Persistent homology and directional statistics

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(joint work with Peter Kim)

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1. Introduction

- Point Cloud Data from densities
- Densities on S^1

Point Cloud Data

Point Cloud Data (PCD) is a sequence of sampled points $X = (X_1, X_2, \ldots, X_n)$.

Usually the PCD is assumed to have been sampled uniformly from some manifold.

From this PCD one can use persistent homology to determine Betti barcodes.

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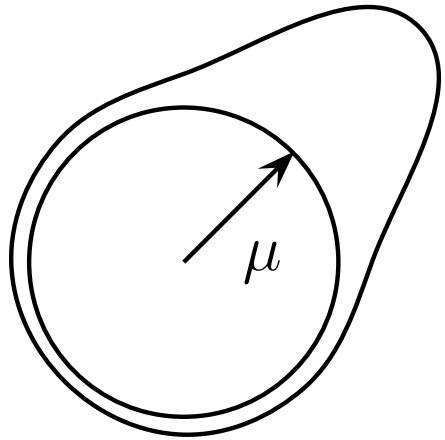
From this PCD one can use persistent homology to determine Betti barcodes.

Our point of view: We will assume that the PCD is sampled according some density on a manifold.

Densities on S^1

We will focus on unimodal densities on S^1, called von Mises distributions.



$$f_{\mu,\kappa}(x) = c(\kappa)e^{\kappa x^t \mu}.$$

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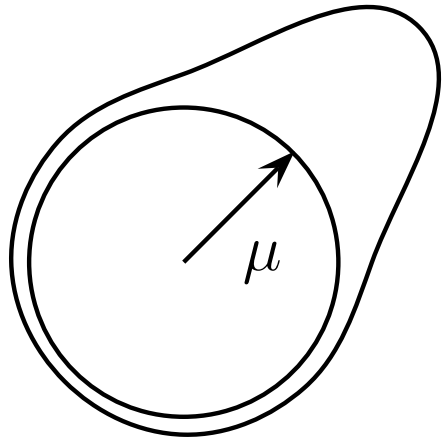
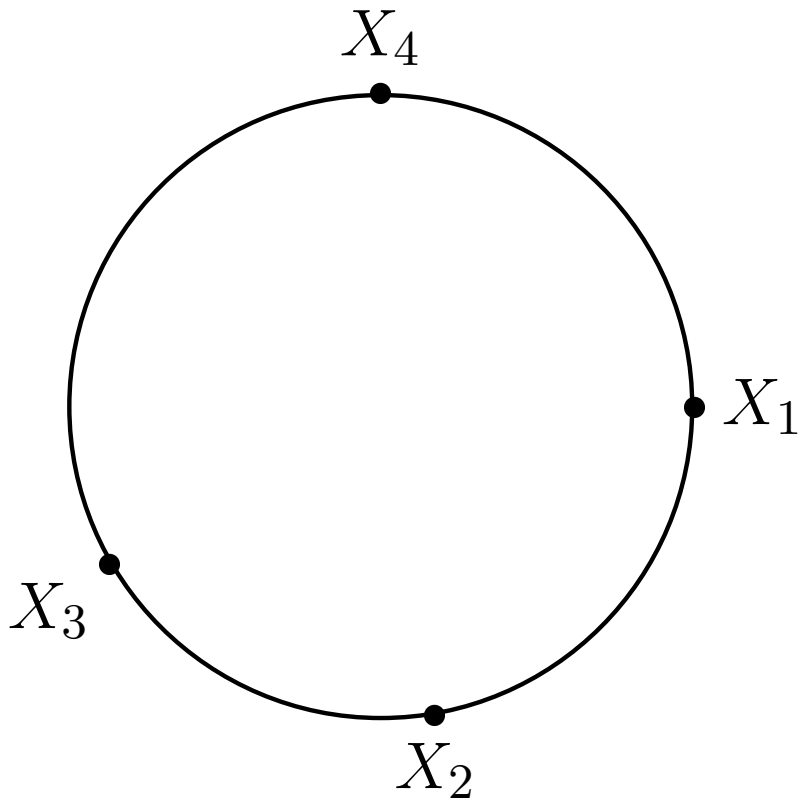


$$f_{\mu,\kappa}(x) = c(\kappa)e^{\kappa x^t \mu}.$$

Sample $X = (X_1, \ldots, X_n) \subset S^1$ according to $f_{\mu,\kappa}$

$\xrightarrow{\text{persistent homology}}$ Betti barcodes

2. Persistent Homology



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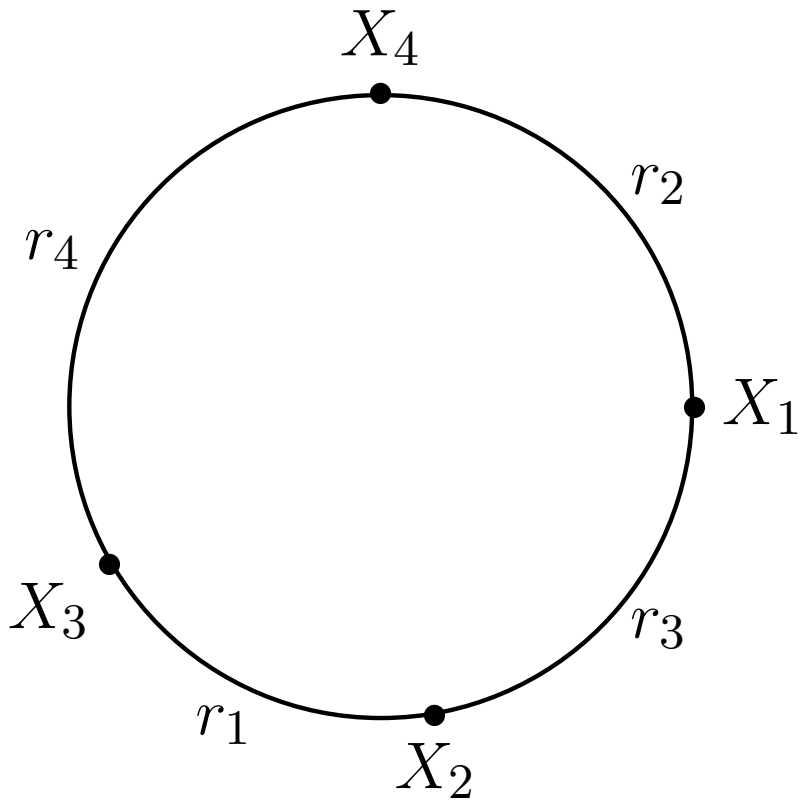
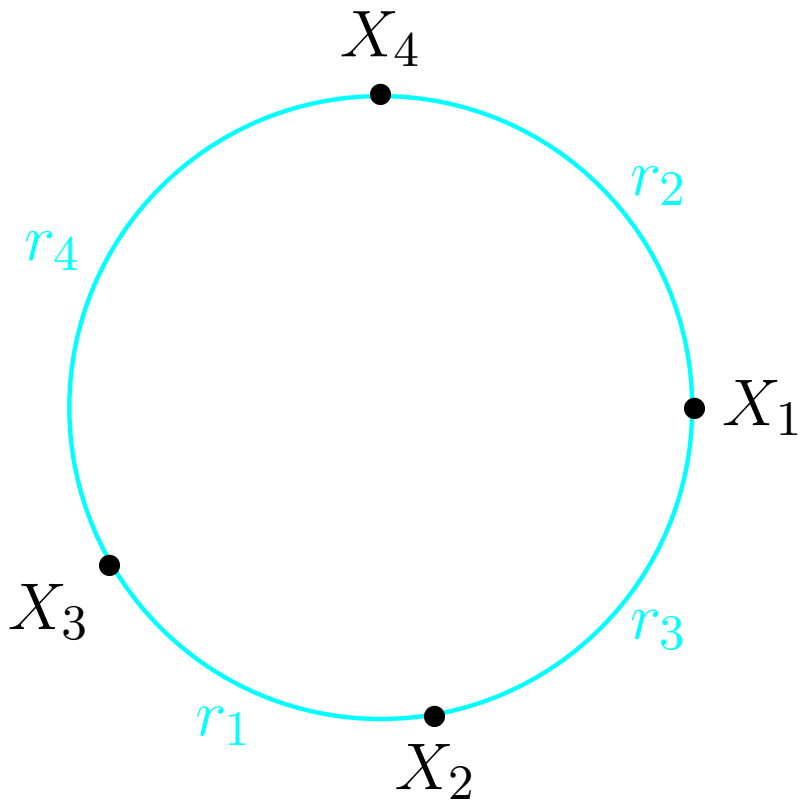


The Rips complex



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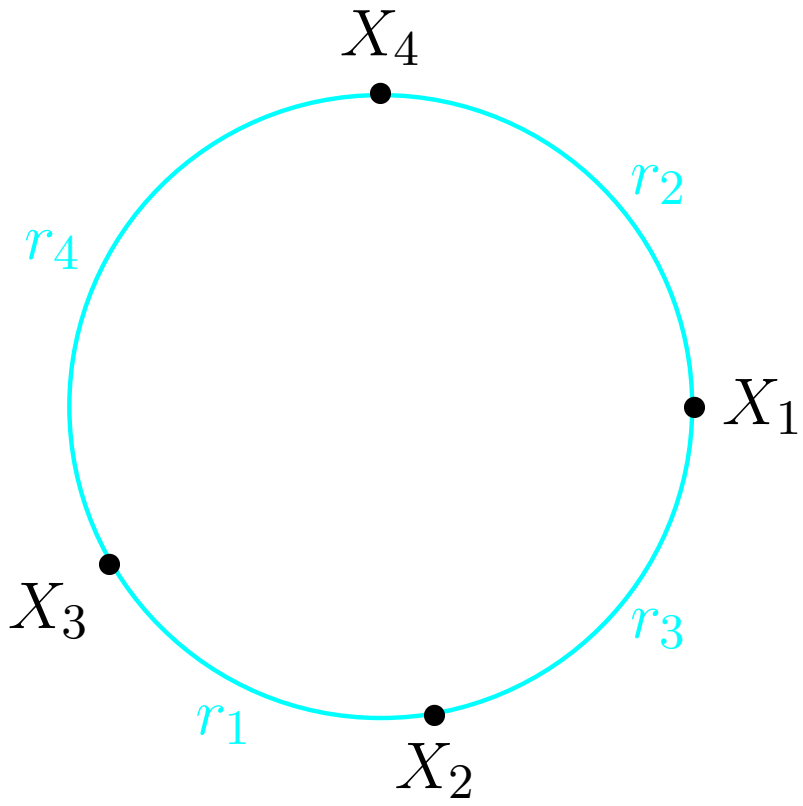

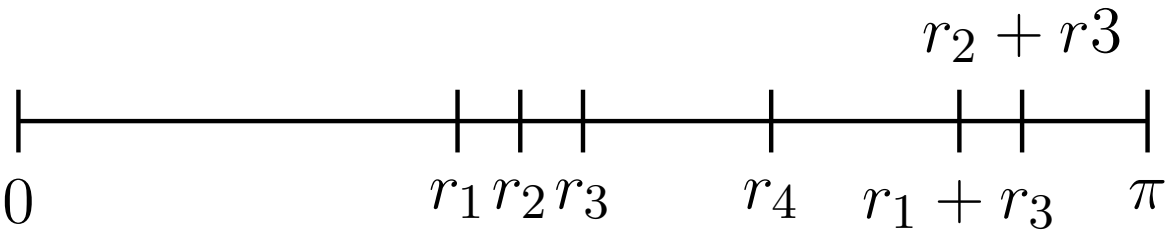

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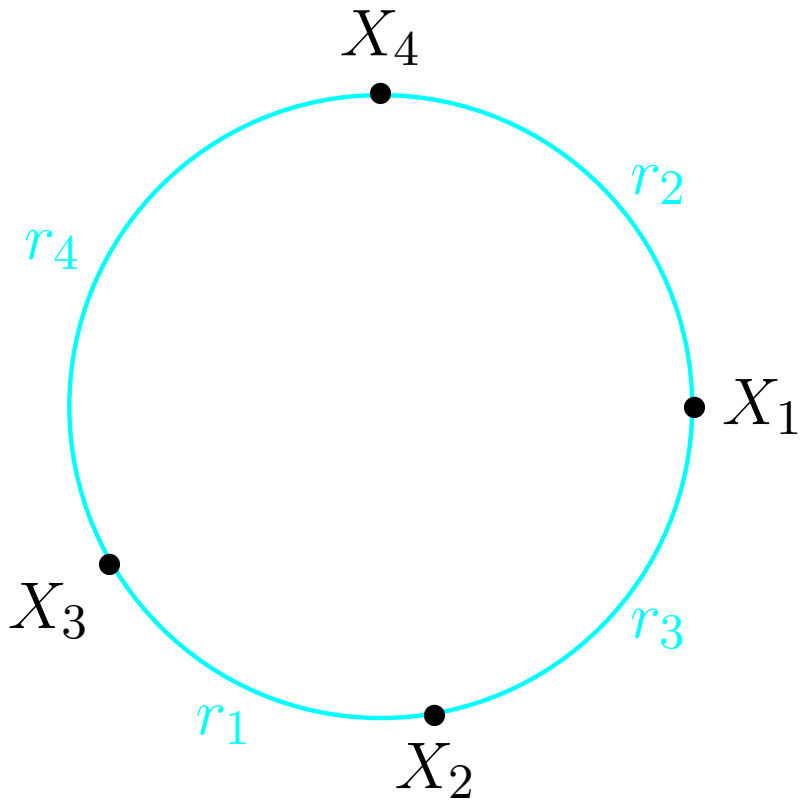

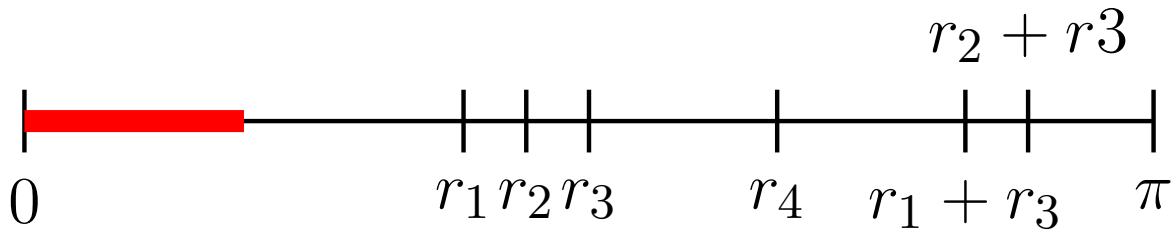

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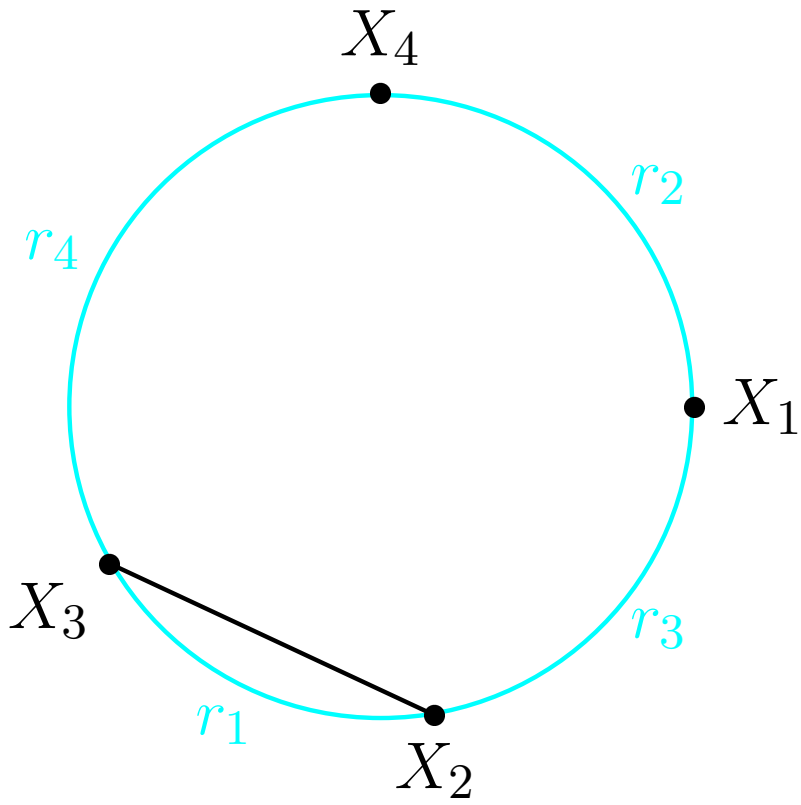

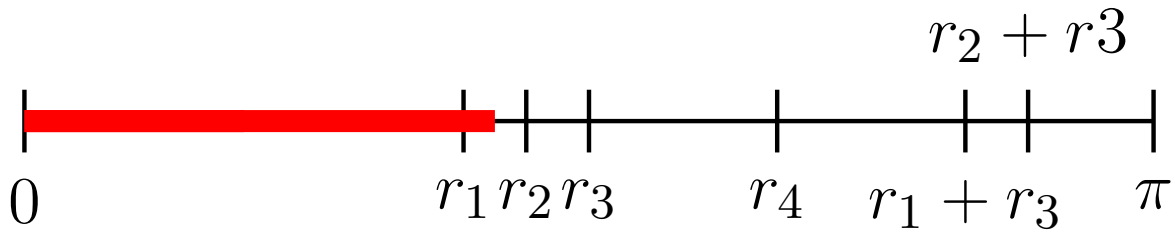

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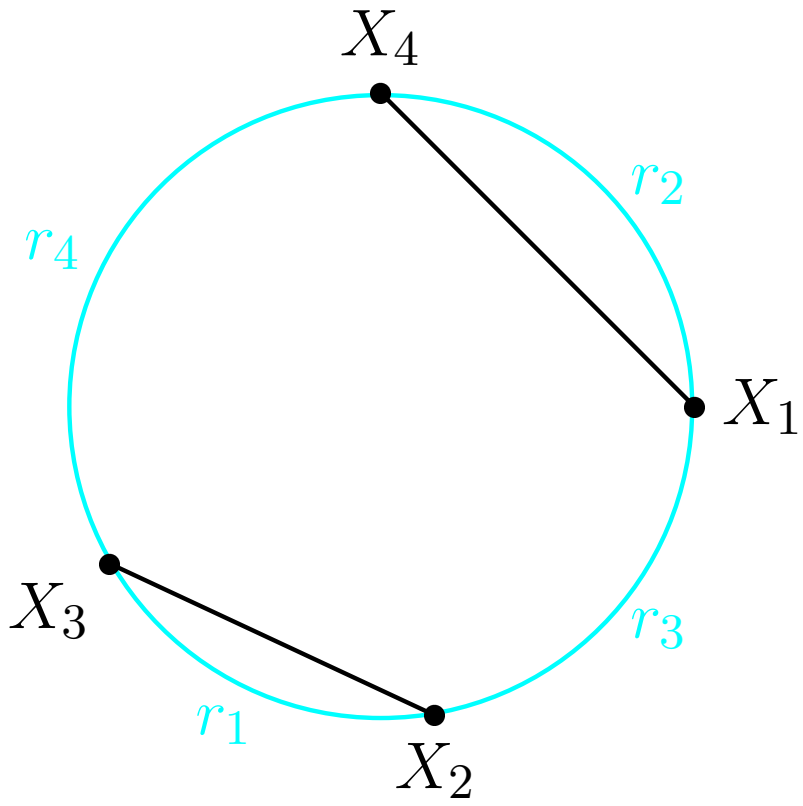

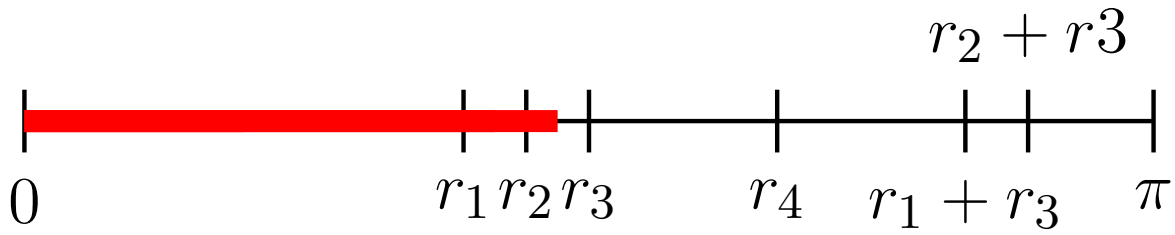

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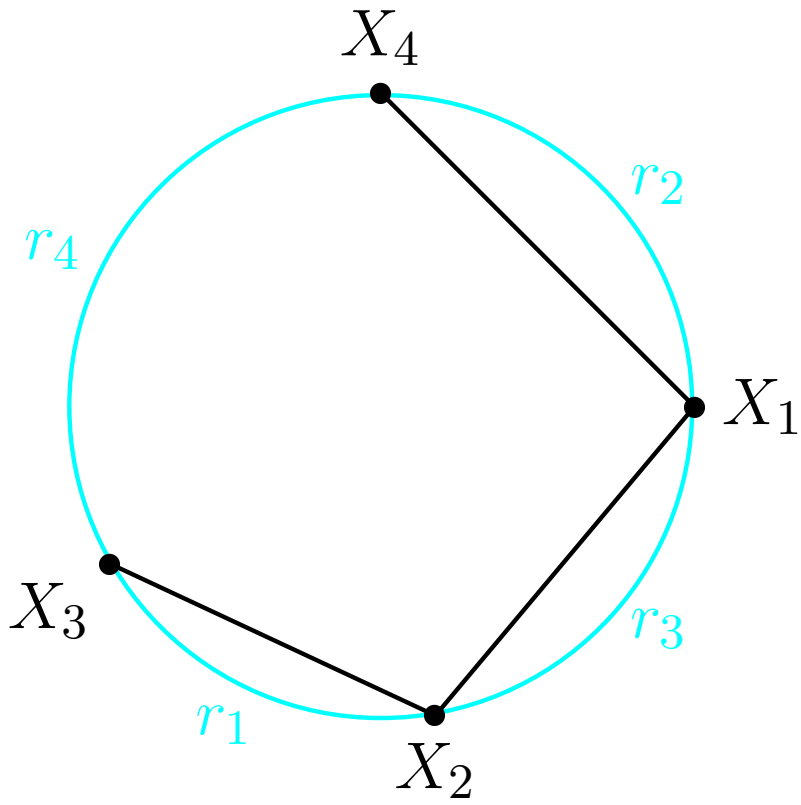

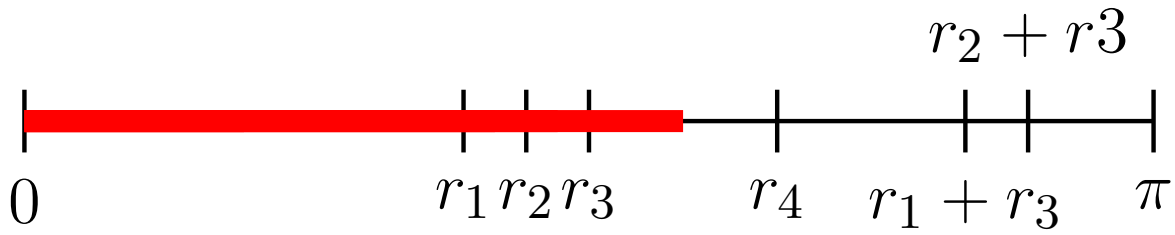

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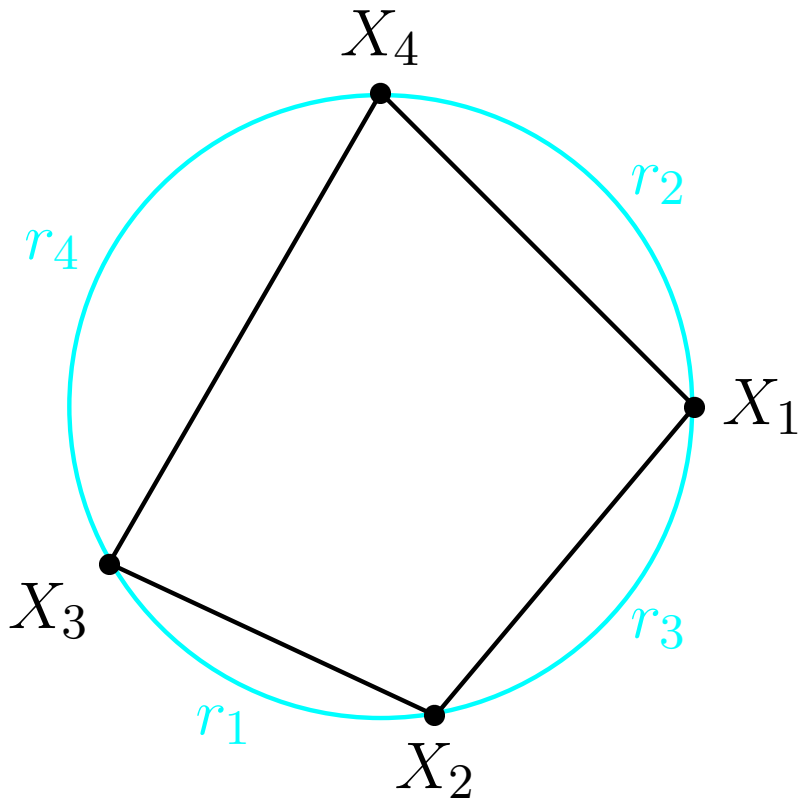

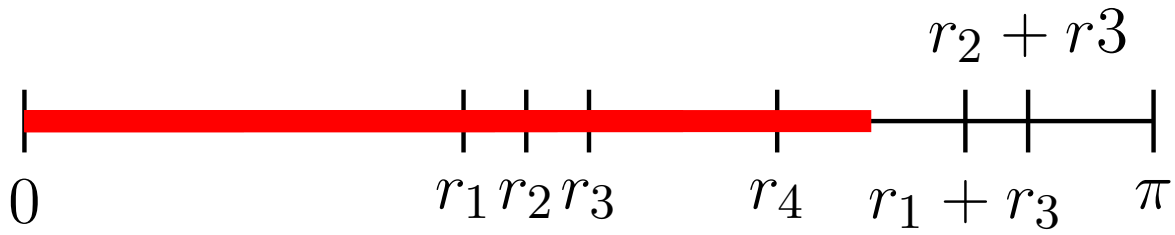

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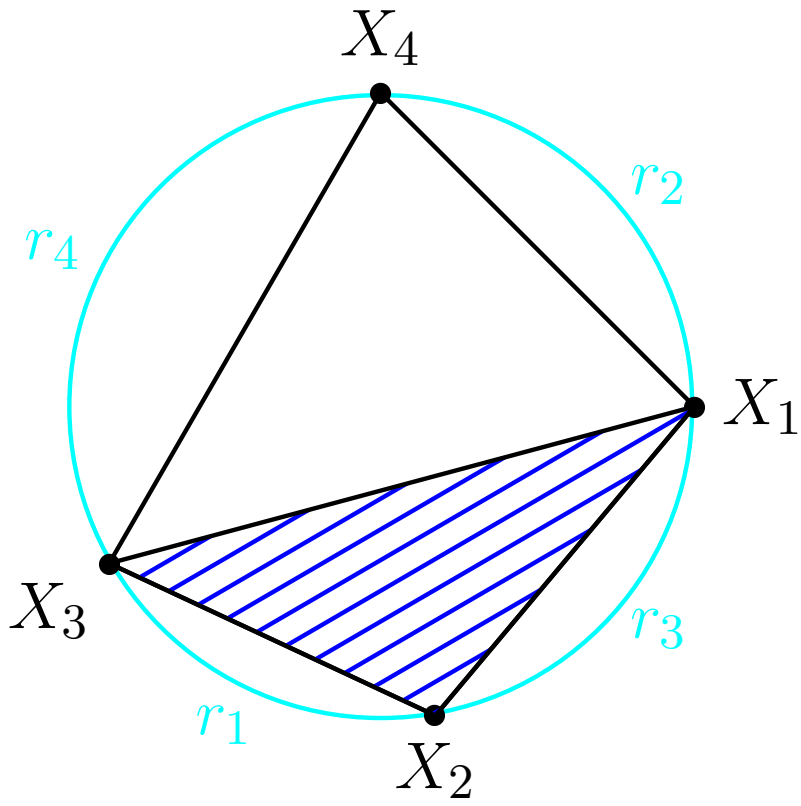

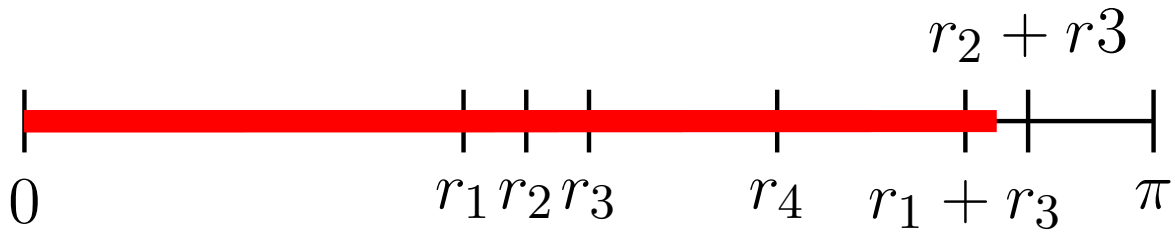

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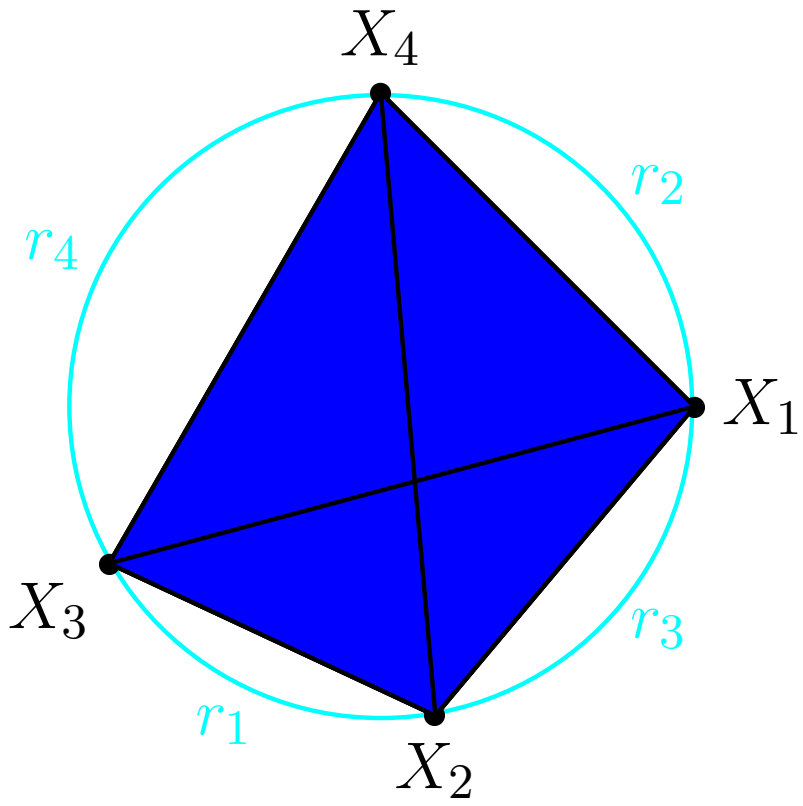
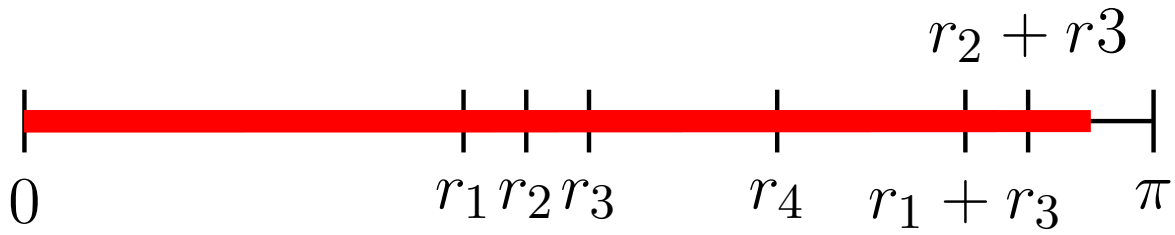
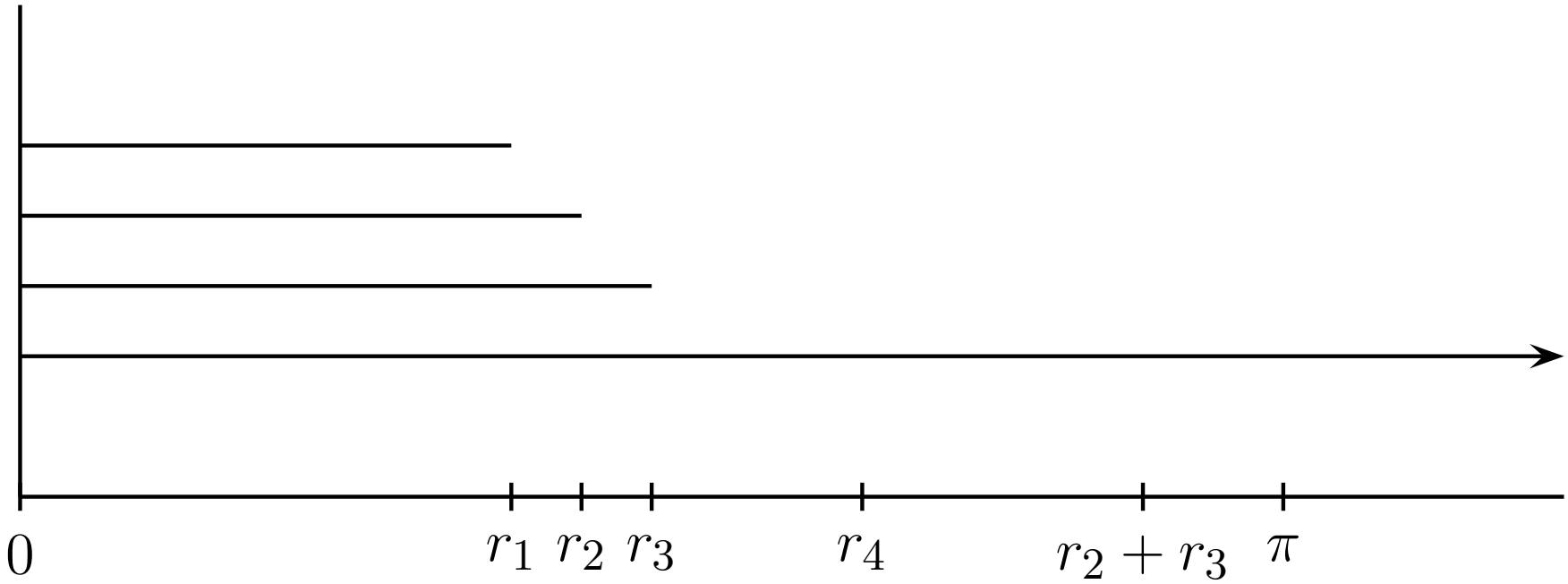




Betti 0-barcode



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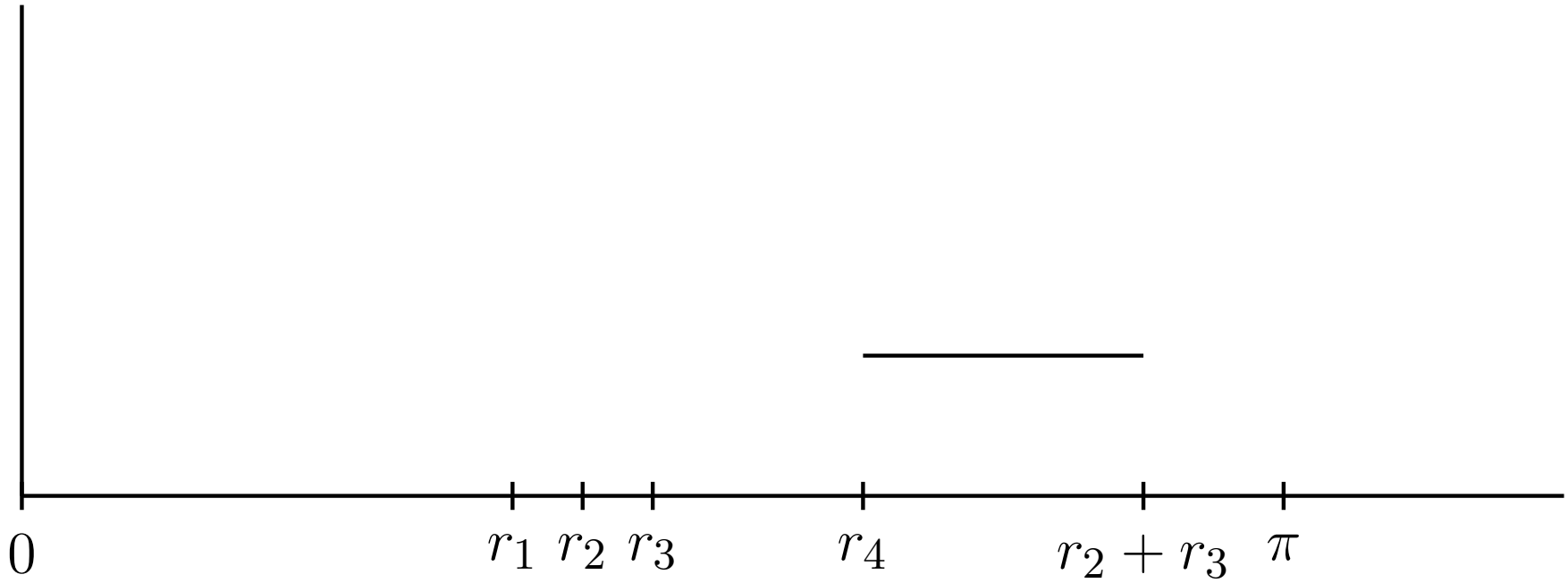

A different view

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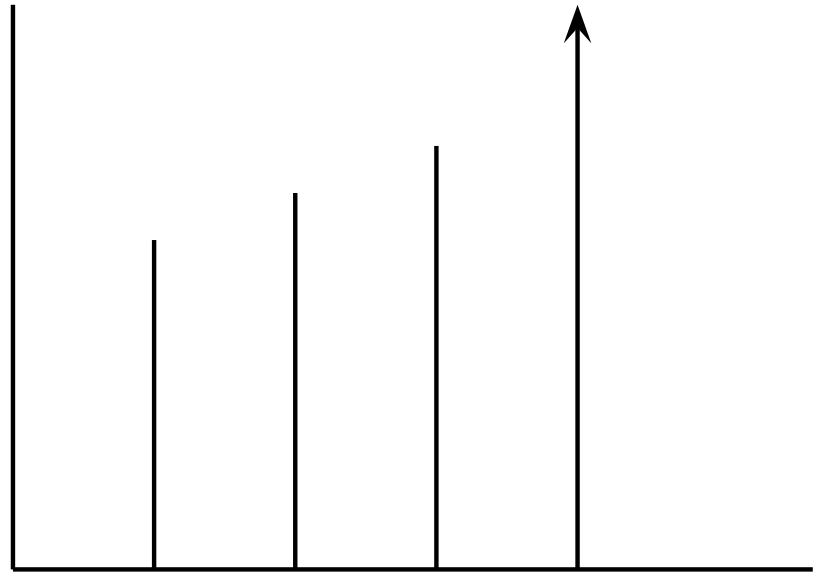

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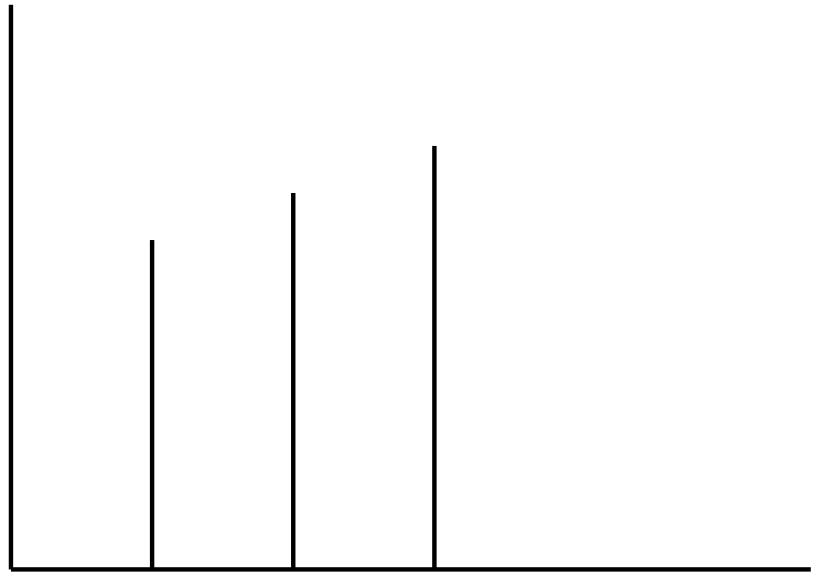

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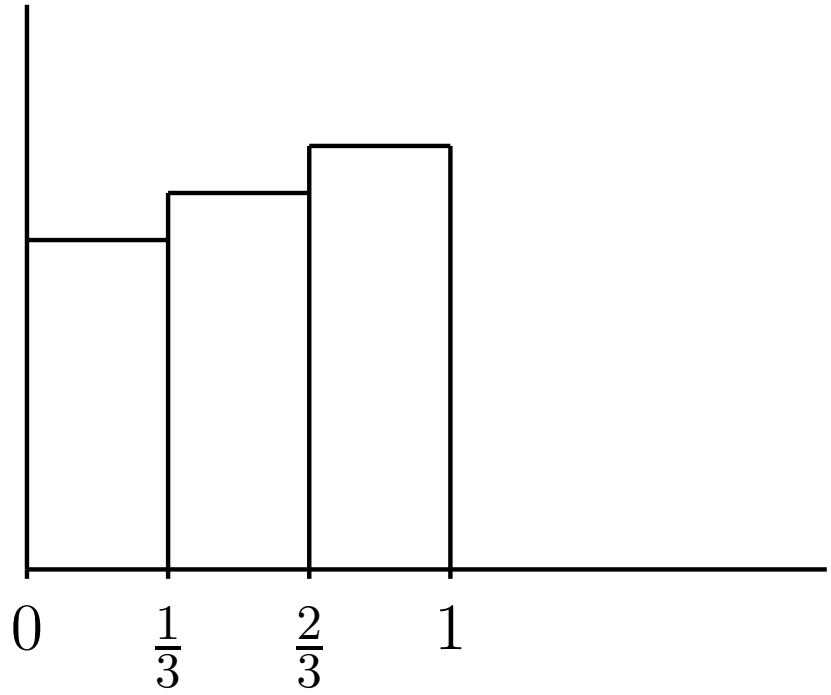

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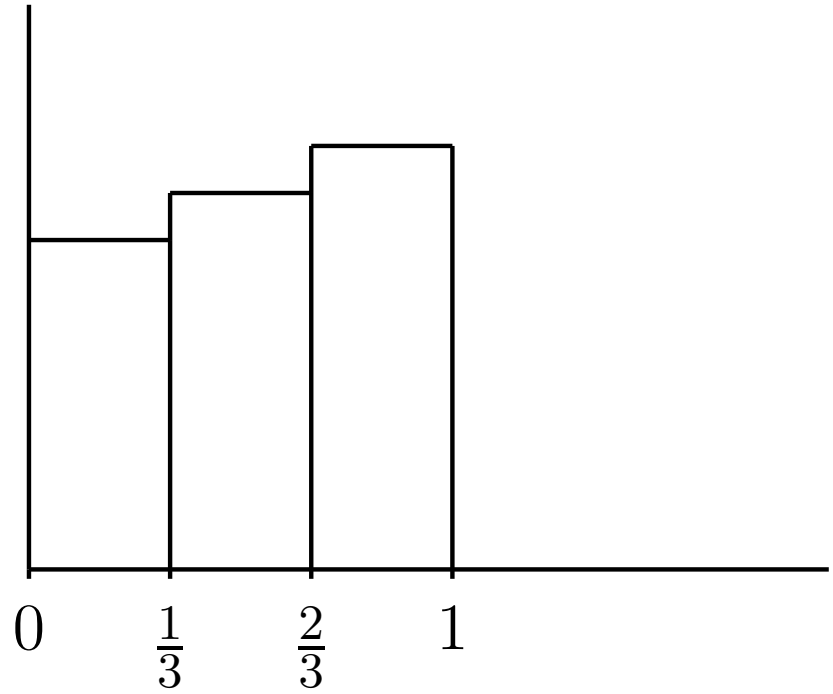

This gives us a function ${}_nB_0 : [0,1] \to \mathbf{R}_{\geq 0}$,
where ${}_nB_0(x)$ is the length on the k-th longest 0-homology interval, where $k = \lceil (n-1)x \rceil$.

Normalization

To see the effect of κ as $n \to \infty$, we normalize the above function.

$$ {}_n\bar{B}_0(x) = \frac{{}_nB_0(x)}{\int_0^1 {}_nB_0(t)dt} $$

We call this the Betti-0 function.

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Order $\{U_2, \ldots U_n\}$ to obtain

$$0 < U_{n:1} < U_{n:2} < \ldots < U_{n:n-1} < 1.$$

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For $1 \leq k \leq n$, let

$$S_k = U_{n:k} - U_{n:k-1}.$$

These are called the spacings.

Ordered Spacings

Order S:

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Recall: ${}_nB_0(x)$ equals the length of the k-th longest homology interval, where $k = \lceil (n-1)x \rceil$.

Therefore,

$${}_nB_0(x) \propto S_{n:\lceil (n-1)x \rceil}.$$

Uniform spacings

If X is sampled uniformly from S^1 then each U_k is uniform on $[0, 1]$ and $S = (S_1, \ldots, S_n)$ are called uniform spacings.

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Lemma: $S = (S_1, \ldots, S_n)$ has a constant density on the standard $(n-1)$-simplex Δ^{n-1}

$$S_k \geq 0, \quad \sum_{k=1}^{n} S_i = 1.$$

Expected spacings

For $1 \leq k \leq n$, $ES_{n:k}$ can be found be integrating over Δ^{n-1}.

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Proposition:

$$ES_{n:k} = \frac{1}{n} \sum_{j=n+1-k}^{n} \frac{1}{j}$$

Expected Betti-0 function

Recall that ${}_nB_0(x) \propto S_{n:\lceil (n-1)x \rceil}$.

Thus,

$$\begin{aligned} E({}_nB_0)(x) &\propto ES_{n:\lceil (n-1)x \rceil} \\ &= \frac{1}{n} \left(\sum_{j=n+1-\lceil (n-1)x \rceil}^{n} \frac{1}{j} \right) . \end{aligned}$$

Hence,

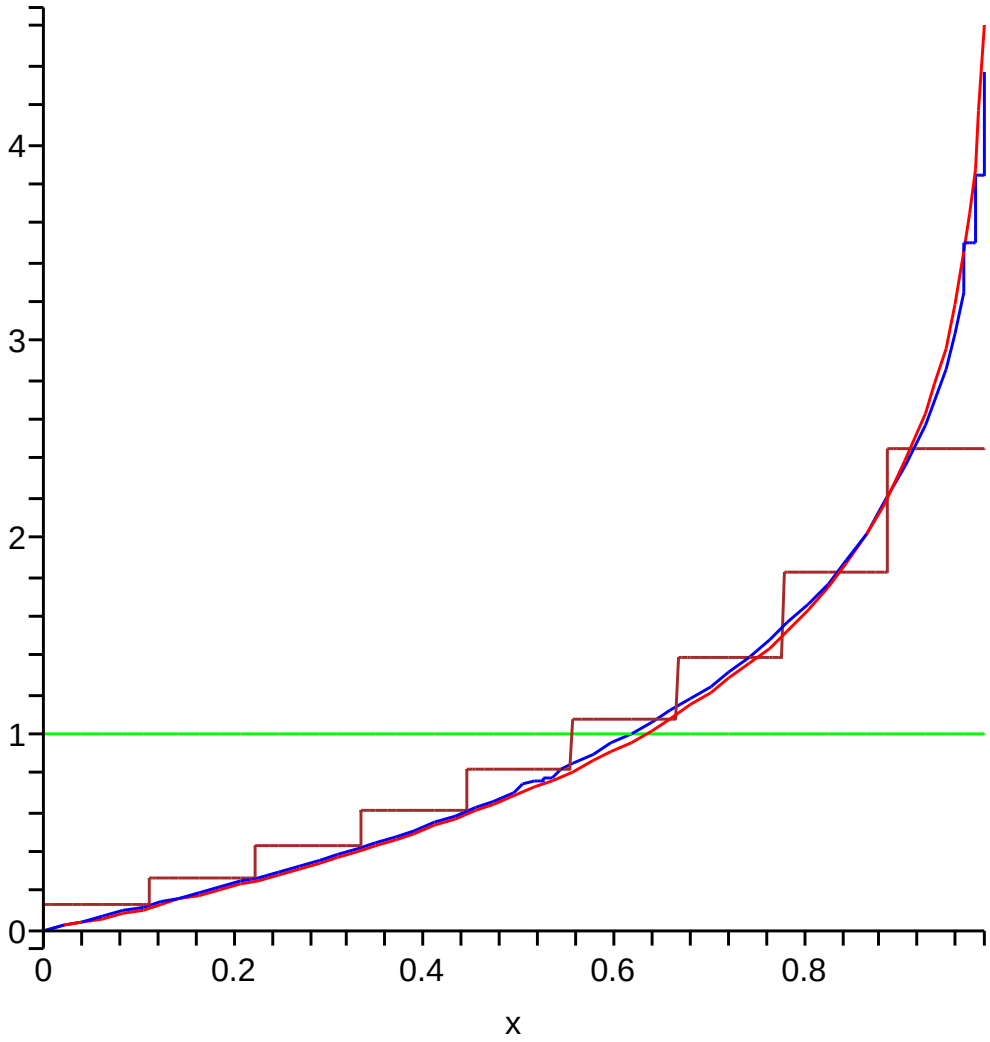
$$E({}_n\bar{B}_0)(x) = \frac{c_n}{n} \left(\sum_{j=n+1-\lceil (n-1)x \rceil}^{n} \frac{1}{j} \right)$$

Limiting Betti-0 function

Proposition[B-K]:

$$E({}_n\bar{B}_0)(x) \to -\log(1-x) \text{ as } n \to \infty.$$

Betti-0 Graphs



Summary

- We would like to be able to understand the persistent homology of PCD sampled from densities.
- To understand the limiting behavior of the Betti-0 barcode, it helps to consider the Betti-0 function.
- For densities on S^1 the persistent homology is determined by the spacings.
- For the uniform density we can explicitly calculate the expected persistent homology.
- The limiting Betti-0 function for the uniform density is $-\log(1-x)$.