Models and van Kampen theorems for directed homotopy theory

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Directed spaces

**Definition**

A **directed space** is a topological space $X$ together with a set $dX$ of continuous maps $[0, 1] \to X$ called **directed paths** satisfying the following:

1. all constant paths are directed paths;
2. directed paths are closed under concatenation; and
3. if $\gamma$ is a directed path and $f : [0, 1] \to [0, 1]$ is a non-decreasing continuous map then $\gamma \circ f$ is a directed path.

A **directed map** $f : (X, dX) \to (Y, dY)$ is a continuous map $f : X \to Y$ such that $dX \subseteq dY$. 

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Directed homotopy theory
Examples of directed spaces

- Any topological space $X$ is a directed space with $dX$ equal to the set of all paths in $X$. 
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- Let $\vec{I}$ be $[0, 1]$ together with all non-decreasing continuous maps $f : [0, 1] \rightarrow [0, 1]$. So directed paths in a directed space $X$ are exactly the directed maps $\vec{I} \rightarrow X$. 
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- Let $\vec{S}^1$ be the unit circle together with all counterclockwise paths.
Constructions on directed spaces

- Given two directed spaces $X$ and $Y$, then
  - $X \sqcup Y$ is a directed space with $d(X \sqcup Y) = dX \sqcup dY$.
  - $X \times Y$ is a directed space with $d(X \times Y) = dX \times dY$ where $(f, g)(t) = (f(t), g(t))$.

- If $X$ is a directed space and $A \subseteq X$, then
  - $A$ is a directed space with $dA$ equal to the subset of paths in $dX$ whose image is in $A$.
  - $X/A$ is a directed space.

- In fact, directed spaces have all limits and colimits.
Concurrent parallel computing

Several processes with shared resources
A concurrent system

Example

2 processes using 2 shared resources \(a\) and \(b\) which can only be used by one process at a time

Notation

\[ P_x - \text{a process locks resource } x \]
\[ V_x - \text{a process releases resource } x \]

Program

The first process: \(Pa \quad Pb \quad Vb \quad Va\)
The second process: \(Pb \quad Pa \quad Va \quad Vb\)
The Swiss flag
The Swiss flag

Example
**Problem:** Uncountably many states and execution paths.
Directed homotopies

**Definition**

A homotopy between directed maps $f, g : B \to C$ is a directed map $H : B \times \vec{I} \to C$ restricting to $f$ and $g$. Write $H : f \rightsquigarrow g$.

**Definition**

Directed maps $f, g$ are homotopic if there is a chain of homotopies

$$f \rightsquigarrow f_1 \xleftarrow{\sim} f_2 \rightsquigarrow \ldots \xleftarrow{\sim} f_n \rightsquigarrow g.$$
Equivalence classes of directed paths

**Definition**

Directed paths are *homotopy equivalent* if they are so relative to their endpoints.

**Remark**

*Directed paths up to homotopy are significantly different from paths up to homotopy!*
There are paths which are not homotopic to directed paths.
A room with two barriers

Two directed paths which are homotopic as paths, but not as directed paths.
The fundamental group, and the fundamental groupoid

**Definition**

- For $x \in X$, the fundamental group $\pi_1(X, x)$ is the set of homotopy classes of paths beginning and ending at $x$.
- The fundamental groupoid $\pi_1(X)$, is a category with
  - objects: points in $X$
  - morphisms: homotopy classes of paths

**Remark**

*The existence of composition with associativity and identity is built into the definition of a category.*
The fundamental group, and the fundamental groupoid

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The existence of composition with associativity and identity is built into the definition of a category.
The fundamental category

**Definition**

The fundamental category $\vec{\pi}_1(X)$ has
- objects: the points in $X$
- morphisms: homotopy classes of directed paths
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- objects: the points in $X$
- morphisms: homotopy classes of directed paths

### Problem

*The fundamental category is enormous.*
Full subcategories of the fundamental category

Plan
We would like to derive a “small” category from the fundamental category that still contains useful information.

Definition
Given $A \subseteq X$, let $\pi_1(X, A)$ have
- objects: points in $A$
- morphisms: homotopy classes of paths in $X$
The fundamental bipartite graph

**Definition**

For \((X, dX)\) write \(x \leq y\) if there exists a dipath \(\gamma\) with \(\gamma(0) = 1\) and \(\gamma(1) = y\). This gives \(X\) a preorder.
The fundamental bipartite graph

**Definition**
For \((X, dX)\) write \(x \leq y\) if there exists a dipath \(\gamma\) with \(\gamma(0) = 1\) and \(\gamma(1) = y\). This gives \(X\) a preorder.

**Definition**
Let \(\text{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}\).
Let \(\text{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}\).

**Definition (B)**
The fundamental bipartite graph of \(X\) is \(\vec{\pi}_1(X, \text{Min}(X) \cup \text{Max}(X))\).
Example of the fundamental bipartite graph
**Definition**

A *future retract* of $\pi_1(X)$ moves each $x \in X$ along a directed path in $X$ to a point $x^+$ which “has the same future”.

\[ P^+ : \pi_1(X) \to \pi_1(X, A) \]
**Definition**

A past retract of $\vec{\pi}_1(X)$ moves each $x \in X$ backwards along a directed path in $X$ to a point $x^-$ which “has the same past”.

$$P^- : \vec{\pi}_1(X) \to \vec{\pi}_1(X, A)$$
Retracts for triples

For $A \subseteq B \subseteq X$, one can similarly define future retracts

$$P^+ : \bar{\pi}_1(X, B) \to \bar{\pi}_1(X, A).$$

This functor is a left adjoint to the inclusion functor.

Dually, one has past retracts.
Definition (B)

An extremal model is a chain of future retracts and past retracts

\[ \tilde{\pi}_1(X) \xrightarrow{P_1^+} \tilde{\pi}_1(X, X_1) \xrightarrow{P_2^-} \tilde{\pi}_1(X, X_2) \xrightarrow{P_3^+} \ldots \xrightarrow{P_n^±} \tilde{\pi}_1(X, A), \]

such that \( \text{Min}(X) \cup \text{Max}(X) \subseteq A \).
Extremal models

Definition (B)

An **extremal model** is a chain of future retracts and past retracts

\[\bar{\pi}_1(X) \xrightarrow{P_1^+} \bar{\pi}_1(X, X_1) \xrightarrow{P_2^-} \bar{\pi}_1(X, X_2) \xrightarrow{P_3^+} \ldots \xrightarrow{P_n^\pm} \bar{\pi}_1(X, A),\]

such that \(\text{Min}(X) \cup \text{Max}(X) \subseteq A\).

Proposition (B)

An extremal model induces an injection of fundamental bipartite graphs.

Theorem (B)

If \(X\) is a compact and \(\leq\) is a partial order, then extremal models induce an isomorphism of fundamental bipartite graphs.
Let $X$ be a path–connected space with $dX$ all paths. Choose $x \in X$.

There is a unique functor $\vec{\pi}_1(X) \to \vec{\pi}_1(X, x)$.

This functor is a future retract, a past retract, and a minimal extremal model.

It coincides with the functor from the fundamental groupoid to the fundamental group.
Examples of extremal models
An extremal model for the Swiss flag

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Directed homotopy theory
Examples of extremal models
Let $x \in \tilde{S}^1$.

There is a future retract

$$P^+ : \tilde{\pi}_1(\tilde{S}^1) \to \tilde{\pi}_1(\tilde{S}^1, x) \cong (\mathbb{N}, +).$$

It is a minimal extremal model.
Van Kampen Theorem for the fundamental category

**Theorem (Grandis 2003, Goubault 2003)**

Assume $X = \text{Int}(X_1) \cup \text{Int}(X_2)$ and let $X_0 = X_1 \cap X_2$. Then the pushout of directed spaces:

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X
\end{array}
$$

induces a pushout of fundamental categories:

$$
\begin{array}{ccc}
\vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\
\downarrow & & \downarrow \\
\vec{\pi}_1(X_2) & \longrightarrow & \vec{\pi}_1(X)
\end{array}
$$
Compatible subspaces

\[ X = \text{Int}(X_1) \cup \text{Int}(X_2) \] and \[ A = \text{Int}(A_1) \cup \text{Int}(A_2). \]

\[ X_0 = X_1 \cap X_2 \] and \[ A_0 = A_1 \cap A_2. \]

\[ A_k \subseteq X_k, \quad k = 0, 1, 2. \]

We have the following pushout in the arrow category of pospaces.

![Diagram of pushout](attachment:pushout_diagram.png)
Furthermore, assume that we have compatible retracts:

\[ \tilde{\pi}_1(X_1, A_1) \leftarrow \tilde{\pi}_1(X_0, A_0) \rightarrow \tilde{\pi}_1(X_2, A_2) \]

\[ \tilde{\pi}_1(X_1) \leftarrow \tilde{\pi}_1(X_0) \leftarrow \tilde{\pi}_1(X_2) \]

\[ P_1^+ \]

\[ P_0^+ \]

\[ P_2^+ \]
Van Kampen theorem for full subcategories

Theorem (B)

The above inclusions induce the following pushout in $\mathbf{Cat}$.

$$
\begin{array}{ccc}
\vec{\pi}_1(X_0, A_0) & \longrightarrow & \vec{\pi}_1(X_2, A_2) \\
\downarrow & & \downarrow \\
\vec{\pi}_1(X_1, A_1) & \longrightarrow & \vec{\pi}_1(X, A)
\end{array}
$$
Van Kampen theorem for full subcategories

**Theorem (B)**

Furthermore, they induce the following pushout in the arrow category on $\mathbf{Cat}$.

$$
\begin{array}{ccc}
\tilde{\pi}_1(X_0, A_0) & \to & \tilde{\pi}_1(X_2, A_2) \\
\downarrow & & \downarrow \\
\tilde{\pi}_1(X_1, A_1) & \to & \tilde{\pi}_1(X, A) \\
\downarrow & & \downarrow \\
\tilde{\pi}_1(X_0) & \to & \tilde{\pi}_1(X_2) \\
\tilde{\pi}_1(X_1) & \to & \tilde{\pi}_1(X) \\
\end{array}
$$
Theorem (B)

Finally, there is an induced retraction $P^+$, which is a pushout.

\[
\begin{align*}
\vec{\pi}_1(X_0, A_0) & \xrightarrow{P_0^+} \vec{\pi}_1(X_1, A_1) & \xrightarrow{P_1^+} \vec{\pi}_1(X_0) & \xrightarrow{P^+} \vec{\pi}_1(X_1) \\
& \xrightarrow{P_2^+} \vec{\pi}_1(X_2, A_2) & \xrightarrow{} \vec{\pi}_1(X, A) & \xrightarrow{} \vec{\pi}_1(X)
\end{align*}
\]
Compatible triples
Compatible retracts

\[ \vec{\pi}_1(X_0, A_0) \xrightarrow{P_0^+} \vec{\pi}_1(X_2, A_2) \]

\[ \vec{\pi}_1(X_0, B_0) \xrightarrow{P_2^+} \vec{\pi}_1(X_2, B_2) \]

\[ \vec{\pi}_1(X_1, A_1) \xrightarrow{P_1^+} \vec{\pi}_1(X_1, B_1) \]
Theorem (B)

The inclusions above induce the following pushout in the arrow category on $\text{Cat}$.

\[
\begin{array}{ccc}
\pi_1(X_0, A_0) & \longrightarrow & \pi_1(X_2, A_2) \\
\pi_1(X_1, A_1) & \longrightarrow & \pi_1(X, A) \\
\pi_1(X_0, B_0) & \longrightarrow & \pi_1(X_2, B_2) \\
\pi_1(X_1, B_1) & \longrightarrow & \pi_1(X, B)
\end{array}
\]
A van Kampen theorem for future (past) retracts

**Theorem (B)**

*There is an induced retraction $P^+$, which is a pushout.*

\[
\begin{align*}
\pi_1(X_0, A_0) & \xrightarrow{P_0^+} \pi_1(X, A) & \xrightarrow{P_2^+} \pi_1(X_2, A_2) \\
\pi_1(X_0, B_0) & \xrightarrow{P_1^+} \pi_1(X, B) & \xrightarrow{P^+} \pi_1(X_2, B_2)
\end{align*}
\]
A van Kampen theorem for extremal models

**Theorem (B)**

The pushout of compatible extremal models is an extremal model.
Van Kampen for extremal models example.
Van Kampen for extremal models example
Van Kampen for extremal models example
Summary

- Directed spaces provide a good mathematical model for concurrent parallel computing.
- Directed paths up to homotopy are different from paths up to homotopy.
- The homotopy classes of directed paths assemble into the fundamental category.
- Minimal extremal models provide a way to generalize the fundamental group to directed spaces.
- There is a van Kampen theorem for extremal models.
Applications

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.

- E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.
The fundamental bipartite graphs detects deadlocks and captures the essential schedules. Is this part of a homology theory?

What can we do with higher directed homotopy?