Quillen and Concurrency

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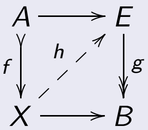
Oct. 30, 2006. Computational Applications of Algebraic Topology @ MSRI

Quillen's model categories

Definition

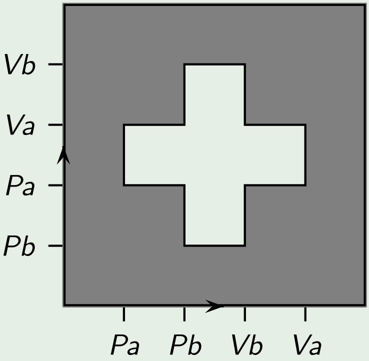
A model category is a category **M** with three distinguished classes of morphisms: WE, Cof, and Fib that are closed under composition and contain the identity maps, such that:

1. **M** has all finite limits and colimits;
2. if two out of three of f, g and $g \circ f$ are in WE, then so is the third;
3. WE, Cof, and Fib are closed under retracts;
4. 

if either f or g are in WE
5. factorization.

Concurrent systems

Example



Concurrent systems

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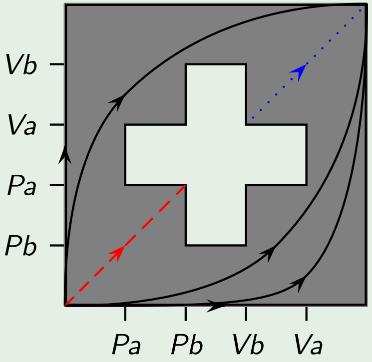

Pospaces

Definition

A pospace $(X, \leq)$ is a topological space together with a partial order. A dimap is a continuous map f such that $x \leq x'$ implies $f(x) \leq f(x')$. Let **poTop** denote the category of pospaces and dimaps.

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Lemma

There is an adjunction **Top** $\leftrightarrows$ **poTop**.

Proposition

poTop *has all finite limits and colimits.*

Directed homotopy, homotopy eq, fib, cof

Definition

Two dimaps $f, g : (X, \leq) \to (Y, \leq)$ are dihomotopic if there is a dimap $H : (X, \leq) \times (I, \Delta) \to (Y, \leq)$ from f to g.
A dihomotopy equivalence is a dimap with a dihomotopy inverse.

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Definition (Kahl)

A difibration is a dimap that has the right lifting property with respect to all dimaps

$$i : (X, \leq) \to (X, \leq) \times (I, \Delta), \quad i(x) = (x, 0).$$

A dicofibration is a dimap that has the left lifting property with respect to all difibrations that are dihomotopy equivalences.

A model category for **poTop**

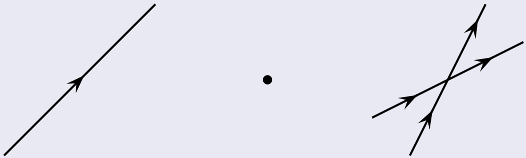
Theorem (Kahl, 2006)

poTop *together with the dihomotopy equivalences, dicofibrations, and difibrations is a model category. All pospaces are fibrant and cofibrant. Two dimaps are homotopic iff they are dihomotopic.*

Dihomotopy equivalence is too coarse

Problem

All of the following spaces are dihomotopy equivalent:



Idea (B, 2004)

Use relative dihomotopy.

Relative pospaces

Let $(C, \leq)$ be a pospace. We can define dihomotopy equivalences relative to $(C, \leq)$, $(C, \leq)$-difibrations, and $(C, \leq)$-dicofibrations.

Theorem (Kahl, 2006)

*The category $(C, \leq) \downarrow$ **poTop** is a fibration category and a cofibration category.*

Flows

Definition

- A flow consists of a set X^0 of states and a topological space $PX = \sqcup_{\alpha,\beta \in X^0} P_{\alpha,\beta}X$ of nonconstant execution paths, source and target maps, $s, t : PX \to X^0$ and a composition law $* : P_{\alpha,\beta}X \times P_{\beta,\gamma}X \to P_{\alpha,\gamma}X$.
- A morphism of flows is a set map $f^0 : X^0 \to Y^0$ and a continuous map $Pf : PX \to PY$ preserving the structure.
- Let **Flow** denote the category of flows and morphisms of flows.

S-homotopies

Definition

A morphism of flows f such that f^0 is a bijection is a (weak) S-homotopy equivalence if Pf is a (weak) homotopy equivalence.

Theorem (Gaucher, 2003)

There is a model structure on **Flow** *such the the weak equivalences are the weak S-homotopies.*

T-homotopies

Theorem (Gaucher)

$\vec{C}^3$ and $\vec{I}$ are not connected by a sequence of S- and T- homotopy equivalences.

T-homotopies

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Theorem (Gaucher, 2006)

Generalized T-homotopy equivalences preserve the underlying homotopy type.

Theorem (Gaucher, 2005)

There does not exist a model structure on **Flow** *such that the weak equivalences are the weak S- and generalized T- homotopy equivalences.*

Local pospaces

Definition

- An order atlas is a open cover of pospaces with compatible partial orders.
- Order atlases are equivalent if they have a common refinement.
- A local pospace is a topological space together with an equivalence class of order atlases.
- A morphism of local pospaces is a continuous map that respects the orders.
- Let **LPS** denote the category of local pospaces and morphisms of pospaces.

Presheaves and Simplicial Presheaves

Definition

- The category of presheaves $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ has as objects contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$ and has as morphisms natural transformations.
- There is a Yoneda embedding $\mathbf{C} \hookrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.
- The category of simplicial presheaves is the category $\mathbf{sSet}^{\mathbf{C}^{\mathrm{op}}}$.
- There is a Yoneda embedding $\mathbf{C} \hookrightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \hookrightarrow \mathbf{sSet}^{\mathbf{C}^{\mathrm{op}}}$.

Grothendieck topology and sheaves

Definition

A Grothendieck topology on **C** assigns each $M \in \mathbf{C}$ a ‘cover’.

Example

LPS has a Grothendieck topology whose basis is given by the open dicovers.

Definition

A sheaf is a presheaf that is compatible with the Grothendieck topology.

Model structures on $\mathbf{sSet}^{\mathbf{C}^{\mathrm{op}}}$

Definition

- projective structure [Bousfield-Kan, 1972]: WE and Fib are given objectwise
- injective structure [Joyal, 1984]: WE and Cof are given objectwise

Theorem

Both of the above give (proper, simplicial, cellular) model categories.

Model structures on $\mathbf{sSet}^{\mathbf{C}^{op}}$, part 2

Theorem (Jardine, 1987, 1996)

Let $\mathbf{C}$ *be a small category with a Grothendieck topology. Then* $\mathbf{sSet}^{\mathbf{C}^{op}}$ *the category of simplicial presheaves on* $\mathbf{C}$ *has a (proper, simplicial) model structure in which*

- *the cofibrations are the monomorphisms, and*
- *the weak equivalences are the local weak equivalences.*

Furthermore, if the $\mathbf{Shv}(\mathbf{C})$ *has enough points then the local weak equivalences are the stalkwise equivalences.*

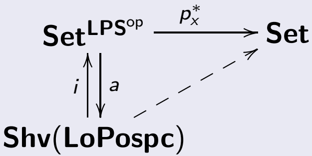
Points in **LPS**

Let $Z \in \mathbf{LPS}$ and let $x \in Z$. Define

$$p_x^* : \mathbf{Set}^{\mathbf{LPS}^{\mathrm{op}}} \longrightarrow \mathbf{Set}$$
$$F \mapsto \operatorname*{colim}_{x \in L \overset{\subset}{\mathrm{open}} Z} F(L)$$

Theorem (B-Worytkiewicz)

p_x descends to a point in $\mathbf{Shv}(\mathbf{LoPospc})$.



and provides $\mathbf{Shv}(\mathbf{LoPospc})$ *with enough points.*

Let $Z \in \mathbf{LPS}$.

Definition

The category of directed étale bundles over Z has

- objects: dimaps $E \to Z$ which are local homeomorphisms
- morphisms: maps $E_1 \to E_2$ such that

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

  commutes

Theorem (B-W)

There is an equivalence of categories between $\mathbf{Shv(Z)}$ *and* $\mathbf{Etale(Z)}$.

Stalkwise equivalences

Definition

- Given a point $p^* : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{Set}$ and $F \in \mathbf{Set}^{\mathbf{C}^{op}}$, $\mathrm{stalk}_p(F) = p^*(F)$.
- The simplicial stalk functor $(-)_p : \mathbf{sSet}^{\mathbf{C}^{op}} \to \mathbf{sSet}$ is given by $P \mapsto \{\mathrm{stalk}_p(P_n)\}_{n \geq 0}$.
- $f : P \to Q \in \mathbf{sSet}^{\mathbf{C}^{op}}$ is a stalkwise equivalence iff $f_p \in sSet$ is a weak equivalence for all points p.

Theorem (B-W)

The stalkwise equivalences in $\mathbf{sSet}^{\mathbf{LPS}^{op}}$ coming from **LPS** *via the Yoneda embedding, are the isomorphisms.*

Model category for **LPS**

Theorem (B-W, 2006)

The category $\mathbf{sSet}^{\mathbf{LPS}^{\mathrm{op}}}$ *has a proper, cellular, simplicial model structure in which*

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are the stalkwise equivalences, and*
- *the fibrations are the morphisms which have the right lifting property with respect to all trivial cofibrations.*

Furthermore among morphisms coming from **LPS** *(using the Yoneda embedding* $\mathbf{LPS} \hookrightarrow \mathbf{sSet}^{\mathbf{LPS}^{\mathrm{op}}}$*), the weak equivalences are precisely the isomorphisms.*

The relative case, localized

Theorem (B-W, 2006)

Let $\mathcal{I} = \{\bar{y}(f) \mid f$ is a directed homotopy equivalence rel $A\}$. Then the category $\bar{\mathbf{y}}(\mathbf{A}) \downarrow \mathrm{sPre}(\mathbf{LPS})$ has a left proper, cellular model structure in which

- *the cofibrations are the monomorphisms,*
- *the weak equivalences are the $\mathcal{I}$-local equivalences, and*
- *the fibrations are those morphisms which have the right lifting property with respect to monomorphisms which are $\mathcal{I}$-local equivalences.*