

Models and van Kampen theorems for directed homotopy theory

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Directed spaces

Definition

A **directed space** is a topological space X together with a set dX of continuous maps $[0, 1] \rightarrow X$ called **directed paths** satisfying the following:

- 1 all constant paths are directed paths;
- 2 directed paths are closed under concatenation; and
- 3 if γ is a directed path and $f : [0, 1] \rightarrow [0, 1]$ is a non-decreasing continuous map then $\gamma \circ f$ is a directed path.

A **directed map** $f : (X, dX) \rightarrow (Y, dY)$ is a continuous map $f : X \rightarrow Y$ such that $dX \subseteq dY$.

Examples of directed spaces

- Any topological space X is a directed space with dX equal to the set of all paths in X .

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- Let \vec{I} be $[0, 1]$ together with all non-decreasing continuous maps $f : [0, 1] \rightarrow [0, 1]$.
So directed paths in a directed space X are exactly the directed maps $\vec{I} \rightarrow X$.

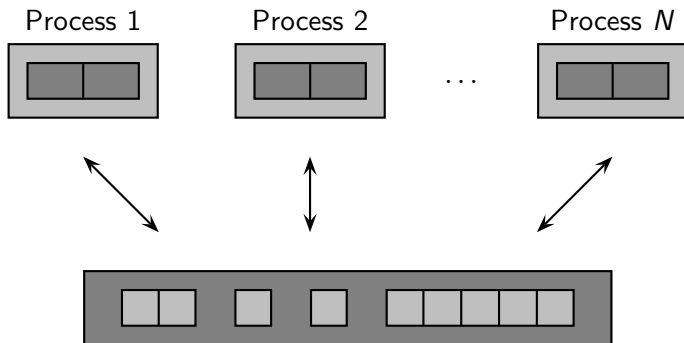
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- Let \vec{I} be $[0, 1]$ together with all non-decreasing continuous maps $f : [0, 1] \rightarrow [0, 1]$.
So directed paths in a directed space X are exactly the directed maps $\vec{I} \rightarrow X$.
- Let \vec{S}^1 be the unit circle together with all counterclockwise paths.

Constructions on directed spaces

- Given two directed spaces X and Y , then
 - $X \sqcup Y$ is a directed space with $d(X \sqcup Y) = dX \sqcup dY$.
 - $X \times Y$ is a directed space with $d(X \times Y) = dX \times dY$ where $(f, g)(t) = (f(t), g(t))$.
- If X is a directed space and $A \subseteq X$, then
 - A is a directed space with dA equal to the subset of paths in dX whose image is in A .
 - X/A is a directed space .
- In fact, directed spaces have all limits and colimits.

Concurrent parallel computing



Several processes with shared resources

A concurrent system

Example

2 processes using 2 shared resources a and b which can only be used by one process at a time

Notation

P_x - a process locks resource x

V_x - a process releases resource x

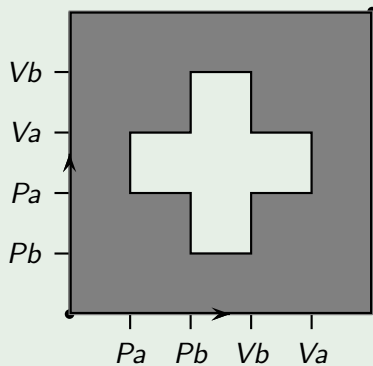
Program

The first process: $P_a \quad P_b \quad V_b \quad V_a$

The second process: $P_b \quad P_a \quad V_a \quad V_b$

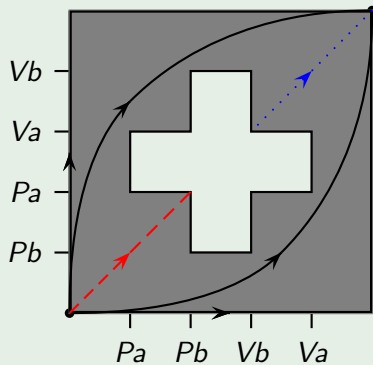
The Swiss flag

Example



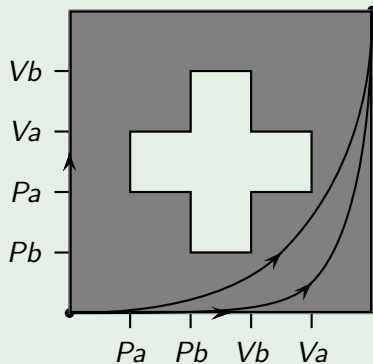
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Example



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Example

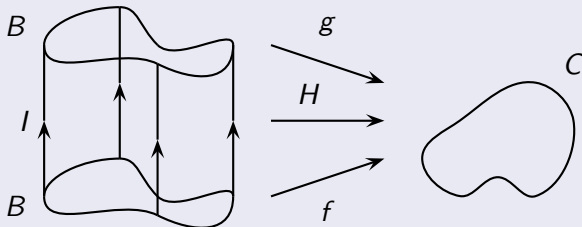


Problem: Uncountably many states and execution paths.

Directed homotopies

Definition

A **homotopy** between directed maps $f, g : B \rightarrow C$ is a directed map $H : B \times \vec{I} \rightarrow C$ restricting to f and g . Write $H : f \xrightarrow{\sim} g$.



Definition

Directed maps f, g are homotopic if there is a chain of homotopies

$$f \xrightarrow{\sim} f_1 \xleftarrow{\sim} f_2 \xrightarrow{\sim} \dots \xleftarrow{\sim} f_n \xrightarrow{\sim} g.$$

Equivalence classes of directed paths

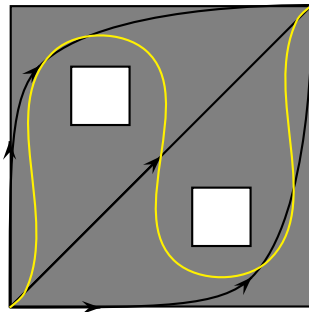
Definition

Directed paths are **homotopy equivalent** if they are so relative to their endpoints.

Remark

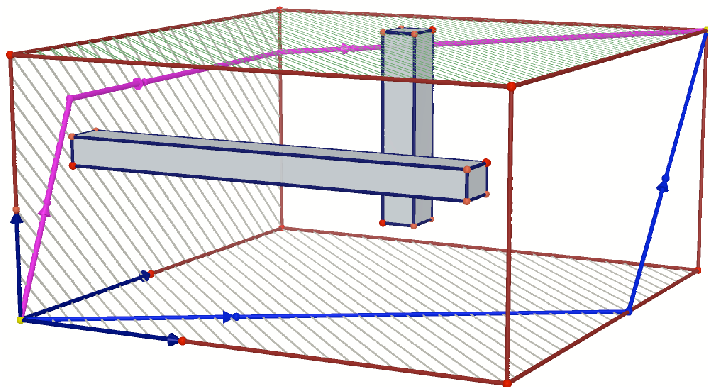
Directed paths up to homotopy are significantly different from paths up to homotopy!

Directed paths



There are paths which are not homotopic to directed paths.

A room with two barriers



Two directed paths which are homotopic as paths, but not as directed paths.

The fundamental group, and the fundamental groupoid

Definition

- For $x \in X$, the **fundamental group** $\pi_1(X, x)$ is the set of homotopy classes of paths beginning and ending at x .
- The **fundamental groupoid** $\pi_1(X)$, is a category with
 - objects: points in X
 - morphisms: homotopy classes of paths

Remark

The existence of composition with associativity and identity is built into the definition of a category.

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The **fundamental category** $\vec{\pi}_1(X)$ has

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Problem

The fundamental category is enormous.

Full subcategories of the fundamental category

Plan

We would like to derive a “small” category from the fundamental category that still contains useful information.

Definition

Given $A \subseteq X$, let $\vec{\pi}_1(X, A)$ have

- objects: points in A
- morphisms: homotopy classes of paths in X

The fundamental bipartite graph

Definition

For (X, dX) write $x \leq y$ if there exists a dipath γ with $\gamma(0) = x$ and $\gamma(1) = y$. This gives X a preorder.

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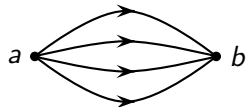
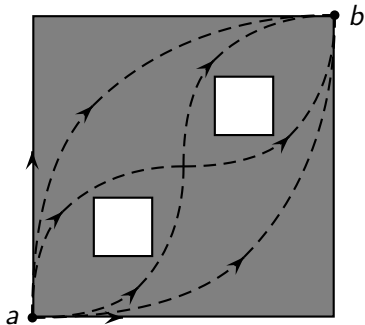
Let $\text{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}$.

Let $\text{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}$.

Definition (B)

The **fundamental bipartite graph** of X is $\vec{\pi}_1(X, \text{Min}(X) \cup \text{Max}(X))$.

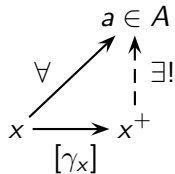
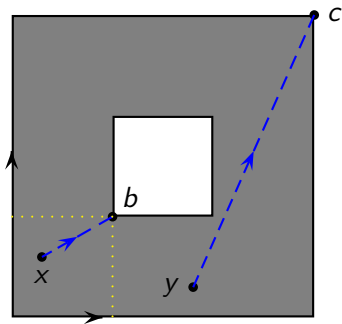
Example of the fundamental bipartite graph



Future retracts

Definition

A **future retract** of $\vec{\pi}_1(X)$ moves each $x \in X$ along a directed path in X to a point x^+ which “has the same future”.

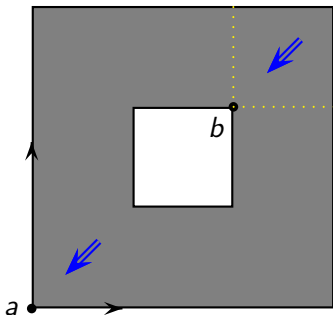


$$P^+ : \vec{\pi}_1(X) \rightarrow \vec{\pi}_1(X, A)$$

Past retracts

Definition

A **past retract** of $\vec{\pi}_1(X)$ moves each $x \in X$ backwards along a directed path in X to a point x^- which “has the same past”.



$$\begin{array}{ccc}
 & & x^- \xrightarrow{[\gamma_x]} x \\
 & \uparrow \exists! & \nearrow \forall \\
 & a \in A &
 \end{array}$$

$$P^- : \vec{\pi}_1(X) \rightarrow \vec{\pi}_1(X, A)$$

Retracts for triples

For $A \subseteq B \subseteq X$, one can similarly define future retracts

$$P^+ : \vec{\pi}_1(X, B) \rightarrow \vec{\pi}_1(X, A).$$

This functor is a left adjoint to the inclusion functor.

Dually, one has past retracts.

Extremal models

Definition (B)

An **extremal model** is a chain of future retracts and past retracts

$$\vec{\pi}_1(X) \xrightarrow{P_1^+} \vec{\pi}_1(X, X_1) \xrightarrow{P_2^-} \vec{\pi}_1(X, X_2) \xrightarrow{P_3^+} \dots \xrightarrow{P_n^\pm} \vec{\pi}_1(X, A),$$

such that $\text{Min}(X) \cup \text{Max}(X) \subseteq A$.

Extremal models

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An **extremal model** is a chain of future retracts and past retracts

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such that $\text{Min}(X) \cup \text{Max}(X) \subseteq A$.

Proposition (B)

An extremal model induces an injection of fundamental bipartite graphs.

Theorem (B)

If X is a compact and \leq is a partial order, then extremal models induce an isomorphism of fundamental bipartite graphs.

Extremal models and the fundamental group

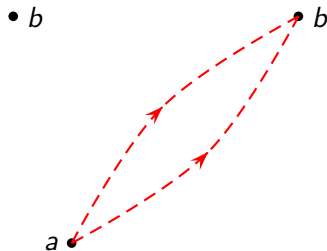
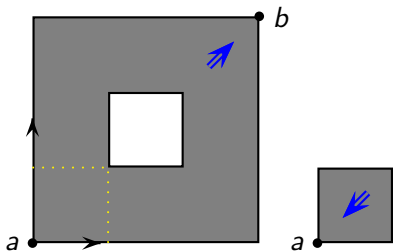
Let X be a path-connected space with dX all paths.
Choose $x \in X$.

There is a unique functor $\vec{\pi}_1(X) \rightarrow \vec{\pi}_1(X, x)$.

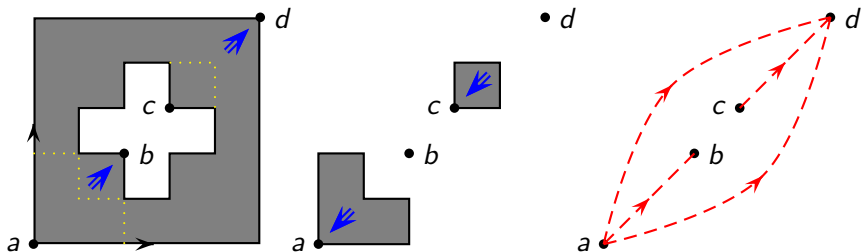
This functor is a future retract, a past retract, and a minimal extremal model.

It coincides with the functor from the fundamental groupoid to the fundamental group.

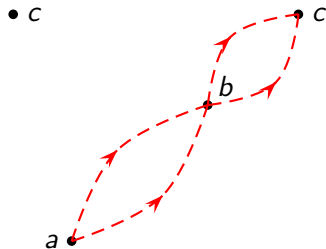
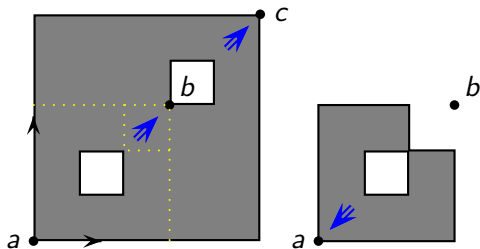
Examples of extremal models



An extremal model for the Swiss flag



Examples of extremal models



An extremal model of \vec{S}^1

Let $x \in \vec{S}^1$.

There is a future retract

$$P^+ : \vec{\pi}_1(\vec{S}^1) \rightarrow \vec{\pi}_1(\vec{S}^1, x) \cong (\mathbb{N}, +).$$

It is a minimal extremal model.

Van Kampen Theorem for the fundamental category

Theorem (Grandis 2003, Goubault 2003)

Assume $X = \text{Int}(X_1) \cup \text{Int}(X_2)$ and let $X_0 = X_1 \cap X_2$. Then the pushout of directed spaces :

$$\begin{array}{ccc}
 X_0 & \longrightarrow & X_1 \\
 \downarrow & & \downarrow \\
 X_2 & \dashrightarrow & X
 \end{array}$$

induces a pushout of fundamental categories:

$$\begin{array}{ccc}
 \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_1) \\
 \downarrow & & \downarrow \\
 \vec{\pi}_1(X_2) & \dashrightarrow & \vec{\pi}_1(X)
 \end{array}$$

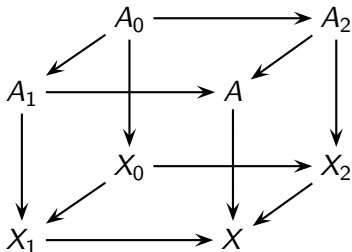
Compatible subspaces

$X = \text{Int}(X_1) \cup \text{Int}(X_2)$ and $A = \text{Int}(A_1) \cup \text{Int}(A_2)$.

$X_0 = X_1 \cap X_2$ and $A_0 = A_1 \cap A_2$.

$A_k \subseteq X_k$, $k = 0, 1, 2$.

We have the following pushout in the arrow category of pospaces.



Compatible retracts

Furthermore, assume that we have compatible retracts:

$$\begin{array}{ccccc}
 & & \vec{\pi}_1(X_0, A_0) & \longrightarrow & \vec{\pi}_1(X_2, A_2) \\
 & \swarrow & \uparrow P_0^+ & & \uparrow P_2^+ \\
 \vec{\pi}_1(X_1, A_1) & & & & \\
 \uparrow P_1^+ & & \vec{\pi}_1(X_0) & \longrightarrow & \vec{\pi}_1(X_2) \\
 \vec{\pi}_1(X_1) & \swarrow & & &
 \end{array}$$

Van Kampen theorem for full subcategories

Theorem (B)

The above inclusions induce the following pushout in **Cat**.

$$\begin{array}{ccc} \vec{\pi}_1(X_0, A_0) & \longrightarrow & \vec{\pi}_1(X_2, A_2) \\ \downarrow & & \downarrow \\ \vec{\pi}_1(X_1, A_1) & \longrightarrow & \vec{\pi}_1(X, A) \end{array}$$

Van Kampen theorem for full subcategories

Theorem (B)

Furthermore, they induce the following pushout in the arrow category on **Cat**.

$$\begin{array}{ccccc}
 & \vec{\pi}_1(X_0, A_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) & \\
 & \swarrow & & \searrow & \\
 \vec{\pi}_1(X_1, A_1) & \xrightarrow{\quad} & \vec{\pi}_1(X, A) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) \\
 \downarrow & \dashrightarrow & \downarrow & \dashrightarrow & \downarrow \\
 \vec{\pi}_1(X_1) & \xrightarrow{\quad} & \vec{\pi}_1(X_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2) \\
 \downarrow & \dashrightarrow & \downarrow & \dashrightarrow & \downarrow \\
 \vec{\pi}_1(X_1) & \xrightarrow{\quad} & \vec{\pi}_1(X) & \xrightarrow{\quad} & \vec{\pi}_1(X_2)
 \end{array}$$

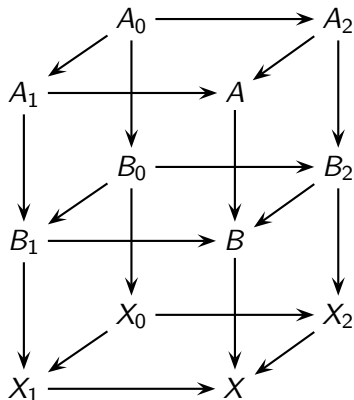
van Kampen theorem for future retracts

Theorem (B)

Finally, there is an induced retraction P^+ , which is a pushout.

$$\begin{array}{ccccc}
 & & \vec{\pi}_1(X_0, A_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) \\
 & \swarrow & \uparrow & & \swarrow \\
 \vec{\pi}_1(X_1, A_1) & \text{---} & \vec{\pi}_1(X, A) & \text{---} & \vec{\pi}_1(X_2, A_2) \\
 & \uparrow P_1^+ & \uparrow P_0^+ & & \uparrow P_2^+ \\
 & & \vec{\pi}_1(X_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2) \\
 & \swarrow & \uparrow P^+ & & \swarrow \\
 \vec{\pi}_1(X_1) & \text{---} & \vec{\pi}_1(X) & \text{---} & \vec{\pi}_1(X_2)
 \end{array}$$

Compatible triples



Compatible retracts

$$\begin{array}{ccccc}
 & & \vec{\pi}_1(X_0, A_0) & \longrightarrow & \vec{\pi}_1(X_2, A_2) \\
 & \swarrow & \uparrow P_0^+ & & \uparrow P_2^+ \\
 \vec{\pi}_1(X_1, A_1) & & & & \\
 \uparrow P_1^+ & & \vec{\pi}_1(X_0, B_0) & \longrightarrow & \vec{\pi}_1(X_2, B_2) \\
 \vec{\pi}_1(X_1, B_1) & \swarrow & & &
 \end{array}$$

A van Kampen theorem for full subcategories

Theorem (B)

The inclusions above induce the following pushout in the arrow category on **Cat**.

$$\begin{array}{ccccc}
 & \vec{\pi}_1(X_0, A_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) & \\
 & \swarrow & & \swarrow & \\
 \vec{\pi}_1(X_1, A_1) & \xrightarrow{\quad} & \vec{\pi}_1(X, A) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \vec{\pi}_1(X_1, B_1) & \xrightarrow{\quad} & \vec{\pi}_1(X, B) & & \vec{\pi}_1(X_2, B_2)
 \end{array}$$

The diagram shows a pushout in the arrow category on **Cat**. The objects are $\vec{\pi}_1(X_i, A_i)$ and $\vec{\pi}_1(X_i, B_i)$ for $i=0,1,2$, and $\vec{\pi}_1(X, A)$ and $\vec{\pi}_1(X, B)$. The arrows are:

- Solid arrows: $\vec{\pi}_1(X_0, A_0) \rightarrow \vec{\pi}_1(X_2, A_2)$, $\vec{\pi}_1(X_0, A_0) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X_0, A_0) \rightarrow \vec{\pi}_1(X_0, B_0)$, $\vec{\pi}_1(X_0, B_0) \rightarrow \vec{\pi}_1(X, B)$, $\vec{\pi}_1(X_1, A_1) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X_1, A_1) \rightarrow \vec{\pi}_1(X_1, B_1)$, $\vec{\pi}_1(X_1, B_1) \rightarrow \vec{\pi}_1(X, B)$, $\vec{\pi}_1(X_2, A_2) \rightarrow \vec{\pi}_1(X_2, B_2)$.
- Dashed arrows: $\vec{\pi}_1(X_1, A_1) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X_1, B_1) \rightarrow \vec{\pi}_1(X, B)$, $\vec{\pi}_1(X_2, A_2) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X_2, B_2) \rightarrow \vec{\pi}_1(X, B)$.
- A vertical dotted arrow points from $\vec{\pi}_1(X, A)$ to $\vec{\pi}_1(X, B)$.

A van Kampen theorem for future (past) retracts

Theorem (B)

There is an induced retraction P^+ , which is a pushout.

$$\begin{array}{ccccc}
 & & \vec{\pi}_1(X_0, A_0) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) \\
 & \swarrow & \uparrow & & \swarrow \\
 \vec{\pi}_1(X_1, A_1) & \xrightarrow{\quad} & \vec{\pi}_1(X, A) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, A_2) \\
 \uparrow P_1^+ & & \uparrow P_0^+ & & \uparrow P_2^+ \\
 \vec{\pi}_1(X_0, B_0) & \xrightarrow{\quad} & \vec{\pi}_1(X, B) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, B_2) \\
 \uparrow & & \uparrow P^+ & & \uparrow \\
 \vec{\pi}_1(X_1, B_1) & \xrightarrow{\quad} & \vec{\pi}_1(X, B) & \xrightarrow{\quad} & \vec{\pi}_1(X_2, B_2)
 \end{array}$$

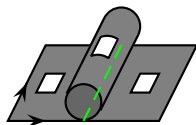
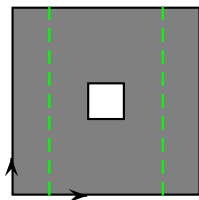
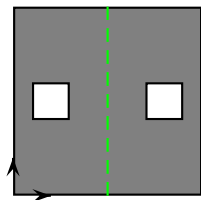
The diagram illustrates a commutative structure involving fundamental groups. It consists of two rows of nodes. The top row nodes are $\vec{\pi}_1(X_0, A_0)$, $\vec{\pi}_1(X, A)$, and $\vec{\pi}_1(X_2, A_2)$. The bottom row nodes are $\vec{\pi}_1(X_0, B_0)$, $\vec{\pi}_1(X, B)$, and $\vec{\pi}_1(X_2, B_2)$. Vertical arrows connect $\vec{\pi}_1(X_0, A_0) \rightarrow \vec{\pi}_1(X_0, B_0)$, $\vec{\pi}_1(X, A) \rightarrow \vec{\pi}_1(X, B)$, and $\vec{\pi}_1(X_2, A_2) \rightarrow \vec{\pi}_1(X_2, B_2)$. Horizontal arrows connect $\vec{\pi}_1(X_0, A_0) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X, A) \rightarrow \vec{\pi}_1(X_2, A_2)$, $\vec{\pi}_1(X_0, B_0) \rightarrow \vec{\pi}_1(X, B)$, and $\vec{\pi}_1(X, B) \rightarrow \vec{\pi}_1(X_2, B_2)$. Dashed arrows connect $\vec{\pi}_1(X_1, A_1) \rightarrow \vec{\pi}_1(X_0, A_0)$, $\vec{\pi}_1(X_1, A_1) \rightarrow \vec{\pi}_1(X, A)$, $\vec{\pi}_1(X_1, B_1) \rightarrow \vec{\pi}_1(X_0, B_0)$, and $\vec{\pi}_1(X_1, B_1) \rightarrow \vec{\pi}_1(X, B)$. A vertical dashed arrow labeled P^+ points from $\vec{\pi}_1(X, B)$ to $\vec{\pi}_1(X, A)$. Vertical arrows labeled P_1^+ , P_0^+ , and P_2^+ point from the bottom row to the top row.

A van Kampen theorem for extremal models

Theorem (B)

The pushout of compatible extremal models is an extremal model.

Van Kampen for extremal models example



Summary

- Directed spaces provide a good mathematical model for concurrent parallel computing.
- Directed paths up to homotopy are different from paths up to homotopy.
- The homotopy classes of directed paths assemble into the fundamental category.
- Minimal extremal models provide a way to generalize the fundamental group to directed spaces.
- There is a van Kampen theorem for extremal models.

Applications

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.
- E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.

Open problems

- The fundamental bipartite graphs detects deadlocks and captures the essential schedules. Is this part of a homology theory?
- What can we do with higher directed homotopy?