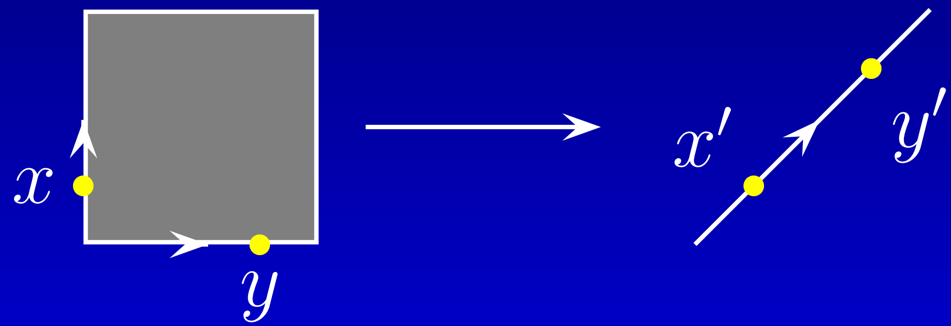
the left hand po-space is not dihomotopy equivalent to $\vec{I}$.

# Another example

Recall that there is a dihomotopy equivalence



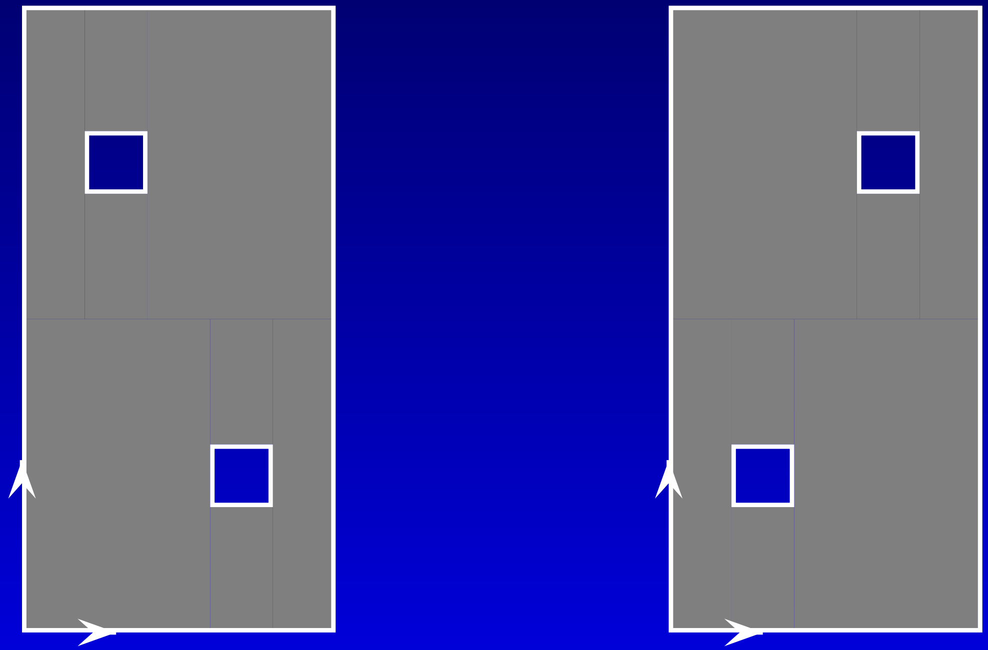

However there is no dihomotopy equivalence



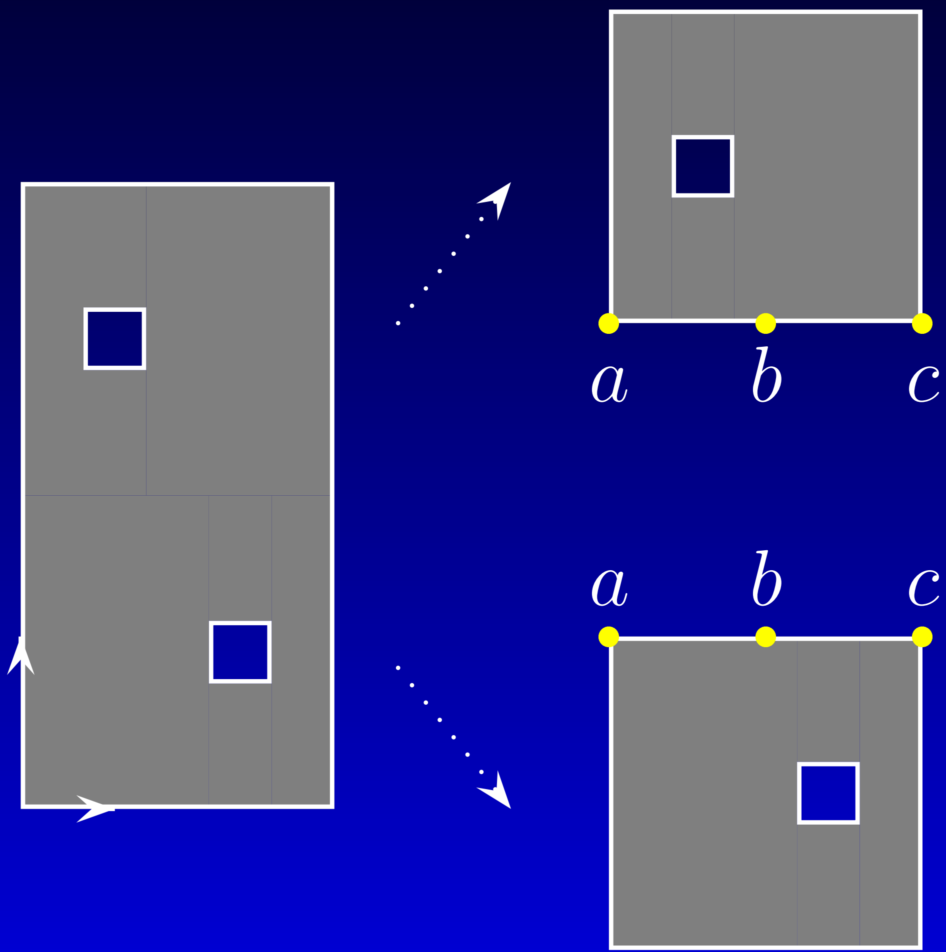

Equivalence depends on the context!

# Compound examples

We would like to find equivalent po-spaces to the following examples by analyzing them piece-by-piece.



# Equivalences of pieces



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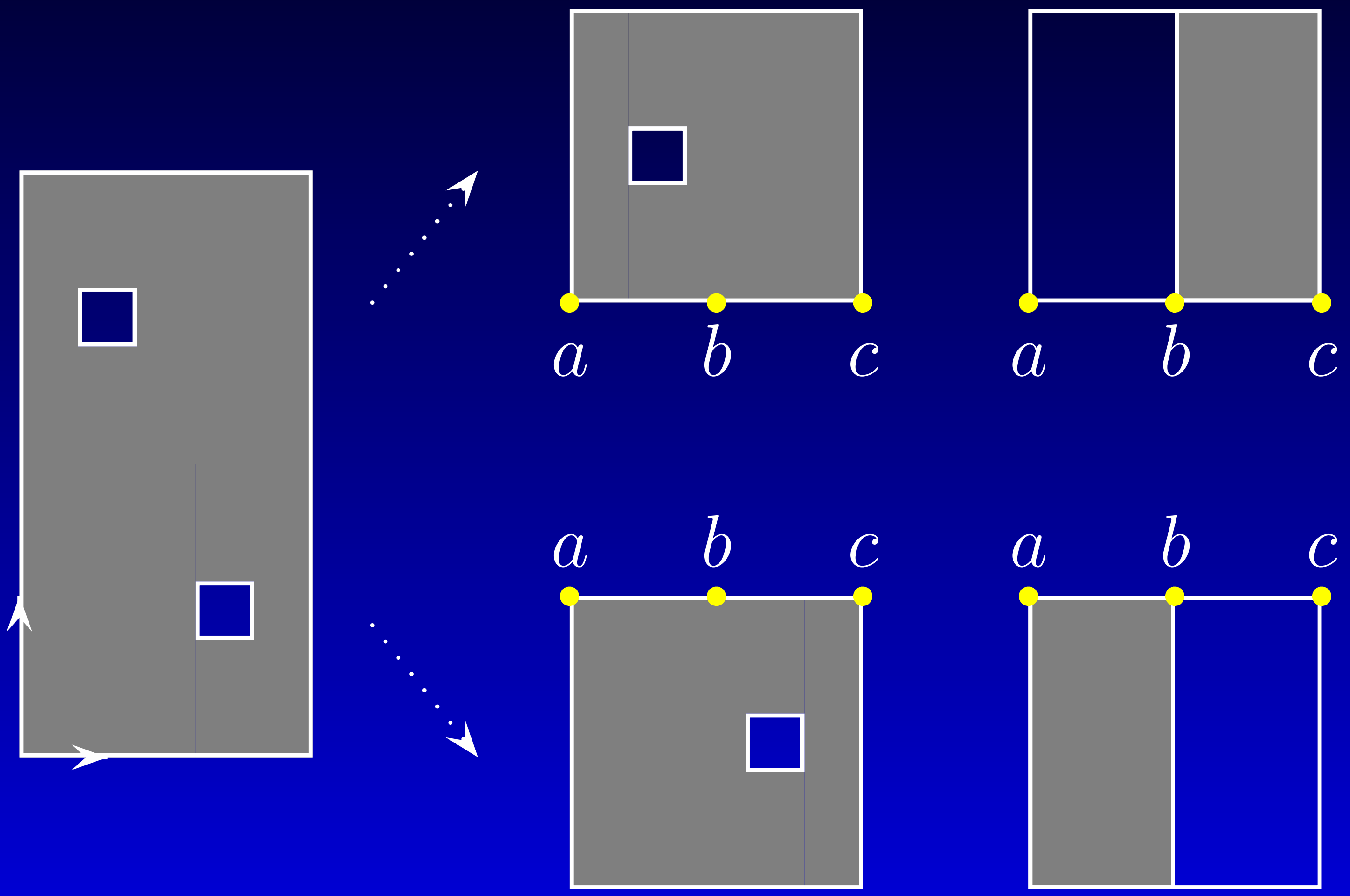
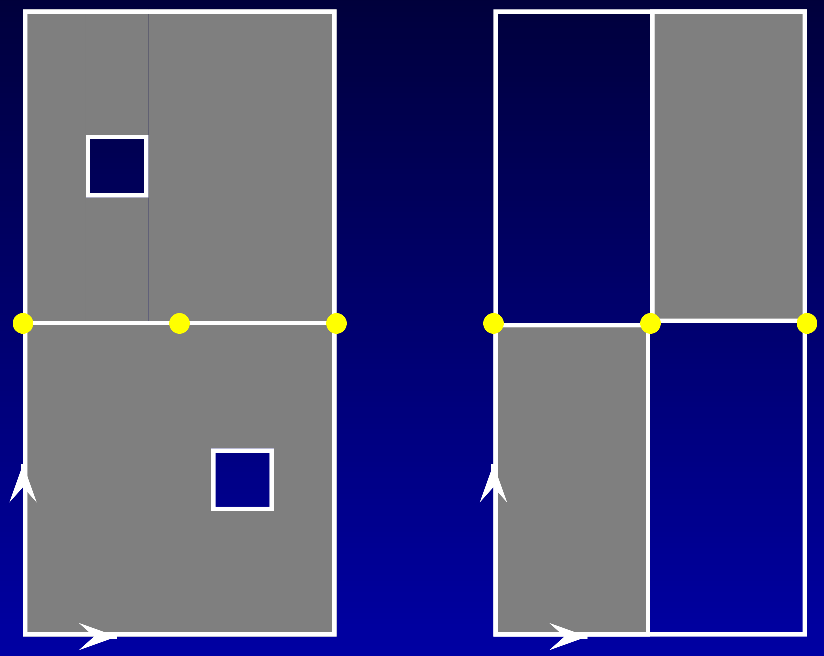


# Patching the pieces together



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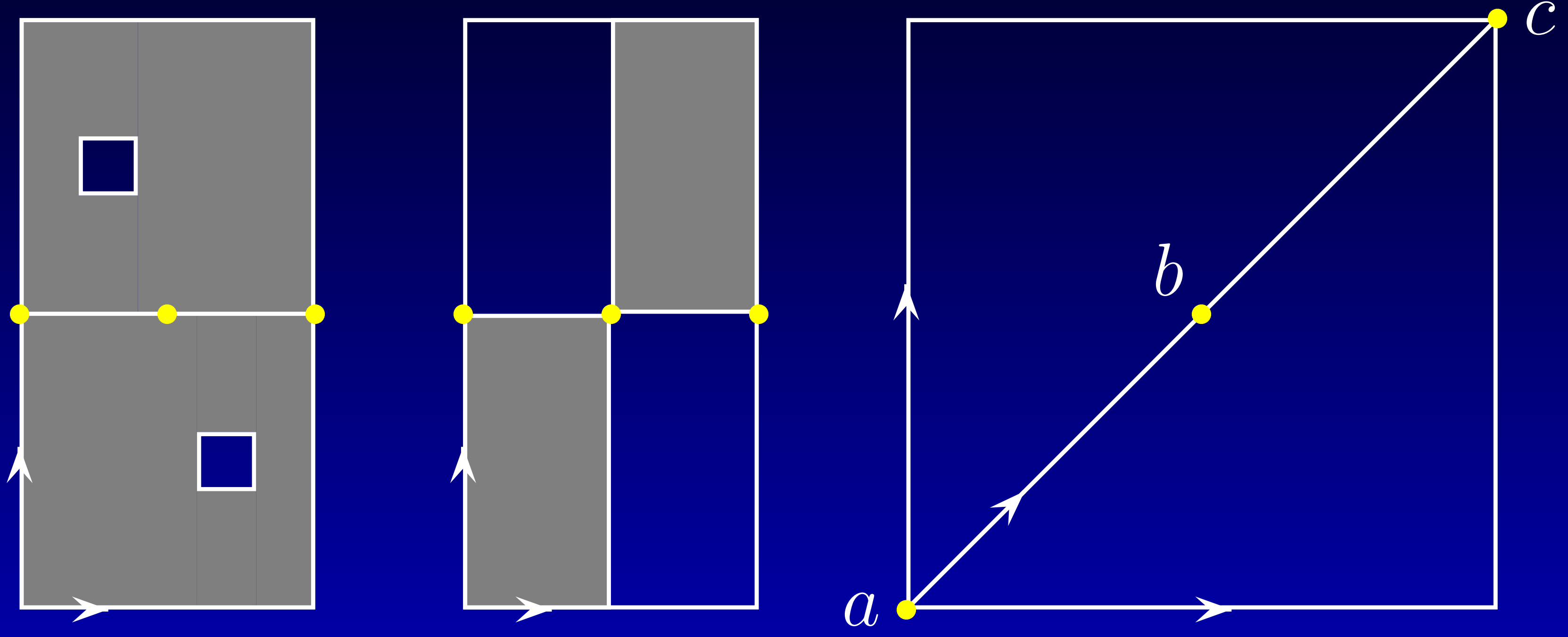
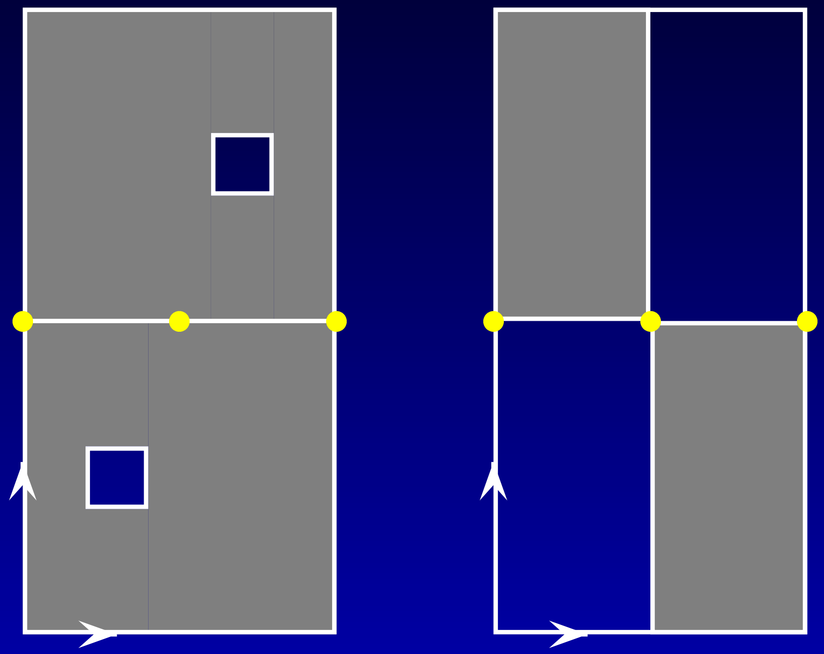


# Second example



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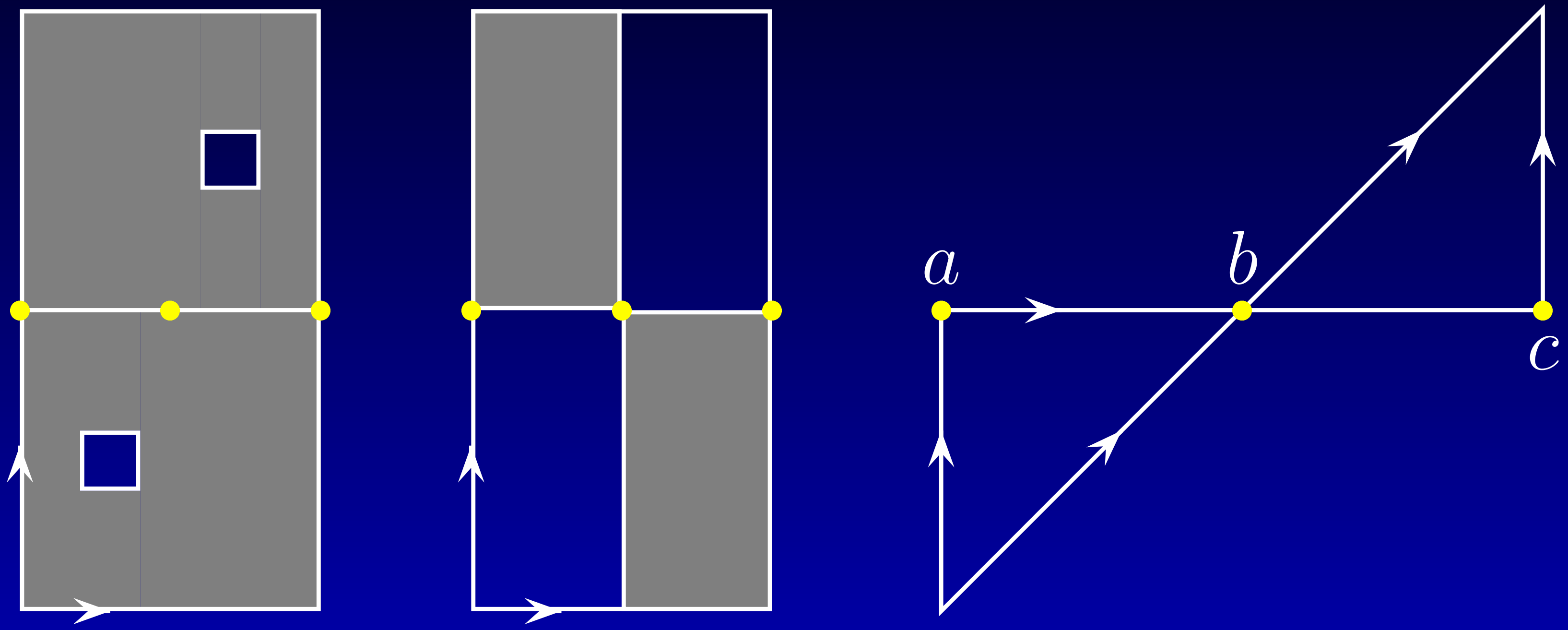


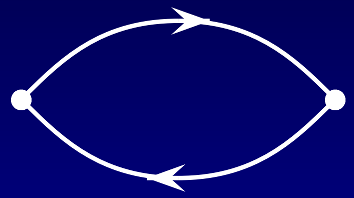
# 3. Modeling systems with loops

(joint work with Kris Worytkiewicz)

- Local po-spaces
- Model categories
- Equivalences for concurrent systems which may have loops

# A more general model

We would like to model execution loops such as



These cannot be modeled by po-spaces.

However they can be modeled by local po-spaces.

# Local po-spaces

Definition:

- An order atlas is a open cover of po-spaces with compatible partial orders.
- A local po-space is a topological space together with an equivalence class of order atlases.
- A morphism of local po-spaces is a continuous map which respects the orders.

# Equivalences of local po-spaces

Just as with po-spaces, we can define local po-spaces under some context $A$, and we consider morphisms which respect the context.

We can also define dihomotopy equivalences using context exactly the same way as with po-spaces.

# Enter some machinery

A powerful framework for studying equivalences is given by model categories.

Definition:
A model category is a category (with all small limits and colimits) and with three distinguished classes of morphisms: weak equivalences, cofibrations, and fibrations satisfying four simple axioms.

The structure of a model category allows one to apply the machinery of homotopy theory.

# Model category axioms

M0. $\mathbf{C}$ has all small limits and small colimits

M1. 2 out of 3

M2. retracts

M3. lifting property

M4. factorization

# A model for concurrent systems

Theorem [B-Worytkiewicz]:
The category of local po-spaces under a context $A$ embeds into a model category such that

- the weak equivalences are the dihomotopy equivalences rel $A$,
- the cofibrations are the monomorphisms, and
- pushouts of weak equivalence with cofibrations are weak equivalences

# Sheaves, Simplicial presheaves

Definition:

- The category of presheaves $\mathbf{Set}^{\mathbf{LoPospc}^{\mathrm{op}}}$ has as objects contravariant functors from $\mathbf{LoPospc}$ to $\mathbf{Set}$ and has as morphisms natural transformations

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- Similarly there is a category of simplicial presheaves $\mathbf{sSet}^{\mathbf{LoPospc}^{\mathrm{op}}}$
- There is a Yoneda embedding $\mathbf{LoPospc} \hookrightarrow \mathbf{Set}^{\mathbf{LoPospc}^{\mathrm{op}}} \hookrightarrow \mathbf{sSet}^{\mathbf{LoPospc}^{\mathrm{op}}}$
- The sheaves $\mathbf{Shv}(\mathbf{LoPospc})$ are the presheaves which are compatible with the topology

# Sketch of the proof

Theorem[Jardine]:
Let $\mathbf{C}$ be a small category with a Grothendieck topology. Then $\mathbf{sSet}^{\mathbf{C}^{op}}$ the category of simplicial presheaves on $\mathbf{C}$ has a (proper, simplicial) model structure in which

- the cofibrations are the monomorphisms, and
- the weak equivalences are the local weak equivalences.

Furthermore, if the Grothendieck topos $Shv(\mathbf{C})$ has enough points then the local weak equivalences are the stalkwise equivalences.

# Grothendieck topology

Proposition[B-W]: **LoPospc** has a Grothendieck topology given by open directed covers.

# Enough points

Let $Z$ be a local po-space. Definition: The category of directed étale bundles over $Z$ has

- objects: dimaps $E \to Z$ which are local homeomorphisms
- morphisms: maps $E_1 \to E_2$ such that

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

commutes

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Theorem[B-W]: There is an equivalence between $\mathbf{Shv(Z)}$ and $\mathbf{Etale(Z)}$.

Corollary: $\mathbf{Shv(LoPospc)}$ has enough points.

# Stalkwise equivalences

Using the above results, it follows from Jardine's theorem that $\mathbf{sSet}^{\mathbf{LoPospc}^{\mathrm{op}}}$ has a model structure in which the weak equivalences are the stalkwise equivalences.

# Stalkwise equivalences

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Proposition[B-W]: The stalkwise equivalences coming from $\mathbf{LoPospc}$ are the isomorphisms.

# Adding context

Given context $A \in \mathbf{LoPospc}$ there is a induced model structure on

$$\mathbf{A} \downarrow \mathbf{sSet}^{\mathbf{LoPospc}^{\mathbf{op}}}$$

Finally one can localize with respect to the dihomotopy equivalences rel $A$ to obtain the main theorem. $\square$

# 4. Summary

- It would be useful to have a robust notion of equivalence in models of concurrency.
- To allow a piece-by-piece analysis we would like equivalences that remain equivalences even after additions are made to the model.
- Using context provides such equivalences.

# Summary

- Using po-spaces, local po-spaces, context, and model categories, we have a good mathematical framework for studying concurrent systems.
- In particular, using equivalences, this framework should allow for a piece-by-piece analysis of concurrent systems.

# Future Work

Theoretical: Consider all possible contexts in a single framework. (Stay tuned for Kris' talk!)

Practical: Use the current theoretical framework for analyzing real-world examples.

# Acknowledgments

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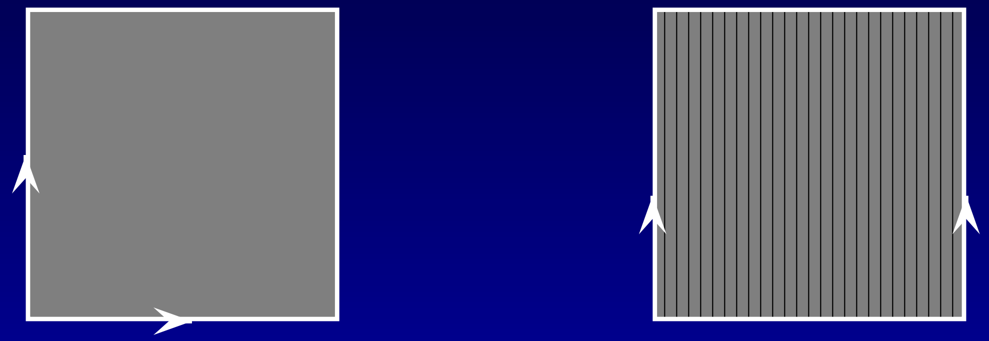
This talk and associated preprints are available at
http://igat.epfl.ch/bubenik/

# Non-discrete context

Take $\vec{I} \times \vec{I}$ and $I \times \vec{I}$



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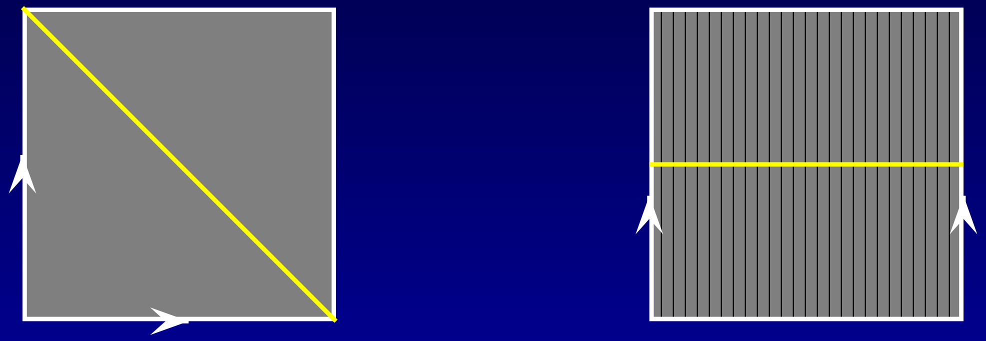

and glue them together along the yellow lines.