A mathematical model for concurrent parallel computing

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The end of Moore’s Law?

Introduction

Models

Concln

A mathematical model for concurrent parallel computing
Example: Multi-core processors

Intel

AMD
Example: Internet database
Classical non-parallel computing

A process with its own private resources
Concurrent parallel computing

Several processes with shared resources
A mathematical model

Concurrent systems can be modeled by spaces in which only some certain paths are allowed.

**Definition**

A **directed space** is a topological space $X$ with a distinguished set of paths, $\gamma : [0, 1] \to X$, called **directed paths**.

**Example**

- A line segment with direction arrows.
- A square with arrows indicating direction.
- A circle with arrows, depicting a cycle.
A concurrent system

Example
2 processes using 2 shared resources $a$ and $b$ which can only be used by one process at a time

Notation
$P_x$ - a process locks resource $x$
$V_x$ - a process releases resource $x$

Program
The first process: $Pa$ $Pb$ $Vb$ $Va$
The second process: $Pb$ $Pa$ $Va$ $Vb$
The Swiss flag

Example

A mathematical model for concurrent parallel computing
The Swiss flag

Example

A mathematical model for concurrent parallel computing
Problem: The state space is infinite.
The essential schedules of the Swiss flag

Example

This is a subspace of the Swiss flag.
Develop a framework for concurrency where equivalences are accounted for.

We would like equivalences that allow a piece-by-piece analysis. This will make the analysis of large programs tractable.
Develop a framework for concurrency where equivalences are accounted for.

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*Idea*

*Use algebraic topology.*
Undirected equivalences

**Definition**

Given continuous maps $f, g : B \rightarrow C$, 

$B$ \quad \rightarrow \quad f \quad \rightarrow \quad B$ 

$g \quad \rightarrow \quad \quad \quad \quad \quad \quad \quad \rightarrow \quad C$ 

$B$ 

$C$
Undirected equivalences

Definition

Given continuous maps $f, g : B \to C$,
Undirected equivalences

Definition

Given continuous maps $f, g : B \to C$, a homotopy between $f$ and $g$ is a continuous map $H : B \times I \to C$ restricting to $f$ and $g$. This is an equivalence relation. Write $H \cdot [f] \simeq [g]$. 
Undirected equivalences

Definition
Spaces $B, C$ are homotopy equivalent if there are maps $f : B \leftrightarrow C : g$ such that

$$g \circ f \simeq \operatorname{Id}_B \text{ and } f \circ g \simeq \operatorname{Id}_C.$$
Directed spaces

**Definition**

- A **directed space** is a topological space $X$ with a distinguished set of **directed paths**.
- A **directed map** is a continuous map $f : X \rightarrow Y$ between directed spaces such that if $\gamma : [0,1] \rightarrow X$ is a directed path in $X$, then $f(\gamma)$ is a directed path in $Y$.

**Remark**

*Subspaces, products, and quotients of directed spaces inherit a directed space structure.*
A homotopy between directed maps \( f, g : B \to C \) is a directed map \( H : B \times \vec{I} \to C \) restricting to \( f \) and \( g \). Write \( H : f \to g \).
Directed equivalences

Definition

1. Write $f \simeq g$ if there is a chain of dihomotopies
   \[ f \rightarrow f_1 \leftarrow f_2 \rightarrow \ldots \leftarrow f_n \rightarrow g. \]

2. Po-Spaces $B, C$ are dihomotopy equivalent if there are dimaps $f : B \Leftrightarrow C : g$ such that
   \[ g \circ f \simeq \text{Id}_B \quad \text{and} \quad f \circ g \simeq \text{Id}_C. \]
The fundamental category \( \pi_1(X) \) has

- objects: the points in \( X \)
- morphisms: homotopy classes of directed paths
The fundamental category

**Definition**

The fundamental category $\pi_1(X)$ has
- objects: the points in $X$
- morphisms: homotopy classes of directed paths

**Problem**

*The fundamental category is enormous.*
Full subcategories of the fundamental category

Plan

We would like to derive a “small” category from the fundamental category that still contains useful information.

Definition

Given $A \subseteq X$, let $\vec{\pi}_1(X, A)$ have

- objects: points in $A$
- morphisms: homotopy classes of paths in $X$
The fundamental bipartite graph

Definition

For \((X, dX)\) write \(x \leq y\) if there exists a dipath \(\gamma\) with \(\gamma(0) = 1\) and \(\gamma(1) = y\). This gives \(X\) a preorder.
The fundamental bipartite graph

**Definition**

For \((X, dX)\) write \(x \leq y\) if there exists a dipath \(\gamma\) with \(\gamma(0) = 1\) and \(\gamma(1) = y\). This gives \(X\) a preorder.

**Definition**

Let \(\text{Min}(X) = \{a \in X \mid a' \leq a \implies a' = a\}\).
Let \(\text{Max}(X) = \{b \in X \mid b \leq b' \implies b = b'\}\).

**Definition (B)**

The **fundamental bipartite graph** of \(X\) is \(\vec{\pi}_1(X, \text{Min}(X) \cup \text{Max } X)\).
Example of the fundamental bipartite graph

\begin{itemize}
\item[a] \quad \text{a}
\item[b] \quad \text{b}
\end{itemize}

\begin{itemize}
\item[a] \quad \text{a}
\item[b] \quad \text{b}
\end{itemize}
**Future retracts**

**Definition**

A future retract of $\vec{\pi}_1(X)$ moves each $x \in X$ along a directed path in $X$ to a point $x^+$ which “has the same future”.

$$P^+ : \vec{\pi}_1(X) \to \vec{\pi}_1(X, A)$$
Past retracts

Definition

A past retract of $\pi_1(X)$ moves each $x \in X$ backwards along a directed path in $X$ to a point $x^-$ which “has the same past”.

$$P^- : \pi_1(X) \to \pi_1(X, A)$$
Extremal models

Definition (B)

An extremal model is a chain of future retracts and past retracts

\[ \pi_1(X) \xrightarrow{P_1^+} \pi_1(X, X_1) \xrightarrow{P_2^-} \pi_1(X, X_2) \xrightarrow{P_3^+} \ldots \xrightarrow{P_n^\pm} \pi_1(X, A), \]

such that \( \text{Min}(X) \cup \text{Max}(X) \subseteq A. \)
**Extremal models**

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**Proposition (B)**

An extremal model induces an injection of fundamental bipartite graphs.

**Theorem (B)**

*If* \( X \) *is a compact and* \( \leq \) *is a partial order, then extremal models induce an isomorphism of fundamental bipartite graphs.*
Examples of extremal models

- a
- b

- a
- b

- a
- b
An extremal model for the Swiss flag
Examples of extremal models

- a
- b
- c

- a
- b
- c

- a
- b
- c

A mathematical model for concurrent parallel computing
An extremal model of $\tilde{S}^1$

Let $x \in \tilde{S}^1$.

There is a future retract

$$P^+ : \tilde{\pi}_1(\tilde{S}^1) \to \tilde{\pi}_1(\tilde{S}^1, x) \cong (\mathbb{N}, +).$$

It is a minimal extremal model.
Piecewise analysis

Theorem (B)

*Extremal models can be constructed in a piecewise manner.*
Van Kampen for extremal models example
Van Kampen for extremal models example
Van Kampen for extremal models example
Directed spaces provide a good mathematical model for concurrent parallel computing.

Using directed homotopies one can hope to cope with the “state space explosion”.

The homotopy classes of directed paths assemble into the fundamental category.

Minimal extremal models provide a way to generalize the fundamental group to directed spaces.

Extremal models can be constructed in a piecewise manner.
Applications

- L. Fajstrup, E. Goubault, and M. Raussen (1998) used geometry and directed topology to give an algorithm for detecting deadlocks, unsafe regions and inaccessible regions for po-spaces such as the Swiss flag, in any dimension.

- E. Goubault and E. Haucourt (2005) reduced the fundamental category to “components” to develop a static analyzer (ALCOOL) of concurrent parallel programs.