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DESCRIPTIONS OF DEFORMATION

CONSIDER A DIFFERENTIAL VOLUME OF A CONTINUUM, WHICH WE'LL REFER TO AS A PARTICLE. BY ADOPTING A CARTESIAN COORDINATE SYSTEM THIS PARTICLE AS WELL AS ANY OTHER PARTICLE IN THE CONTINUUM HAS SET OF COORDINATES AT TIME

$$t = 0$$

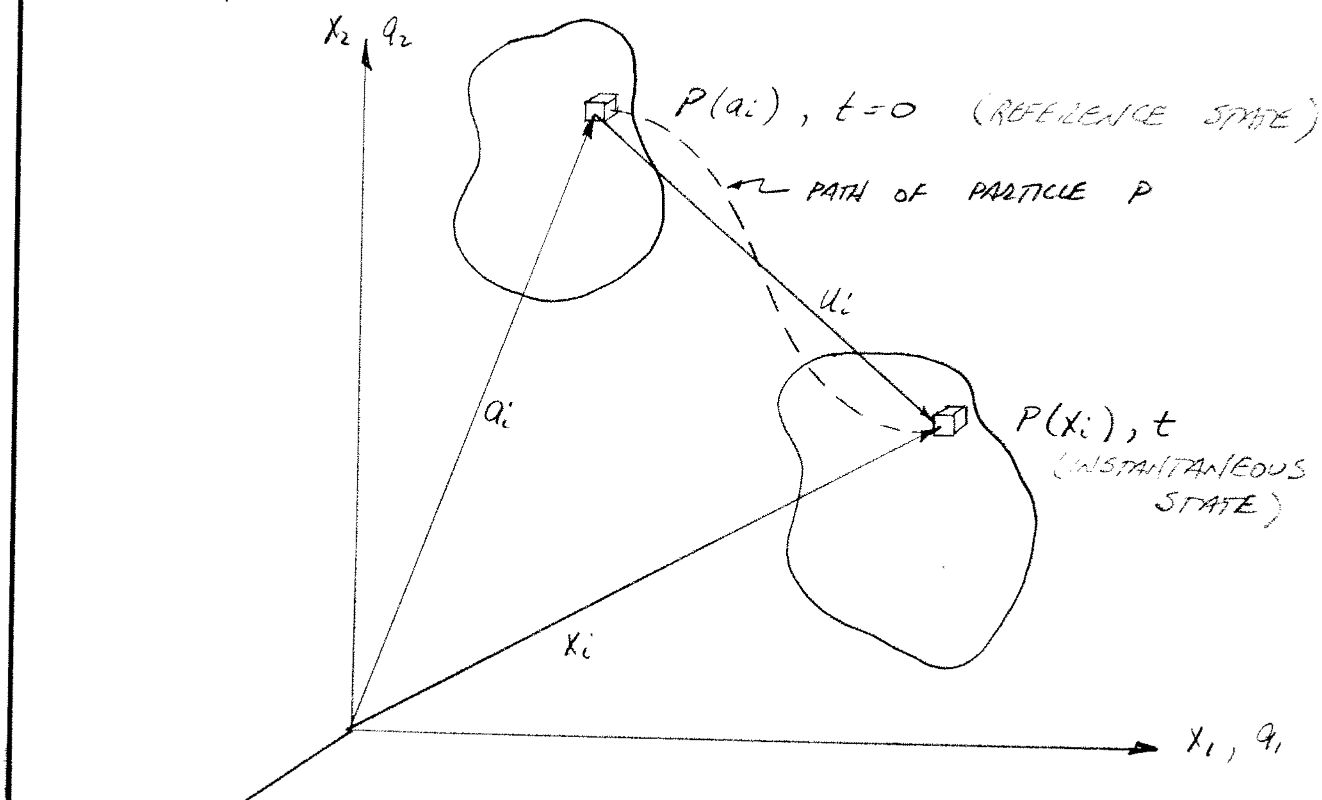
LET THE POSITION (OR COORDINATES) OF THE PARTICLE BE

$$a_i = (a_1, a_2, a_3)$$

AND AT ANY OTHER TIME t LET THE POSITION OF THE PARTICLE BE

$$X_i = (x_1, x_2, x_3)$$

GRAPHICALLY



NOTE: HERE a_i STANDS FOR THE INITIAL COORDINATES WHICH ARE TAKEN WITH RESPECT TO THE SAME AXES AS THE INSTANTANEOUS COORDINATES X_i

X_3, a_3

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THE VECTOR u_i IS DEFINED AS

$$u_i = \text{DISPLACEMENT VECTOR (OF PARTICLE P)}$$
$$= x_i - a_i$$

IF THE DISPLACEMENT VECTOR IS KNOWN FOR EVERY PARTICLE IN THE CONTINUUM, THEN THE DEFORMED BODY CAN BE CONSTRUCTED FROM THE ORIGINAL BODY. THUS A DEFORMATION CAN BE DESCRIBED BY THE DISPLACEMENT FIELD, AND THE BODY'S ORIGINAL POSITION. IN A SENSE

$$x_i = x_i(a_1, a_2, a_3)$$
$$= x_i(a_i)$$

IN CONTINUUM MECHANICS WE WILL ASSUME THAT DEFORMATION IS CONTINUOUS. THIS EXCLUDES PHENOMENON SUCH AS CRACK GROWTH IN FRACTURE MECHANICS. THUS DEFORMATION IS A TRANSFORMATION THAT WE WILL CONSIDER AS ONE-TO-ONE, SINGLE VALUED, WITH A UNIQUE INVERSE

$$a_i = a_i(x_1, x_2, x_3)$$

FOR THIS TO BE TRUE

1. x_i MUST POSSESS CONTINUOUS FIRST PARTIAL DERIVATIVES WITH RESPECT TO a_i AND
2. THE JACOBIAN DETERMINANT DEFINED AS

$$J = DET\left\{\frac{\partial x_i}{\partial a_j}\right\} = DET\begin{bmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \frac{\partial x_1}{\partial a_3} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_2}{\partial a_3} \\ \frac{\partial x_3}{\partial a_1} & \frac{\partial x_3}{\partial a_2} & \frac{\partial x_3}{\partial a_3} \end{bmatrix}$$

$$> 0$$

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THE DISPLACEMENT VECTOR u_i CAN BE SPECIFIED EITHER AS A FUNCTION OF a_i,

$$u_i(a_1, a_2, a_3) = x_i(a_1, a_2, a_3) - a_i$$

WHICH WE WILL TERM THE <u>LAGRANGIAN</u> DESCRIPTION OF DEFORMATION. OR THE DISPLACEMENT VECTOR CAN BE SPECIFIED AS A FUNCTION OF x_i,

$$u_i(x_1, x_2, x_3) = x_i - a_i(x_1, x_2, x_3)$$

WHICH WE WILL TERM THE <u>EULERIAN</u> DESCRIPTION OF DEFORMATION. THIS WELL ESTABLISHED NOMENCLATURE WILL BE RETAINED IN THE FOLLOWING DEVELOPMENT, EVEN THOUGH IT CANNOT BE JUSTIFIED ON HISTORICAL GROUNDS. (SEE, FOR INSTANCE, C. TRUESDELL, JOURNAL OF RATIONAL MECHANICAL ANALYSIS, 1, (1952), 125, FOOTNOTE 5 ON PAGE 139).

NOTE THE FOLLOWING

1. THE LAGRANGIAN DESCRIPTION REFERS MOTION (E.G, DEFORMATION, VELOCITY, ACCELERATION, ...) TO A REFERENCE CONFIGURATION. IN ELASTICITY THE REFERENCE CONFIGURATION IS USUALLY CHOSEN AS THE INITIAL UNSTRESSED STATE, THE CONFIGURATION TO WHICH THE BODY WILL RETURN WHEN IT IS UNLOADED. IN FLUID FLOW AND IN CONTINUUM MECHANICS IN GENERAL, THE REFERENCE CONFIGURATION MAY BE CHOSEN ARBITRARILY

2. THE EULERIAN DESCRIPTION FIXES ATTENTION ON A GIVEN REGION OF SPACE AND TAKES DEFORMATION TO BE DEPENDENT UPON THE SPATIAL COORDINATES x_i. THIS TYPE OF SPATIAL DESCRIPTION IS ESPECIALLY USEFUL IN FLUID MECHANICS.

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EXAMPLE: TO CONTRAST THE LAGRANGIAN AND EULERIAN DESCRIPTIONS OF DEFORMATION, CONSIDER THE FOLLOWING TRANSFORMATION IS A FUNCTION OF TIME AND IS DEFINED BY

$$x_i = x_i\,(a_1, a_2, a_3, t)$$

WHERE

$$x_1 = a_1 t^2 + 2a_2 t + a_1$$

$$x_2 = 2a_1 t^2 + a_2 t + a_2$$

$$x_3 = (1/2)\, a_3 t + a_3$$

THIS TRANSFORMATION POSSESSES A UNIQUE INVERSE

$$a_1 = \frac{x_1(t+1) - 2x_2 t}{-3t^3 + t^2 + t + 1}$$

$$a_2 = \frac{x_2(t^2+1) - 2x_1 t^2}{-3t^3 + t^2 + t + 1}$$

$$a_3 = \frac{2x_3}{t+2}$$

THE DISPLACEMENT VECTOR IN TERMS OF THE LAGRANGIAN DESCRIPTION IS

$$u_1 = x_1 - a_1 = a_1 t^2 + 2a_2 t$$

$$u_2 = x_2 - a_2 = 2a_1 t^2 + a_2 t$$

$$u_3 = x_3 - a_3 = (1/2)\, a_3 t$$

NOTE THAT A TIME $t = 0$

$$u_i = (0, 0, 0)$$

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IN TERMS OF AN EULERIAN DESCRIPTION

$$u_1 = x_1 - a_1 = x_1 - \frac{x_1(t+1) - 2x_2 t}{-3t^3 + t^2 + t + 1}$$

$$u_2 = x_2 - a_2 = x_2 - \frac{x_2(t^2+1) - 2x_1 t^2}{-3t^3 + t^2 + t + 1}$$

$$u_3 = x_3 - a_3 = x_3 - \frac{2x_3}{t+2}$$

ONCE AGAIN, FOR $t=0$

$$u_i = (0, 0, 0)$$

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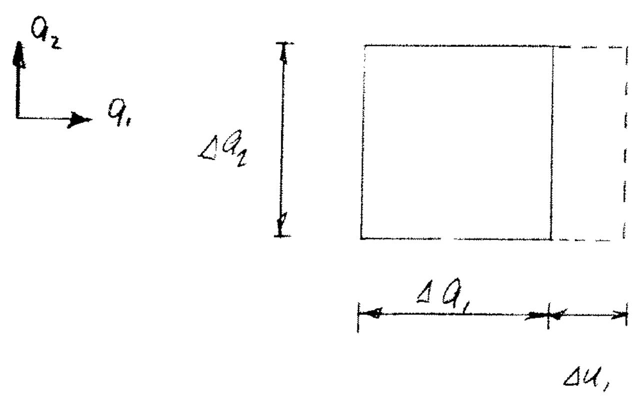
CONCEPTS OF STRAIN AND RELATIVE DISPLACEMENTS (2D)

RECALL FROM UNDERGRADUATE MECHANICS OF SOLIDS WE DEFINED UNIAXIAL STRAIN IN THE FOLLOWING MANNER USING A DIFFERENTIAL CUBE



NOTE: THE SYMBOL "a" CORRESPONDS TO CAPITAL "X" IN MALVERN'S TEXT. THE SYMBOL "a" WILL BE USED FOR CLARITY TO DISTINGUISH BETWEEN UPPERCASE "X" AND LOWERCASE "X."

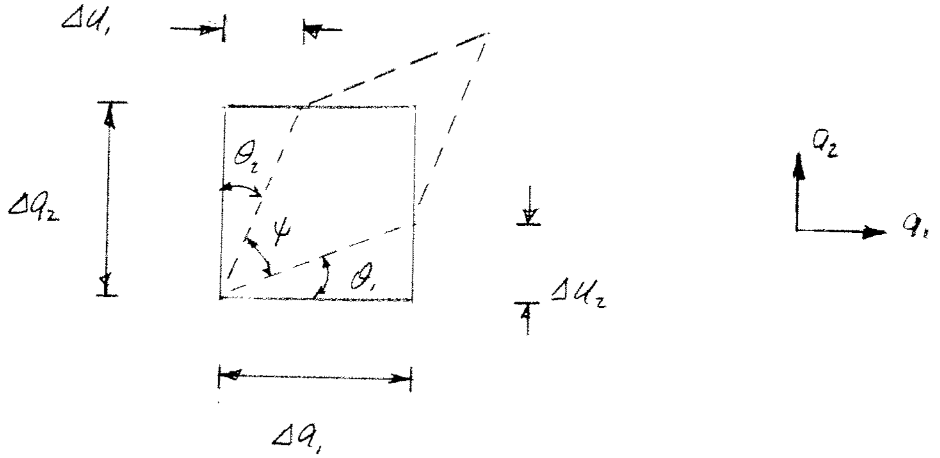
THE STRAIN IN THE 1-DIRECTION IS DEFINED BY THE LIMIT

$$\epsilon_{11} = \lim_{\Delta a_1 \to 0} \left\{ \frac{\Delta u_1}{\Delta a_1} \right\}$$

$$= \frac{\partial u_1}{\partial a_1} \longrightarrow u_i = u_i(a_i)$$

$$u_1 = u_1(a_1)$$

IN A SIMILAR FASHION WE USE THE SAME DIFFERENTIAL CUBE TO DEFINE SHEAR STRAIN.



HERE

$$\epsilon_{12} = \left(\frac{1}{2}\right)\left\{ \frac{\pi}{2} - \psi \right\}$$

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$$\epsilon_{12} = \left(\frac{1}{2}\right)\left\{ \theta_1 + \theta_2 \right\}$$

$$= \left(\frac{1}{2}\right) \lim_{\substack{\Delta q_1 \to 0 \\ \Delta q_2 \to 0}} \left\{ \frac{\Delta u_2}{\Delta q_1} + \frac{\Delta u_1}{\Delta q_2} \right\}$$

$$= \left(\frac{1}{2}\right)\left\{ \frac{\partial u_2}{\partial q_1} + \frac{\partial u_1}{\partial q_2} \right\}$$

NOTE THAT THE APPROXIMATIONS

$$\theta_1 = \frac{\Delta u_2}{\Delta q_1} \qquad \theta_2 = \frac{\Delta u_1}{\Delta q_2}$$

ONLY HOLD FOR SMALL VALUES OF θ, I.E.

$$\theta = \tan\theta$$

ONLY FOR SMALL VALUES OF θ. THIS IS AN IMPORTANT POINT WHICH IS DISCUSSED IN MORE DETAIL LATER. THIS GIVES RISE TO THE PHRASE

"SMALL STRAIN THEORY"

WHEN THIS LIMITATION CAN NOT BE MET AND THE THEORY IS DEVELOPED TO ACCOMODATE "LARGE" STRAINS, THE THEORY IS REFERRED TO AS

"FINITE DEFORMATION THEORY"

THE REASON FOR INTRODUCING THE (1/2) TO THE DEFINITION OF ϵ_{12} IS A MATTER OF CONVENIENCE. WHEN THE CONCEPT OF STRAIN IS EXTENDED TO THREE DIMENSIONS THE STRAIN COMPONENTS WILL CORRESPOND TO THE COMPONENTS OF A SECOND ORDER TENSOR.

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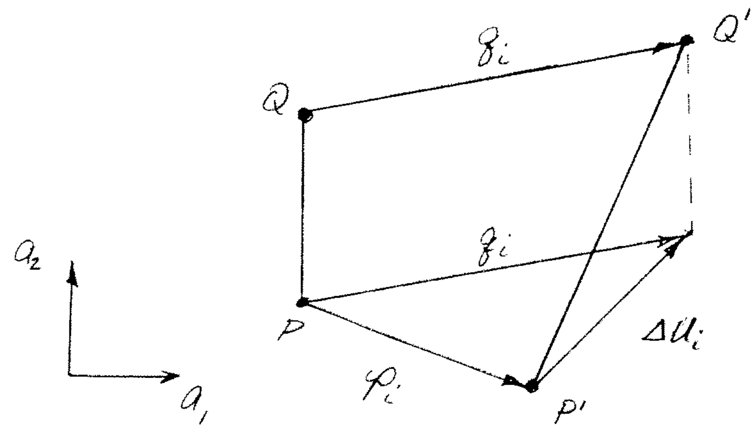
NOW CONSIDER TWO NEIGHBORING POINTS, P AND Q, IN A CONTINUUM SUBJECTED TO A DISPLACEMENT. THE LINE SEGMENT PQ IS THUS SUBJECTED TO A RELATIVE DISPLACEMENT AS FOLLOWS



THUS THE DISPLACEMENT OF Q RELATIVE TO P IS

$$\Delta u_i = q_i - p_i$$

WITH

$$|\overline{PQ}| = \Delta S$$

THEN

$$\frac{\Delta u_i}{\Delta S} = \text{THE RELATIVE DISPLACEMENT PER UNIT LENGTH}$$

AND IN THE LIMIT

$$\lim_{\Delta S \to 0} \left\{ \frac{\Delta u_i}{\Delta S} \right\} = \frac{du_i}{dS}$$

SINCE

$$u_i = u_i(a_i)$$

THEN BY THE CHAIN RULE

$$\frac{du_i}{dS} = \left(\frac{\partial u_i}{\partial a_j}\right)\left(\frac{da_j}{dS}\right)$$

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THUS THE COMPONENTS OF THE UNIT RELATIVE DISPLACEMENT IN TWO DIMENSIONS ARE

$$\frac{du_1}{ds} = \left(\frac{\partial u_1}{\partial a_1}\right)\left(\frac{da_1}{ds}\right) + \left(\frac{\partial u_1}{\partial a_2}\right)\left(\frac{da_2}{ds}\right)$$

$$\frac{du_2}{ds} = \left(\frac{\partial u_2}{\partial a_1}\right)\left(\frac{da_1}{ds}\right) + \left(\frac{\partial u_2}{\partial a_2}\right)\left(\frac{da_2}{ds}\right)$$

HERE WE CAN THINK OF

$$\left(\frac{da_1}{ds}, \frac{da_2}{ds}\right)$$

AS COMPONENTS OF A UNIT VECTOR IN THE DIRECTION OF LINE SEGMENT PQ. IN MATRIX FORMAT

$$\begin{bmatrix} \frac{du_1}{ds} \\ \frac{du_2}{ds} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} \\ \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} \end{bmatrix}}_{\text{CONTAINS A WEALTH OF WEALTH OF INFORMATION}} \begin{bmatrix} \frac{da_1}{ds} \\ \frac{da_2}{ds} \end{bmatrix}$$

THUS

$\frac{du_i}{ds}$ = A FIRST ORDER TENSOR (I.E., A VECTOR) DEFINING THE DISPLACEMENT PER UNIT LENGTH OF Q RELATIVE TO P

$\frac{\partial u_i}{\partial a_j}$ = A SECOND ORDER TENSOR REFERRED TO AS THE DISPLACEMENT GRADIENT TENSOR

$\frac{da_i}{ds}$ = A FIRST ORDER TENSOR (I.E., A VECTOR) DEFINING A UNIT VECTOR IN THE DIRECTION OF LINE SEGMENT PQ

IN LIGHT OF THE THIRD DEFINITION IT IS INTERESTING TO NOTE (AS WE WILL SEE SHORTLY) THAT THIS SAME ARGUMENT CAN BE CONSTRUCTED BY DEFINING A UNIT VECTOR IN THE DIRECTION OF LINE SEGMENT P'Q', THAT IS IN TERMS OF THE INSTANTANEOUS POSITION OF PQ

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NEXT WE WISH TO DECOMPOSE THE DISPLACEMENT GRADIENT MATRIX, BUT FIRST, IF

$$\epsilon_{11} = \frac{\partial u_1}{\partial a_1}$$

THEN IT FOLLOWS IN A SIMILAR MANNER THAT

$$\epsilon_{22} = \frac{\partial u_2}{\partial a_2}$$

THUS THE FIRST TWO ROWS AND FIRST TWO COLUMNS OF THE DISPLACEMENT GRADIENT MATRIX ARE

$$\begin{bmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} \\ \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon_{11} & \frac{\partial u_1}{\partial a_2} \\ \frac{\partial u_2}{\partial a_1} & \epsilon_{22} \end{bmatrix}$$

UNFORTUNATELY

$$\epsilon_{12} = \left(\frac{1}{2}\right)\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right) \neq \frac{\partial u_1}{\partial a_2}$$

HOWEVER

$$\frac{\partial u_1}{\partial a_2} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right) + \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right)$$

$$= \epsilon_{12} + \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right)$$

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SIMILARLY

$$\frac{\partial u_2}{\partial a_1} = \epsilon_{21} + \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} - \frac{\partial u_1}{\partial a_2}\right)$$

IF WE DEFINE THE FOLLOWING SECOND ORDER TENSOR

$$\Omega_{ij} = \text{ROTATIONAL TENSOR}$$

SUCH THAT THE FIRST TWO ROWS AND FIRST TWO COLUMS OF Ω_{ij} ARE

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} - \frac{\partial u_1}{\partial a_2}\right) & 0 \end{bmatrix}$$

THEN UTILIZING THE FIRST TWO ROWS AND THE FIRST TWO COLUMNS OF THE STRAIN MATRIX, ϵ_{ij}, THAT IS

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix}$$

THE DISPLACEMENT GRADIENT TENSOR CAN BE DECOMPOSED AS FOLLOWS

$$\frac{\partial u_i}{\partial a_j} = \epsilon_{ij} + \Omega_{ij}$$

NOTE THAT THE ROTATIONAL TENSOR IS ANTISYMMETRIC, I.E.,

$$\Omega_{ij} = -\Omega_{ji}$$

AND THE STRAIN MATRIX IS SYMMETRIC. PHYSICAL INTERPRATIONS OF THIS CONCEPT OF DECOMPOSING $\partial u_i/\partial a_j$ INTO SYMMETRIC AND ANTISYMMETRIC TENSORS IS PRESENTED NEXT.

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RELATIVE DISPLACEMENTS - PURE STRAIN & PURE ROTATION

WE WISH TO EXPLORE THE RELATIONSHIP

$$\frac{du_i}{ds} = \frac{\partial u_i}{\partial a_j} \frac{da_j}{ds}$$

NOTE THAT THE DISPLACEMENT GRADIENT TENSOR (THE JACOBIAN)

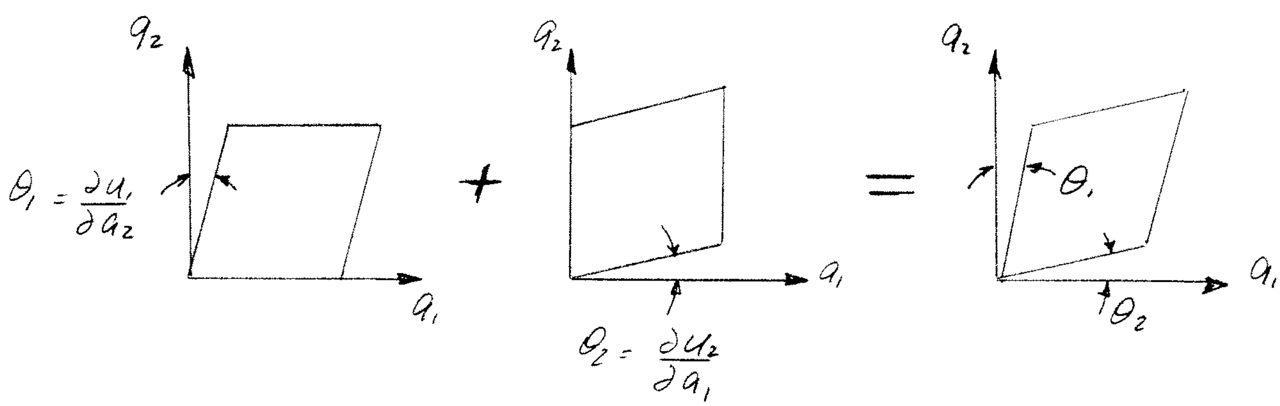
$$\frac{\partial u_i}{\partial a_j}$$

OPERATES ON THE UNIT VECTOR (da_j/ds) YIELDING (du_i/ds). TWO SPECIAL CASES WILL BE STUDIED NEXT

1. PURE SHEAR STRAIN

2. PURE ROTATION

CONSIDER THE SUPERPOSITION OF TWO SIMPLE SHEARS SHOWN IN THE FOLLOWING FIGURE



IF

$$\theta_1 = \theta_2$$

$$\frac{\partial u_1}{\partial a_2} = \frac{\partial u_2}{\partial a_1}$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right) + \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} + \frac{\partial u_1}{\partial a_2}\right) + \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} - \frac{\partial u_1}{\partial a_2}\right)$$

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THE ONLY WAY THIS EQUALITY WILL HOLD IS IF THE SECOND TERM ON EACH SIDE OF THE EQUAL SIGN IS ZERO. THUS

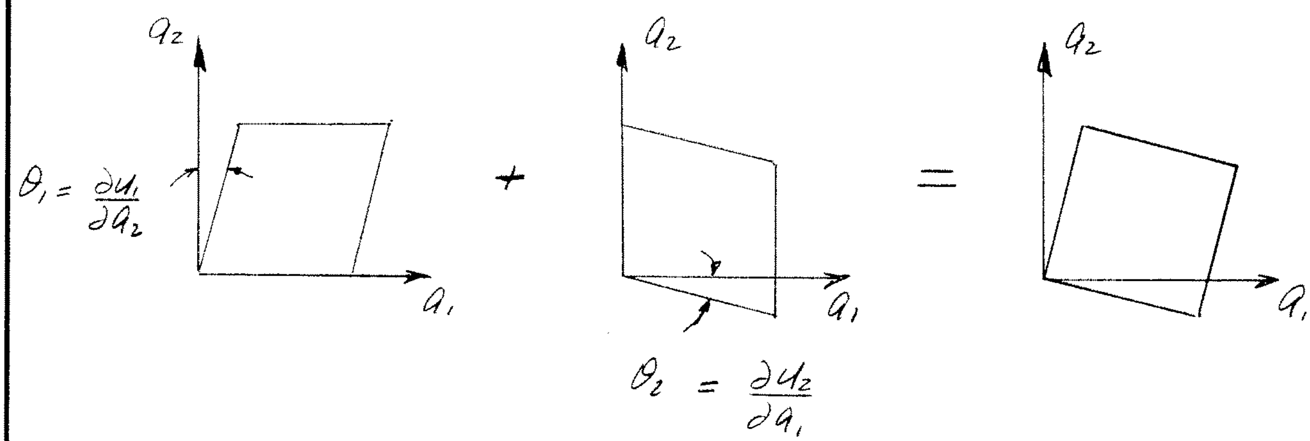
$$\epsilon_{12} + \Omega_{12} = \epsilon_{21} + \Omega_{21}$$

$$\epsilon_{12} + 0 = \epsilon_{21} + 0$$

$$\epsilon_{12} = \epsilon_{21}$$

THUS THE DISPLACEMENT GIVEN IN THE PREVIOUS FIGURE IS A <u>PURE</u> <u>STRAIN</u>.

NOW CONSIDER THE SUPERPOSITION OF TWO SIMPLE SHEARS SUCH THAT



NOTE THAT HERE, IF

$$\theta_1 = -\theta_2$$

$$\frac{\partial u_1}{\partial a_2} = -\frac{\partial u_2}{\partial a_1}$$

THEN WE HAVE A PURE ROTATION, OR

$$\epsilon_{12} + \Omega_{12} = -(\epsilon_{21} + \Omega_{21})$$

$$0 + \Omega_{12} = 0 - \Omega_{21}$$

$$\Omega_{12} = -\Omega_{21}$$

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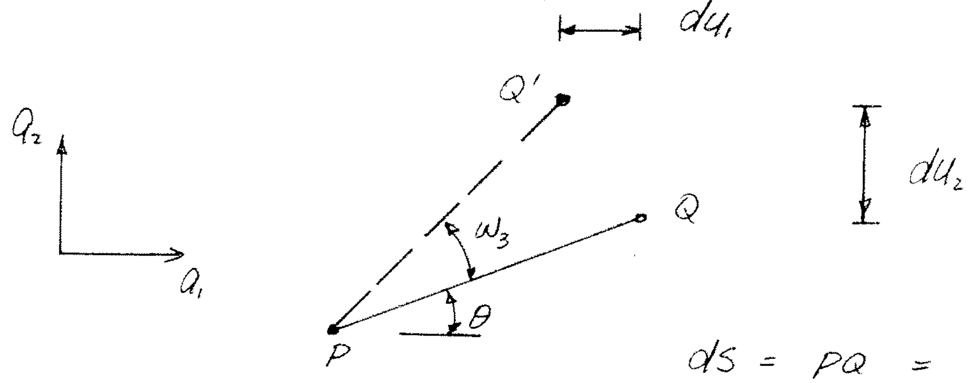
ALTERNATIVELY, THE DISPLACEMENT OF Q RELATIVE TO P, WHERE LINE SEGMENT PQ IS ROTATED THROUGH AN ANGLE ω_3 (NO STRAIN), IS SHOWN GRAPHICALLY AS



$$ds = PQ = PQ'$$

FROM THIS GEOMETRY

$$du_1 = ds\cos(\omega_3+\theta) - ds\cos\theta$$
$$du_2 = ds\sin(\omega_3+\theta) - ds\sin\theta$$

WITH

$$\cos(\omega_3+\theta) = \cos\omega_3\cos\theta - \sin\omega_3\sin\theta$$
$$\sin(\omega_3+\theta) = \sin\omega_3\cos\theta + \cos\omega_3\sin\theta$$

THEN

$$du_1 = ds\left[(\cos\omega_3 - 1)\cos\theta - \sin\omega_3\sin\theta\right]$$

$$du_2 = ds\left[\sin\omega_3\cos\theta + (\cos\omega_3 - 1)\sin\theta\right]$$

AND

$$\frac{du_1}{ds} = (\cos\omega_3 - 1)\cos\theta - \sin\omega_3\sin\theta$$

$$\frac{du_2}{ds} = \sin\omega_3\cos\theta + (\cos\omega_3 - 1)\sin\theta$$

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THEN IN MATRIX FORM

$$\begin{bmatrix} \frac{du_1}{ds} \\ \frac{du_2}{ds} \end{bmatrix} = \begin{bmatrix} \cos\omega_3 - 1 & -\sin\omega_3 \\ \sin\omega_3 & \cos\omega_3 - 1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

NOTE THAT FOR PQ

$$\frac{da_1}{ds} = \cos\theta$$

$$\frac{da_2}{ds} = \sin\theta$$

THUS

$$\begin{bmatrix} \frac{du_1}{ds} \\ \frac{du_2}{ds} \end{bmatrix} = \begin{bmatrix} \cos\omega_3 - 1 & -\sin\omega_3 \\ \sin\omega_3 & \cos\omega_3 - 1 \end{bmatrix} \begin{bmatrix} \frac{da_1}{ds} \\ \frac{da_2}{ds} \end{bmatrix}$$

UNDER THE ASSUMPTION OF SMALL ROTATIONS

$$\cos\omega_3 - 1 = 1 - 1$$

$$= 0$$

$$\sin\omega_3 = \omega_3$$

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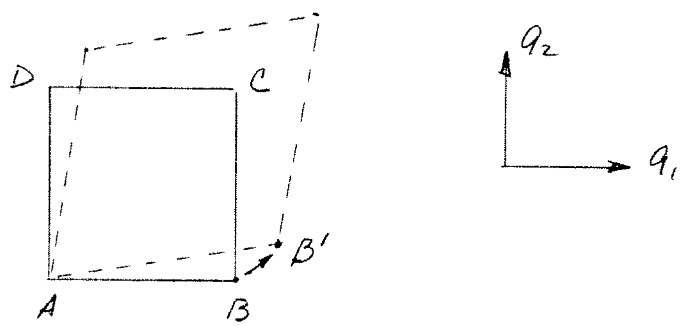
THUS

$$\begin{bmatrix} \frac{du_1}{ds} \\ \frac{du_2}{ds} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 \\ \omega_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{da_1}{ds} \\ \frac{da_2}{ds} \end{bmatrix}$$

THUS IF ALL THE ELEMENTS OF THE STRAIN MATRIX ARE ZERO AND THE ROTATION IS SMALL, THEN THE UNIT DISPLACEMENT OF Q RELATIVE TO P CONSTITUTES A RIGID BODY ROTATION.

FINALLY, IF WE CONSIDER THE FOLLOWING DEFORMATION OF OUR CUBIC ELEMENT



ASSUMING PURE STRAIN CONDITIONS, SUCH THAT

$$\Omega_{ij} = [0]$$

THEN IF WE TAKE

$$\frac{da_1}{ds} = 1$$

$$\frac{da_2}{ds} = 0$$

THEN THE RELATIONSHIP

$$\frac{du_i}{ds} = \frac{\partial u_i}{\partial a_j} \frac{da_j}{ds}$$

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WOULD YIELD

$$\left(\frac{du_i}{ds}\right)_{AB} = \text{UNIT DISPLACEMENT OF POINT B RELATIVE TO A}$$

I.E., IN MATRIX NOTATION

$$\begin{bmatrix} \frac{du_1}{ds} \\ \frac{du_2}{ds} \end{bmatrix}_{AB} = \begin{bmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} \\ \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} \end{bmatrix} \begin{bmatrix} \frac{da_1}{ds} \\ \frac{da_2}{ds} \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

THUS

$$\frac{du_1}{ds} = \epsilon_{11} \qquad \frac{du_2}{ds} = \epsilon_{21}$$

OR UNDER THE CONDITIONS OF PURE STRAIN

$\frac{du_1}{ds} = \epsilon_{11}$ = DIFFERENTIAL DISPLACEMENT OF B RELATIVE TO A IN THE 1-DIRECTION

$\frac{du_2}{ds} = \epsilon_{21}$ = DIFFERENTIAL DISPLACEMENT OF B RELATIVE TO A IN THE 2-DIRECTION

THE IMPORTANCE OF

$$\frac{du_2}{ds} = \epsilon_{21}$$

IS EXPLORED IN THE SECTION ON TORSION IN A CIRCULAR CYLINDER

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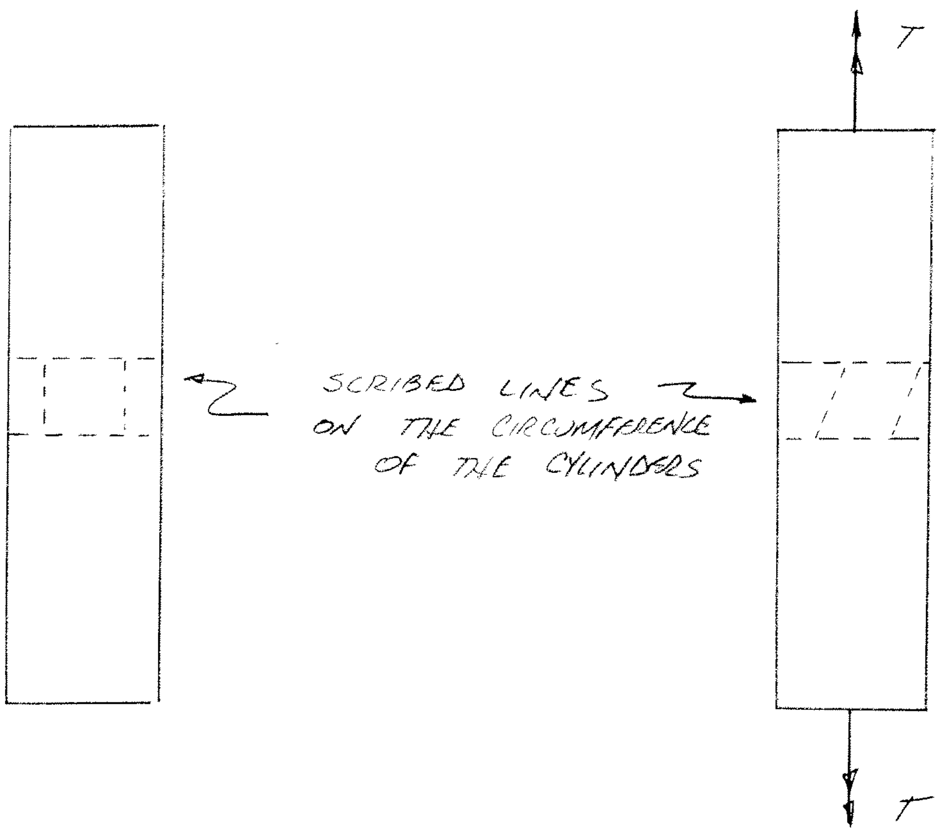
TORSION OF A RIGHT CIRCULAR CYLINDER

ONE OF THE ASSUMPTIONS INCORPORATED INTO THE DEVELOPMENT OF THE EQUATION FOR ANGULAR DISPLACEMENT IN A RIGHT CIRCULAR CYLINDER SUBJECTED TO TORQUE

$$\phi = \frac{TL}{JG}$$

IS THE OBSERVED FACT THAT OUT OF PLANE WARPING DOES NOT OCCUR. GRAPHICALLY



BEFOR APPLICATION OF T

AFTER APPLICATION OF T

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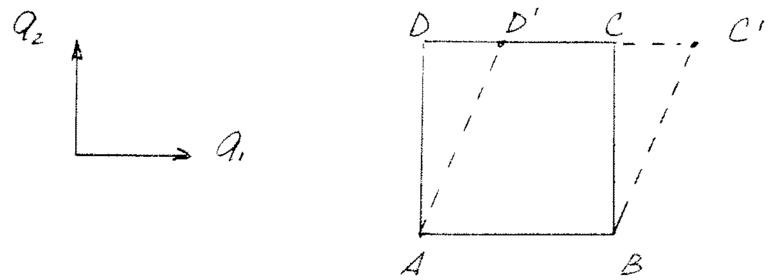
SUPERIMPOSING THE DEFORMED ELEMENT ON THE UNDEFORMED ELEMENT YIELDS



IF THE ELEMENT HAD BEEN SUBJECTED TO A PURE SHEAR STRAIN, THEN THE FIRST TWO ROWS AND THE FIRST TWO COLUMNS OF THE DISPLACEMENT GRADIENT RELATIONSHIP WOULD APPEAR AS

$$\begin{bmatrix} \dfrac{du_1}{ds} \\ \\ \dfrac{du_2}{ds} \end{bmatrix}_{DC} = \begin{bmatrix} 0 & \epsilon_{12} \\ \\ \epsilon_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}$$

THUS

$$\left(\frac{du_2}{ds} \right)_{DC} = \epsilon_{21}$$

HOWEVER WITH THE ABOVE DEFORMATION PATTERN IT IS QUITE CLEAR THAT

$$(du_2)_{DC} \equiv 0$$

THUS

$$\left(\frac{du_2}{ds} \right)_{DC} = 0$$

DOES THIS IMPLY

$$\epsilon_{21} = \epsilon_{12} = 0 \;?$$

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OBVIOUSLY NOT. IN FACT WHAT HAS HAPPENED IS THE ELEMENT HAS UNDERGONE BOTH STRAIN AND RIGID BODY ROTATION. THUS

$$\begin{bmatrix} \dfrac{du_1}{ds} \\ \\ \dfrac{du_2}{ds} \end{bmatrix}_{DC} = \begin{bmatrix} 0 & \epsilon_{12} \\ \\ \epsilon_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_3 \\ \\ \omega_3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}$$

WHERE

$$\frac{du_2}{ds} = \epsilon_{21} + \omega_3 = 0$$

IMPLYING

$$\omega_3 = -\epsilon_{21}$$

THUS A DIFFERENTIAL ELEMENT IN A RIGHT CIRCULAR CYLINDER UNDERGOES BOTH A SHEAR STRAIN AND A RIGID BODY ROTATION EQUAL IN MAGNITUDE AND OPPOSITE IN DIRECTION TO THE SHEAR STRAIN.

THIS EXAMPLE ILLUSTRATES ANOTHER POINT. THE STUDENT SHOULD ALWAYS THINK OF SHEAR STRAINS IN TERMS OF CHANGES IN ANGLES.

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SMALL STRAINS AND ROTATIONS IN THREE DIMENSIONS

GUIDED BY THE PREVIOUS DISCUSSION, WE WISH TO EXTEND THE RELATIONSHIP

$$\frac{du_i}{ds} = \left(\frac{\partial u_i}{\partial a_j}\right)\frac{da_j}{ds}$$

TO THREE DIMENSIONS. RECALL THAT THIS RELATIONSHIP CAN BE WRITTEN AS THE SUM OF A SYMMETRIC AND ANTISYMMETRIC TENSOR IN THE FOLLOWING MANNER

$$\frac{du_i}{ds} = \left(\frac{1}{2}\right)\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_i}{\partial a_j}\right)\frac{da_j}{ds}$$

$$= \left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)\right\}\frac{da_j}{ds}$$

THE COMPONENTS OF THE LINEAR STRAIN TENSOR WERE DEFINED AS

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$

AND THE COMPONENTS OF THE ROTATION TENSOR WERE DEFINED AS

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$$

THUS

$$\frac{du_i}{ds} = \left(E_{ij} + \Omega_{ij}\right)\frac{da_j}{ds}$$

IF A GIVEN DEFORMATION CONSTITUTES RIGID BODY MOTION, THEN

$$E_{ij} \equiv [0]$$

NOTE THAT IN GENERAL A RIGID BODY MOTION CONSISTS OF

1. A TRANSLATION DEFINED BY THE FIRST ORDER TENSOR u_i, AND
2. A ROTATION DEFINED BY THE SECOND ORDER TENSOR Ω_{ij}

THUS FOR RIGID BODY MOTION

$$\frac{du_i}{dS} = \Omega_{ij}\left(\frac{da_j}{dS}\right)$$

WHERE IN FULL NOTATION

$$\Omega_{11} = 0$$

$$\Omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right)$$

$$= -\Omega_{21}$$

$$\Omega_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_3} - \frac{\partial u_3}{\partial a_1}\right)$$

$$= -\Omega_{31}$$

$$\Omega_{22} = 0$$

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$$\Omega_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_3} - \frac{\partial u_3}{\partial a_2}\right)$$

$$= -\Omega_{32}$$

$$\Omega_{33} = 0$$

NOTE THE ENTIRE ROTATION TENSOR HAS ONLY THREE <u>INDEPENDENT</u> COMPONENTS

$$\left(\frac{\partial u_1}{\partial a_2} - \frac{\partial u_2}{\partial a_1}\right)$$

$$\left(\frac{\partial u_1}{\partial a_3} - \frac{\partial u_3}{\partial a_1}\right)$$

$$\left(\frac{\partial u_2}{\partial a_3} - \frac{\partial u_3}{\partial a_2}\right)$$

THESE THREE COMPONENTS FORM A DUAL VECTOR (DUAL IN THE SENSE THAT A SECOND VECTOR CAN BE CONSTRUCTED FROM Ω_{ij} WITH COMPONENTS THAT ARE NEGATIVE VALUES OF THE ABOVE). THIS VECTOR IS CALLED THE ROTATION VECTOR, WITH THE FOLLOWING NOTATION

$$\omega_1 = \Omega_{32}$$

$$\omega_2 = \Omega_{13}$$

$$\omega_3 = \Omega_{21}$$

THE ROTATION VECTOR CAN BE EXPRESSED IN TERMS OF THE ROTATION TENSOR IN THE FOLLOWING TENSOR RELATIONSHIP

$$\omega_i = \left(\frac{1}{2}\right) e_{ijk}\,\Omega_{kj}$$

$$= -\left(\frac{1}{2}\right) e_{ijk}\,\Omega_{jk}$$

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THE DUAL OF THIS ROTATION VECTOR WOULD THEN BE DEFINED AS

$$\tilde{\omega}_i = \text{DUAL OF } \omega_i$$

$$= \left(\frac{1}{2}\right) e_{ijk}\, \Omega_{jk}$$

$$= -\left(\frac{1}{2}\right) e_{ijk}\, \Omega_{kj}$$

THE INVERSE OF THE RELATIONSHIP BETWEEN THE ROTATION VECTOR AND THE ROTATION TENSOR CAN BE CONSTRUCTED IN THE FOLLOWING MANNER

$$\omega_i = \left(\frac{1}{2}\right) e_{ijk}\, \Omega_{kj}$$

$$= \left(\frac{1}{2}\right) e_{i\ell m}\, \Omega_{m\ell}$$

$$\omega_k = \left(\frac{1}{2}\right) e_{k\ell m}\, \Omega_{m\ell}$$

MULTIPLYING BOTH SIDES BY e_{ijk}

$$e_{ijk}\, \omega_k = \left(\frac{1}{2}\right) e_{ijk}\, e_{k\ell m}\, \Omega_{m\ell}$$

$$= \left(\frac{1}{2}\right) e_{kij}\, e_{k\ell m}\, \Omega_{m\ell}$$

$$= -\left(\frac{1}{2}\right)\left(\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell}\right) \Omega_{\ell m}$$

$$= -\left(\frac{1}{2}\right)\left(\delta_{i\ell}\, \delta_{jm}\, \Omega_{\ell m} - \delta_{im}\, \delta_{j\ell}\, \Omega_{\ell m}\right)$$

$$= -\left(\frac{1}{2}\right)\left(-\Omega_{ij} - \Omega_{ji}\right)$$

$$= -\left(\frac{1}{2}\right)\left(-\Omega_{ji} - \Omega_{ji}\right)$$

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$$\Omega_{ji} = e_{ijk}\, \omega_k$$

FOR RIGID BODY ROTATION

$$\frac{du_i}{ds} = \Omega_{ij} \frac{da_j}{ds}$$

$$= -\Omega_{ji} \frac{da_j}{ds}$$

$$= -e_{ijk}\, \omega_k \frac{da_j}{ds}$$

$$= (e_{ikj}\, \omega_k) \frac{da_j}{ds}$$

THIS INDICATES THAT FOR PURE ROTATION, (du_i/ds) IS THE RESULT OF THE CROSS PRODUCT BETWEEN THE DUAL VECTOR ω_k AND (da_j/ds).

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EXAMPLE (2-4.5 B&C) FOR TORSION OF A CYLINDRICAL BAR WITH AN ELLIPTICAL CROSS SECTION, THE COMPONENTS OF DISPLACEMENT ARE GIVEN BY

$$u_1 = -\theta a_2 a_3$$

$$u_2 = \theta a_1 a_3$$

$$u_3 = \left(\frac{b^2 - h^2}{b^2 + h^2}\right) \theta a_1 a_2$$

WHERE a_3 IS THE COORDINATE AXIS ALONG THE LENGTH OF THE BAR, θ IS THE ANGLE OF TWIST (IN RADIANS PER UNIT LENGTH) AND (h, b) ARE THE MAJOR AND MINOR AXES OF THE ELLIPTICAL CROSS SECTION. VERIFY THAT THIS DISPLACEMENT FIELD IS ADMISSABLE.

WE NEED ONLY DEMONSTRATE THE JACOBIAN IS GREATER THAN ZERO TO PROVE THIS DISPLACEMENT FIELD IS ADMISSABLE. WITH

$$J = \mathrm{DET} \begin{bmatrix} 1 + \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \frac{\partial u_1}{\partial a_3} \\ \frac{\partial u_2}{\partial a_1} & 1 + \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} \\ \frac{\partial u_3}{\partial a_1} & \frac{\partial u_3}{\partial a_2} & 1 + \frac{\partial u_3}{\partial a_3} \end{bmatrix}$$

$$= \mathrm{DET} \begin{bmatrix} 1 & -\theta a_3 & -\theta a_2 \\ \theta a_3 & 1 & \theta a_1 \\ \left(\frac{b^2 - h^2}{b^2 + h^2}\right)\theta a_2 & \left(\frac{b^2 - h^2}{b^2 + h^2}\right)\theta a_1 & 1 \end{bmatrix}$$

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$$J = 1 - \left(\frac{b^2 - h^2}{b^2 + h^2}\right)\left[2\theta^3 a_1 a_2 a_3 + \theta^2 a_1^2 - \theta^2 a_2^2 \right]$$
$$+ \theta^2 a_3^2$$
$$> 0$$

THUS

$$\left(1 + \theta^2 a_3^2\right) > \left(\frac{b^2 - h^2}{b^2 + h^2}\right)\left[2\theta^3 a_1 a_2 a_3 + \theta^2 a_1^2 - \theta^2 a_2^2 \right]$$

$$\left(1 + \theta^2 a_3^2\right) > (-1)\left(\frac{h^2 - b^2}{h^2 + b^2}\right)\left[2\theta^3 a_1 a_2 a_3 + \theta^2 a_1^2 - \theta^2 a_2^2 \right]$$

$$\left(1 + \theta^2 a_3^2\right) > \left(\frac{h^2 - b^2}{h^2 + b^2}\right)\left[\theta^2 a_2^2 - \theta^2 a_1^2 - 2\theta^3 a_1 a_2 a_3 \right]$$

$$\left(1 + \theta^2 a_3^2\right)\left(\frac{h^2 + b^2}{h^2 - b^2}\right) > \theta^2\left[a_2^2 - a_1^2 - 2\theta a_1 a_2 a_3 \right]$$

IF WE ASSUME

$$\theta << 1$$

THEN

$$\frac{h^2 + b^2}{h^2 - b^2} > 0$$

WHICH IS SATISFIED ALWAYS SINCE h IS THE MAJOR AXIS OF THE ELLIPSE AND

$$h > b$$

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EXAMPLE: WITH

$$E_{ij} \equiv [0]$$

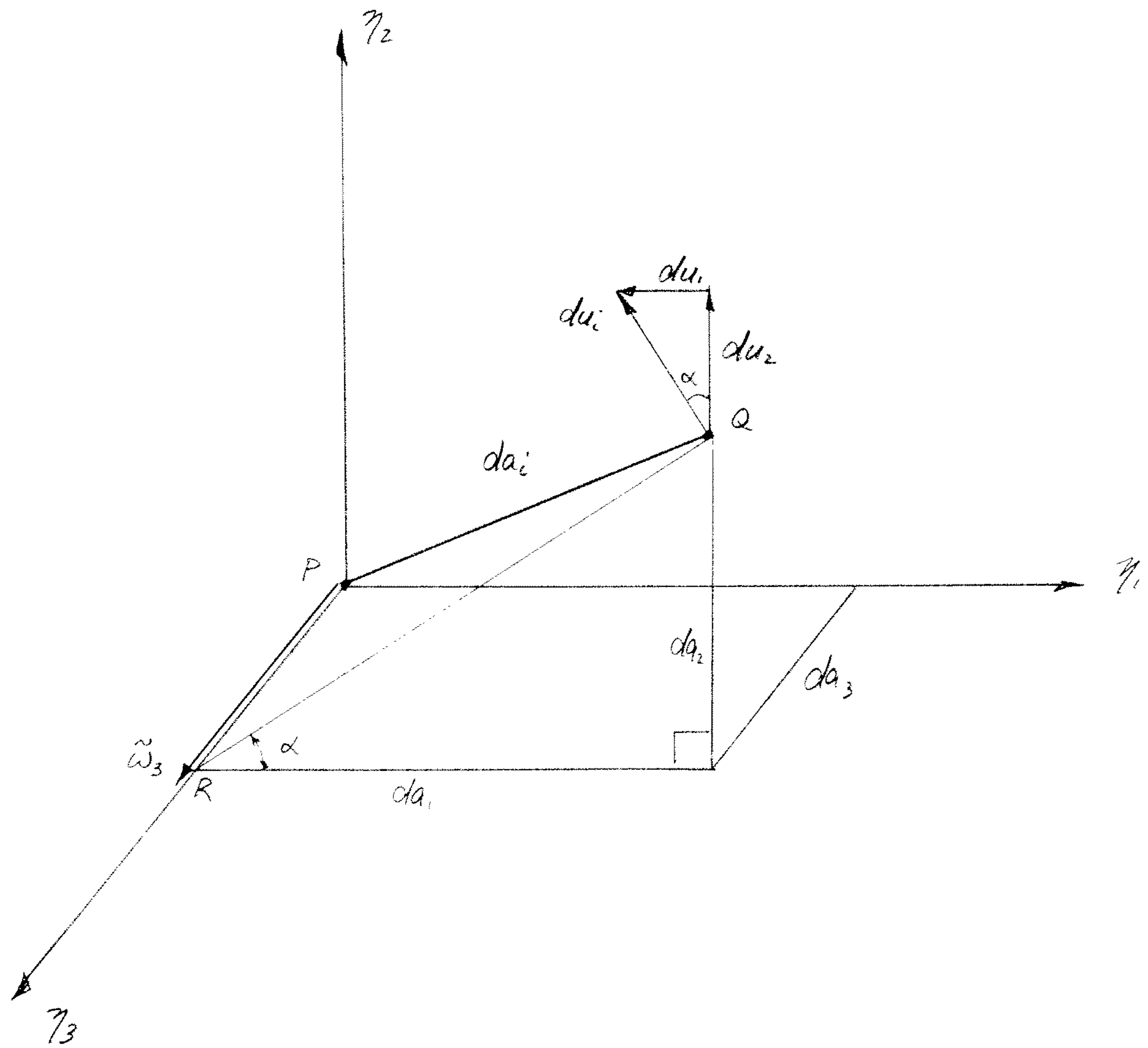
AND

$$\omega_j = (0, 0, \omega_3)$$

GEOMETRICALLY DESCRIBE THE DISPLACEMENT OF A LINE SEGMENT PQ WHERE

$$PQ = da_i$$

INTRODUCE A LOCAL CARTESIAN COORDINATE SYSTEM WITH ORIGIN AT P AND PARALLEL TO THE a_i AXES.



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WITH

$$du_i = e_{ikj}\, \omega_k\, da_j$$

THEN

$$du_1 = \omega_2\, da_3 - \omega_3\, da_2$$
$$= (0)\, da_3 - \omega_3\, da_2$$
$$= -\omega_3\, da_2$$

$$du_2 = \omega_3\, da_1 - \omega_1\, da_3$$
$$= \omega_3\, da_1 - (0)\, da_3$$
$$= \omega_3\, da_1$$

$$du_3 = \omega_1\, da_2 - \omega_2\, da_1$$
$$= (0)\, da_2 - (0)\, da_1$$
$$= 0$$

FROM THE FIGURE

$$|du_i| = \omega_3 (QR)$$

AND

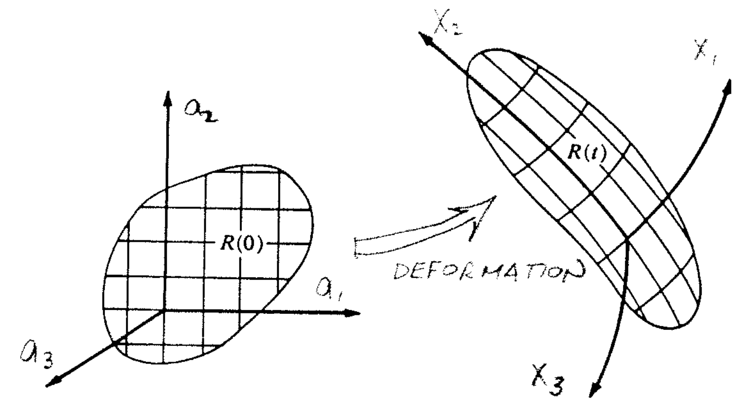
$$du_1 = -|du_i| \sin\alpha \qquad du_2 = |du_i| \cos\alpha$$
$$= -\omega_3 (QR) \sin\alpha \qquad = +\omega_3 (QR) \cos\alpha$$
$$= -\omega_3\, da_2 \qquad = +\omega_3\, da_1$$

THUS TENSOR RELATIONSHIP YIELDS WHAT YOU EXPECT PHYSICALLY FROM THE FIRST PAGE.

PHYSICAL INTERPRETATION OF J

CONSIDER A REGION $R(0)$ OCCUPIED BY A CONTINUUM PRIOR TO THE APPLICATION OF LOADS AND/OR DISPLACEMENTS. AFTER THE APPLICATION OF LOADS AND/OR DISPLACEMENTS, THE CONTINUUM WILL MOVE TO A REGION DEFINED AS $R(t)$. HERE t IS A DUMMY VARIABLE REPRESENTING TIME. THIS UPDATED REGION MAY OR MAY NOT OVERLAP THE REGION $R(0)$.

NOW CONSIDER THAT THE ORIGINAL CONTINUUM HAS BEEN MARKED WITH LINES PARALLEL TO THE a_i AXES. AFTER LOADS AND/OR DISPLACEMENTS HAVE BEEN APPLIED, THESE LINES MARKED ON THE CONTINUUM ORIGINALLY WILL NOW TRACE CURVES.



IN THE ORIGINAL CONFIGURATION THE VOLUME OF A DIFFERENTIAL CUBE FROM SOMEWHERE WITHIN THE CONTINUUM IS

$$dV_0 = da_1 \; da_2 \; da_3$$

IN THE DEFORMED CONFIGURATION

$$dV = (A \times B) \cdot C$$

$$= e_{ijk} \, dx_i \, dx_j \, dx_k$$

THIS IS A RESULT FROM CALCULUS WHERE THE TRIPLE SCALAR PRODUCT DEFINES THE VOLUME OF A BOX (I.E., THE DIFFERENTIAL VOLUME dV). ALSO RECALL THAT IF

$$x_i = x_i(a_i)$$

THEN

$$dx_i = \left(\frac{\partial x_i}{\partial a_j}\right) da_j$$

$$= \left(\frac{\partial x_i}{\partial a_1}\right) da_1 + \left(\frac{\partial x_i}{\partial a_2}\right) da_2 + \left(\frac{\partial x_i}{\partial a_3}\right) da_3$$

IF dx_i IS A VECTOR THAT WAS PARALLEL TO THE a_1 AXIS, THEN IN THE ORIGINAL COORDINATE SYSTEM

$$dx_1 = dx_1(a_1, 0, 0) \equiv da_1$$

$$dx_2 = 0$$

$$dx_3 = 0$$

AND

$$dx_i = dx_i(a_1, 0, 0)$$

THUS

$$dx_i = \left(\frac{\partial x_i}{\partial a_1}\right) da_1 + \left(\frac{\partial x_i}{\partial a_2}\right) da_2 + \left(\frac{\partial x_i}{\partial a_3}\right) da_3$$

$$= \left(\frac{\partial x_i}{\partial a_1}\right) da_1 + (0)\, da_2 + (0)\, da_3$$

$$= \left(\frac{\partial x_i}{\partial a_1}\right) da_1$$

IF dx_j (A DIFFERENT VECTOR FROM dx_i) IS A VECTOR THAT WAS PARALLEL TO THE a_2-AXIS, THEN IN THE ORIGINAL COORDINATE SYSTEM

$$dx_1 = 0$$

$$dx_2 = dx_2(0, a_2, 0) \equiv da_2$$

$$dx_3 = 0$$

AND

$$dx_j = dx_j(0, a_2, 0)$$

THUS

$$dx_j = \left(\frac{\partial x_j}{\partial a_1}\right) da_1 + \left(\frac{\partial x_j}{\partial a_2}\right) da_2 + \left(\frac{\partial x_j}{\partial a_3}\right) da_3$$

$$= (0)\, da_1 + \left(\frac{\partial x_j}{\partial a_2}\right) da_2 + (0)\, da_3$$

$$= \left(\frac{\partial x_j}{\partial a_2}\right) da_2$$

IF dx_k (A VECTOR DIFFERENT FROM dx_i AND dx_j) IS A VECTOR THAT WAS PARALLEL TO THE a_3-AXIS, THEN IN THE ORIGINAL CONFIGURATION

$$dx_1 = 0$$

$$dx_2 = 0$$

$$dx_3 = dx_3(0, 0, a_3) \equiv da_3$$

AND

$$dx_k = dx_k(0, 0, a_3)$$

THUS

$$dx_k = \left(\frac{\partial x_k}{\partial a_1}\right) da_1 + \left(\frac{\partial x_k}{\partial a_2}\right) da_2 + \left(\frac{\partial x_k}{\partial a_3}\right) da_3$$

$$= (0)\, da_1 + (0)\, da_2 + \left(\frac{\partial x_k}{\partial a_3}\right) da_3$$

$$= \left(\frac{\partial x_k}{\partial a_3}\right) da_3$$

BY DEFINING THE VECTORS dx_i, dx_j, AND dx_k IN THIS FASHION, THEN IN THE ORIGINAL CONFIGURATION THESE VECTORS CORRESPOND TO THE VOLUME

$$dV_0 = da_1\, da_2\, da_3$$

AND THIS VOLUME BECOMES

$$dV = e_{ijk}\, dx_i\, dx_j\, dx_k$$

IN THE DEFORMED CONFIGURATION. IF WE EXPAND THE RIGHT HAND SIDE OF THIS EXPRESSION IN TERMS OF THE RELATIONSHIPS DEFINED ABOVE, THEN

$$dV = e_{ijk} \left(\frac{\partial x_i}{\partial a_1}\right) da_1 \left(\frac{\partial x_j}{\partial a_2}\right) da_2 \left(\frac{\partial x_k}{\partial a_3}\right) da_3$$

$$= e_{ijk} \left(\frac{\partial x_i}{\partial a_1}\right)\left(\frac{\partial x_j}{\partial a_2}\right)\left(\frac{\partial x_k}{\partial a_3}\right) da_1\, da_2\, da_3$$

$$= \left[e_{ijk} \left(\frac{\partial x_i}{\partial a_1}\right)\left(\frac{\partial x_j}{\partial a_2}\right)\left(\frac{\partial x_k}{\partial a_3}\right)\right] dV_0$$

OPERATOR INSIDE BRACKETS

AND

$$\frac{dV}{dV_0} = e_{ijk} \left(\frac{\partial x_i}{\partial a_1}\right)\left(\frac{\partial x_j}{\partial a_2}\right)\left(\frac{\partial x_k}{\partial a_3}\right)$$

ALTHOUGH IT MAY NOT BE OBVIOUS, THE RIGHT HAND SIDE IS THE FOLLOWING DETERMINANT

$$e_{ijk}\left(\frac{\partial x_i}{\partial a_1}\right)\left(\frac{\partial x_j}{\partial a_1}\right)\left(\frac{\partial x_k}{\partial a_3}\right) = \mathrm{DET} \begin{bmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \frac{\partial x_1}{\partial a_3} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_2}{\partial a_3} \\ \frac{\partial x_3}{\partial a_1} & \frac{\partial x_3}{\partial a_2} & \frac{\partial x_3}{\partial a_3} \end{bmatrix}$$

$$= \mathrm{DET} \left\{\frac{\partial x_i}{\partial a_j}\right\}$$

$$= J$$

WHICH LEADS US TO THE FOLLOWING RELATIONSHIP

$$\frac{dV}{dV_0} = J$$

NOW THERE IS A CLEAR INTERPRETATION FOR J:

$J > 1$ IMPLIES VOLUME EXPANSION

$0 < J < 1$ IMPLIES VOLUME CONTRACTION

AND IT BECOMES EVIDENT THAT J CAN NEVER BE LESS THAN ZERO.

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EXAMPLE (2-8.2 B&C) THE DISPLACEMENT COMPONENTS FOR A BODY ARE DETERMINED AS

$$u_1 = 2a_1 + a_2$$

$$u_2 = a_3$$

$$u_3 = a_3 - a_2$$

a.) VERIFY THAT THIS DISPLACEMENT VECTOR IS PHYSICALLY POSSIBLE FOR A CONTINUOUSLY DEFORMED BODY.

b.) DETERMINE THE STRAIN IN THE DIRECTION DEFINED BY THE UNIT VECTOR

$$n_i = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

C.) DETERMINE THE DIRECTION COSINES OF THE ELEMENT IN THE UNDEFORMED MEDIUM THAT ENDS UP IN THE a_3 DIRECTION IN THE DEFORMED MEDIUM.

D.) DETERMINE THE CHANGE IN ANGLE BETWEEN THE LINE SEGMENTS WHOSE DIRECTION IN THE UNDEFORMED MEDIUMS ARE CONCIDENT WITH

$$n_i = (1, 0, 0)$$

AND

$$m_i = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

SOLUTION:

a.) TO PROVE THE DISPLACEMENTS ARE ADMISSABLE, SHOW

$$J = DET \begin{vmatrix} 1 + \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \frac{\partial u_1}{\partial a_3} \\ \frac{\partial u_2}{\partial a_1} & 1 + \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} \\ \frac{\partial u_3}{\partial a_1} & \frac{\partial u_3}{\partial a_2} & 1 + \frac{\partial u_3}{\partial a_3} \end{vmatrix} > 0$$

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$$DET(J_{ij}) = \begin{vmatrix} 1+2 & 1 & 0 \\ 0 & 1+0 & 1 \\ 0 & -1 & 1+1 \end{vmatrix} \begin{matrix} 1+2 & 1 \\ 0 & 1+0 \\ 0 & -1 \end{matrix}$$

$$= 3(1)(2) + 0 + 0 - 0 - (-1)(1)(3) - 0$$

$$= 6 + 3$$

$$= 9$$

$$> 0$$ ADMISSABLE

b.) WITH

$$n_i = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

DEFINING THE ORIGINAL DIRECTION, AND WITH

$$\frac{du_i}{ds} = \left(\frac{\partial u_i}{\partial a_j}\right)\left(\frac{da_j}{ds}\right)$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{3} + 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}$$ RELATIVE CHANGE PER UNIT LENGTH

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$$\text{NEW VECTOR} = \begin{bmatrix} 3/\sqrt{3} & + & 1/\sqrt{3} \\ 1/\sqrt{3} & + & 1/\sqrt{3} \\ 0 & + & 1/\sqrt{3} \end{bmatrix} = (\text{RELATIVE CHANGE} + \text{OLD LENGTH})$$

$$= \begin{bmatrix} 4/\sqrt{3} \\ 2/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

DEFINE

$$ds_0 = \left[\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right]^{1/2}$$

$$= 1.0$$

$$ds = \left[\left(\frac{4}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right]^{1/2}$$

$$= \left(\frac{21}{3}\right)^{1/2}$$

$$= \sqrt{7}$$

THUS THE STRAIN ALONG THE DIRECTION DEFINED BY n_i IS

$$\text{STRAIN} = \frac{ds - ds_0}{ds_0}$$

$$= \frac{\sqrt{7} - 1}{1}$$

$$= 1.646$$

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C.) DETERMINE THE DIRECTION COSINES OF A LINE SEGMENT THAT IS CO-LINEAR WITH THE UNIT VECTOR

$$n_j = (0, 0, 1)$$

IN THE DEFORMED CONFIGURATION. THE LINE SEGMENT IS IDENTIFIED AS

$$x_i = x_i(a_j)$$

AFTER DEFORMATION. FROM CALCULUS

$$dx_i = \left(\frac{\partial x_i}{\partial a_j}\right) da_j$$

AND

$$\frac{dx_i}{ds} = \left(\frac{\partial x_i}{\partial a_j}\right) \frac{da_j}{ds}$$

$$= \left[\frac{\partial}{\partial a_j}(a_i + u_i)\right] \frac{da_j}{ds}$$

$$= \left[\delta_{ij} + \frac{\partial u_i}{\partial a_j}\right] \frac{da_j}{ds}$$

THUS

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 1 & 0 \\ 0 & 1+0 & 1 \\ 0 & -1 & 1+1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

THIS LEADS TO THE FOLLOWING SYSTEM OF EQUATIONS

$$0 = 3n_1 + n_2$$

$$0 = n_2 + n_3$$

$$1 = -n_2 + 2n_3$$

FROM THE LAST EQUATION

$$n_2 = 2n_3 - 1$$

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SUBSTITUTING THIS INTO THE SECOND EQUATION YIELDS

$$(2n_3 - 1) + n_3 = 0$$

$$3n_3 = 1$$

$$n_3 = 1/3$$

THUS

$$n_2 = 2(1/3) - 1$$

$$= -1/3$$

FINALLY

$$3n_1 + n_2 = 0$$

$$3n_1 = 1/3$$

$$n_1 = 1/9$$

OR

$$N_i = \left(\frac{1}{9}, -\frac{1}{3}, \frac{1}{3}\right)$$

WHICH ISN'T A UNIT VECTOR. WITH

$$|N_i| = \left[\left(\frac{1}{9}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]^{1/2}$$

$$= \left[\frac{1}{81} + \frac{1}{9} + \frac{1}{9}\right]^{1/2}$$

$$= \left[\frac{1}{81} + \frac{9}{81} + \frac{9}{81}\right]^{1/2}$$

$$= \frac{\sqrt{19}}{9}$$

THUS THE UNIT VECTOR, WHICH IS COMPOSED OF DIRECTION COSINES, IS

$$n_i = \left(\frac{9}{\sqrt{19}}\right) N_i$$

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$$n_i = \frac{9}{\sqrt{19}}\left(\frac{1}{9},\ -\frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(\frac{1}{\sqrt{19}},\ -\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}}\right)$$

d.) WITH

$$A_i = \begin{bmatrix} 4/\sqrt{3} \\ 2/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \longleftarrow \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

AND

$$\frac{du_i}{dS} = \left(\frac{\partial u_i}{\partial a_j}\right)\frac{da_j}{dS} = \left(\frac{\partial u_i}{\partial a_j}\right) n_j$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{NEW VECTOR} = \begin{bmatrix} 2 + 1 \\ 0 + 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = B_i$$

THEN

$$A_i\, B_i = \bar{A} \cdot \bar{B}$$

$$= |\bar{A}|\,|\bar{B}| \cos\beta$$

WITH

$$|\bar{A}| = \left[\left(\frac{4}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right]^{1/2}$$

$$= \left[\frac{16}{3} + \frac{4}{3} + \frac{1}{3} \right]^{1/2}$$

$$= \sqrt{7}$$

$$|\bar{B}| = 3$$

THEN

$$\beta = \cos^{-1}\left[\left(\frac{1}{3\sqrt{7}}\right) A_i B_i \right]$$

$$= \cos^{-1}\left\{ \left(\frac{1}{3\sqrt{7}}\right)\left[3\left(\frac{4}{\sqrt{3}}\right) + 0\left(\frac{2}{\sqrt{3}}\right) + 0\left(\frac{1}{\sqrt{3}}\right)\right]\right\}$$

$$= \cos^{-1}\left\{ \frac{4}{\sqrt{21}} \right\}$$

$$= 29.21^\circ$$

THIS IS THE FINAL ANGLE BETWEEN THE LINE SEGMENTS. WE STILL NEED TO COMPUTE THE ORIGINAL ANGLE BETWEEN THE TWO LINE SEGMENTS IN ORDER TO COMPUTE THE CHANGE IN ANGLE. WITH

$$n_i = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$m_i = (1, 0, 0)$$

THEN

$$\eta = \cos^{-1}\left\{ \frac{1}{(1)(1)} m_i n_i \right\}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{3}} \right)$$

$$= 54.74^\circ$$

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AND

$$\theta = \eta - \beta$$
$$= 54.74 - 29.21^\circ$$
$$= 25.53^\circ$$

NOTE THAT IN THIS PROBLEM THE NORMAL STRAINS AND THE SHEAR STRAINS ARE NOT "SMALL". THIS CAN BE RECTIFIED BY MANIPULATING THE COEFFICIENTS OF THE ORIGINAL DISPLACEMENT FIELD. AS AN EXAMPLE, TAKE

$$u_1 = (2\times 10^{-6})q_1 + (1\times 10^{-6})q_2$$
$$u_2 = (1\times 10^{-6})q_3$$
$$u_3 = (1\times 10^{-6})q_3 - (1\times 10^{-6})q_2$$

AND REDO THIS PROBLEM FOR HOMEWORK.

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EXAMPLE (2-B.3 B&C) THE DISPLACEMENT FIELD FOR A COMPONENT IS SPECIFIED AS

$$u_1 = 2a_1$$

$$u_2 = 3a_2 + a_3$$

$$u_3 = a_3 - a_2$$

a.) VERIFY THAT THIS DISPLACEMENT FIELD IS ADMISSABLE

b.) DETERMINE THE STRAIN IN THE DIRECTION

$$n_i = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$$

AND THE SHEAR STRAIN ASSOCIATED WITH THIS DIRECTION, AND THE DIRECTION

$$m_j = (-1/\sqrt{2},\ 1/\sqrt{2},\ 0)$$

c.) DETERMINE THE DIRECTION COSINES OF THE LINE SEGMENT IN THE UNDEFORMED MEDIUM THAT ENDS UP IN THE a_2 - DIRECTION IN THE DEFORMED MEDIUM.

a.) TO PROVE THAT THE DISPLACEMENT FIELD IS ADMISSABLE, SHOW

$$\text{DET} \begin{bmatrix} 1 + \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \frac{\partial u_1}{\partial a_3} \\ \frac{\partial u_2}{\partial a_1} & 1 + \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} \\ \frac{\partial u_3}{\partial a_1} & \frac{\partial u_3}{\partial a_2} & 1 + \frac{\partial u_3}{\partial a_3} \end{bmatrix}$$

$$> 0$$

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$$J = \begin{vmatrix} 1+2 & 0 & 0 \\ 0 & 1+3 & 1 \\ 0 & -1 & 1+1 \end{vmatrix} \begin{matrix} 1+2 & 0 \\ 0 & 1+3 \\ 0 & -1 \end{matrix}$$

$$= (3)(4)(2) + 0 + 0 - 0 - (-1)(1)(3)$$

$$= 24 + 3$$

$$= 27 > 0$$ <u>ADMISSABLE</u>

b.) WITH

$$n_i = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$$

DEFINING THE ORIGINAL DIRECTION, AND WITH

$$\text{RELATIVE CHANGE} = \frac{du_i}{ds}$$

$$= \left(\frac{\partial u_i}{\partial a_j}\right)\left(\frac{da_j}{ds}\right)$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{3} \\ 3/\sqrt{3} + 1/\sqrt{3} \\ -1/\sqrt{3} + 1/\sqrt{3} \end{bmatrix}$$

RELATIVE CHANGE $= \begin{bmatrix} 2/\sqrt{3} \\ 4/\sqrt{3} \\ 0 \end{bmatrix}$

NEW VECTOR = RELATIVE CHANGE + OLD VECTOR

$$= \begin{bmatrix} 2/\sqrt{3} + 1/\sqrt{3} \\ 4/\sqrt{3} + 1/\sqrt{3} \\ 0 + 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} \\ 5/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

DEFINE

$$dS_0 = \left\{ \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right\}^{1/2}$$

$$= 1.0$$

AND

$$dS = \left\{ \left(\frac{3}{\sqrt{3}}\right)^2 + \left(\frac{5}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right\}^{1/2}$$

$$= \left\{ \frac{9}{3} + \frac{25}{3} + \frac{1}{3} \right\}^{1/2}$$

$$= \left(\frac{35}{3}\right)^{1/2}$$

THUS THE STRAIN IN THE DIRECTION OF n_i IS

$$\text{STRAIN} = \frac{dS - dS_0}{dS_0}$$

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$$STRAIN = \left(\frac{35}{3}\right)^{1/2} - 1$$

$$= 2.416$$

THE ORIGINAL ANGLE BETWEEN n_i AND m_j CAN BE DETERMINED FROM

$$n_i m_j \delta_{ij} = |n_i||m_j| \cos\theta$$

$$\theta = \cos^{-1}\left\{\left[\frac{1}{|n_i||m_j|}\right] n_i m_i\right.$$

$$= \cos^{-1}\left\{\left[\frac{1}{(1)(1)}\right]\left[n_1 m_1 + n_2 m_2 + n_3 m_3\right]\right\}$$

$$= \cos^{-1}\left\{\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}}\right)0\right\}$$

$$= \cos^{-1}(0)$$

$$= 90^\circ$$

LET

$$A_i = \begin{bmatrix} 3/\sqrt{3} \\ 5/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \longrightarrow \text{A VECTOR ASSOCIATED WITH } n_i$$

THEN

$$|A_i| = \left\{\left(\frac{3}{\sqrt{3}}\right)^2 + \left(\frac{5}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2\right\}^{1/2}$$

$$= \left\{\frac{9}{3} + \frac{25}{3} + \frac{1}{3}\right\}^{1/2}$$

$$= \left(\frac{35}{3}\right)^{1/2}$$

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NEXT

$$\text{RELATIVE CHANGE} = \frac{du_i}{ds}$$

$$= \left(\frac{\partial u_i}{\partial q_j}\right)\left(\frac{dq_j}{ds}\right)$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}_{m_j}$$

$$= \begin{bmatrix} -2/\sqrt{2} & \\ 3/\sqrt{2} & + \; 0 \\ -1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{2} \\ 3/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

LET

B_i = NEW VECTOR

$$= \begin{bmatrix} -2/\sqrt{2} & - & 1/\sqrt{2} \\ 3/\sqrt{2} & + & 1/\sqrt{2} \\ -1/\sqrt{2} & + & 0 \end{bmatrix}$$

$$B_i = \begin{bmatrix} -3/\sqrt{2} \\ 4/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \longrightarrow \text{A VECTOR ASSOCIATED WITH } m_j$$

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WITH

$$|B_i| = \left\{ \left(-\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{4}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 \right\}^{1/2}$$

$$= \left(\frac{26}{2}\right)^{1/2}$$

$$= \sqrt{13}$$

NOW

$$A_i B_i = |A_i||B_i| \cos \eta$$

$$\eta = \cos^{-1}\left\{ \left[\frac{1}{|A_i||B_i|}\right] A_i B_i \right\}$$

$$= \cos^{-1}\left\{ \left(\frac{3}{35}\right)^{1/2}\left(\frac{1}{13}\right)^{1/2}\left[\left(\frac{3}{\sqrt{3}}\right)\left(-\frac{3}{\sqrt{2}}\right) + \left(\frac{5}{\sqrt{3}}\right)\left(\frac{4}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{2}}\right) \right] \right\}$$

$$= \cos^{-1}\left\{ \left(\frac{3}{35}\right)^{1/2}\left(\frac{1}{13}\right)^{1/2}\left(\frac{100}{6}\right)^{1/2} \right\}$$

$$= 70.64^\circ$$

FINALLY

$$\text{SHEAR STRAIN} = 90 - 70.64$$

$$= 19.36$$

$$= \left(\frac{\pi}{180^\circ}\right) 19.36$$

$$= 0.338$$

Class Solved Eg MANIK SHAR.

$$\begin{Bmatrix} 0 \\ A \\ 0 \end{Bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$\Rightarrow 3l = 0$

$4m + n = A$

$-m + 2n = 0$

$\Rightarrow m = 2n,$

$\Rightarrow n = \frac{A}{9}, \quad m = \frac{2A}{9}$

Now $l^2 + m^2 + n^2 = 1$

$\Rightarrow 0 + \frac{4A^2}{81} + \frac{A^2}{81} = 1$

$A = \frac{9}{\sqrt{5}}$

$\Rightarrow m = \frac{2}{\sqrt{5}}$

$n = \frac{1}{\sqrt{5}}$

$\therefore$ Direction cosines $= \begin{Bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{Bmatrix}$

It Works

C.) WITH

$$dx_i = \left(\frac{\partial x_i}{\partial a_j}\right) da_j$$

THEN

$$\frac{dx_i}{ds} = \left(\frac{\partial x_i}{\partial a_j}\right) \frac{da_j}{ds}$$

$$= \left(\delta_{ij} + \frac{\partial u_i}{\partial a_j}\right) \frac{da_j}{ds}$$

SINCE

$$x_i = u_i + a_i$$

THUS

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

WHICH LEADS TO THE FOLLOWING SYSTEM OF EQUATIONS IN TERMS OF N_1, N_2, AND N_3

$$0 = 3N_1$$

$$1 = 4N_2 + N_3$$

$$0 = -N_2 + 2N_3$$

THUS

$$N_1 = 0$$

AND

$$N_2 = 2N_3$$

WHICH LEADS TO

$$1 = 4(2N_3) + N_3$$

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OR

$$N_3 = + \frac{1}{9}$$

THUS

$$N_2 = + \frac{2}{9}$$

AND

$$N_i = \begin{bmatrix} 0 \\ \frac{2}{9} \\ \frac{1}{9} \end{bmatrix}$$

WITH

$$|N_i| = \left\{ 0 + \left(+\frac{1}{9}\right)^2 + \left(+\frac{2}{9}\right)^2 \right\}^{1/2}$$

$$= \left(\frac{5}{81}\right)^{1/2}$$

THEN

$$n_i = \begin{bmatrix} 0 \\ \left(\frac{2}{9}\right)\left(\frac{81}{5}\right)^{1/2} \\ \left(\frac{1}{9}\right)\left(\frac{81}{5}\right)^{1/2} \end{bmatrix}$$

$$n_i = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$ (A VECTOR OF DIRECTION COSINES)

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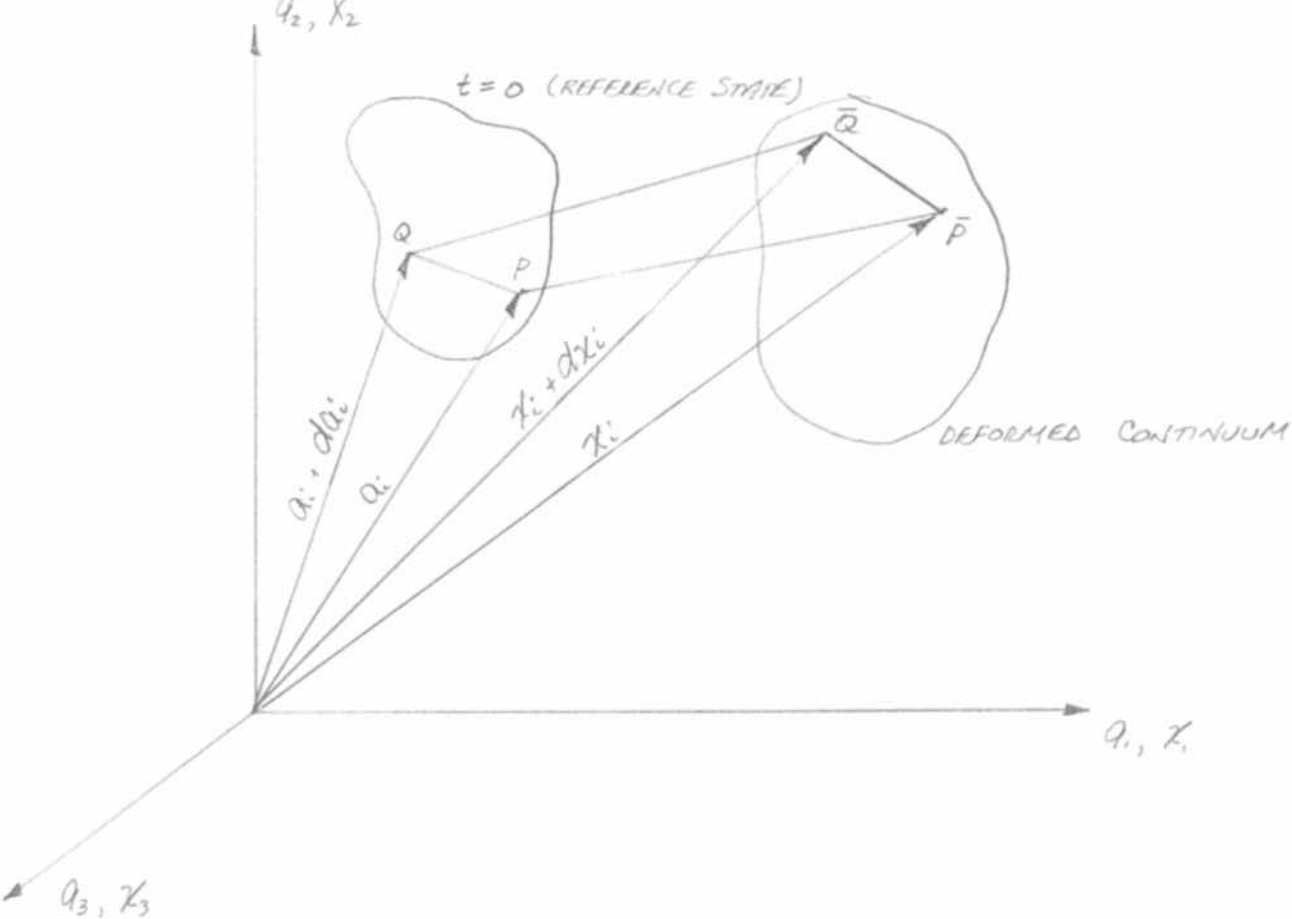
NON-LINEAR STRAIN TENSORS

CONSIDER THE LINE SEGMENT DEFINED BY ENDPOINTS P AND Q IN AN UNDEFORMED CONTINUUM. SUBJECT THIS CONTINUUM TO AN ARBITRARY DEFORMATION AND THE DEFORMED LINE SEGMENT IS NOW DEFINED BY ENDPOINTS $\bar{P}$ AND $\bar{Q}$.



THE FOUR POINTS P, Q, $\bar{P}$ AND $\bar{Q}$ HAVE THE POSITION VECTORS

$P: \; a_i$

$Q: \; a_i + da_i$

$\bar{P}: \; x_i$

$\bar{Q}: \; x_i + dx_i$

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THUS WE CAN IDENTIFY LINE SEGMENT PQ AS da_i AND LINE SEGMENT $\bar{P}\bar{Q}$ AS dx_i. ALSO LET

$$ds_0^2 = (\text{LENGTH OF } PQ)^2$$

$$ds^2 = (\text{LENGTH OF } \bar{P}\bar{Q})^2$$

THEN

$$ds_0^2 = da_i \, da_i$$
$$= \delta_{ij} \, da_j \, da_i$$

$$ds^2 = dx_i \, dx_i$$
$$= \delta_{ij} \, dx_j \, dx_i$$

IF WE ADOPT THE LAGRANGIAN DESCRIPTION OF DEFORMATION

$$u_i = u_i(a_1, a_2, a_3)$$
$$x_i = x_i(a_1, a_2, a_3)$$

WHERE

$$dx_i = \left(\frac{\partial x_i}{\partial a_j}\right) da_j$$

NOW WE CONSIDER THE DIFFERENCE BETWEEN THE SQUARES OF THE LENGTHS OF THE LINE SEGMENTS PQ AND $\bar{P}\bar{Q}$

$$ds^2 - ds_0^2 = dx_i \, dx_i - da_i \, da_i$$
$$= \delta_{ij} \, dx_j \, dx_i - \delta_{ij} \, da_j \, da_i$$
$$= \delta_{kl} \, dx_l \, dx_k - \delta_{ij} \, da_j \, da_i$$

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$$= \delta_{k\ell}\left(\frac{\partial x_k}{\partial a_j}\right) da_j \left(\frac{\partial x_\ell}{\partial a_i}\right) da_i - \delta_{ij}\, da_j\, da_i$$

$$= \delta_{k\ell}\left(\frac{\partial x_k}{\partial a_j}\right)\left(\frac{\partial x_\ell}{\partial a_i}\right) da_j\, da_i - \delta_{ij}\, da_j\, da_i$$

$$= \left\{ \delta_{k\ell}\left(\frac{\partial x_k}{\partial a_j}\right)\left(\frac{\partial x_\ell}{\partial a_i}\right) - \delta_{ij} \right\} da_j\, da_i$$

NOW INTRODUCE THE THE DISPLACEMENT VECTOR WITH

$$x_k = u_k + a_k$$

WHERE

$$\frac{\partial x_k}{\partial a_j} = \frac{\partial u_k}{\partial a_j} + \frac{\partial a_k}{\partial a_j}$$

$$= \frac{\partial u_k}{\partial a_j} + \delta_{kj}$$

THEN

$$ds^2 - ds_0^2 = \left\{ \delta_{k\ell}\left[\frac{\partial u_k}{\partial a_j} + \delta_{kj}\right]\left[\frac{\partial u_\ell}{\partial a_i} + \delta_{\ell i}\right] - \delta_{ij} \right\} da_j\, da_i$$

$$= \left\{ \delta_{k\ell}\left[\left(\frac{\partial u_k}{\partial a_j}\right)\left(\frac{\partial u_\ell}{\partial a_i}\right) + \left(\frac{\partial u_k}{\partial a_j}\right)\delta_{\ell i} + \left(\frac{\partial u_\ell}{\partial a_i}\right)\delta_{kj} \right.\right.$$

$$\left.\left. + \delta_{kj}\,\delta_{\ell i} \right] - \delta_{ij} \right\} da_j\, da_i$$

$$= \left\{ \left(\frac{\partial u_k}{\partial a_j}\right)\delta_{k\ell}\,\delta_{\ell i} + \left(\frac{\partial u_\ell}{\partial a_i}\right)\delta_{k\ell}\,\delta_{kj} \right.$$

$$\left. + \left(\frac{\partial u_k}{\partial a_j}\right)\left(\frac{\partial u_\ell}{\partial a_i}\right)\delta_{k\ell} + \delta_{k\ell}\,\delta_{kj}\,\delta_{\ell i} - \delta_{ij} \right\} da_j\, da_i$$

$$= \left\{ \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \left(\frac{\partial u_\ell}{\partial a_j}\right)\left(\frac{\partial u_\ell}{\partial a_i}\right) \right.$$

$$\left. \delta_{ji} - \delta_{ij} \right\} da_j\, da_i$$

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$$ds^2 - ds_0^2 = \left\{ \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \left(\frac{\partial u_\ell}{\partial a_i}\right)\left(\frac{\partial u_\ell}{\partial a_j}\right) \right\} da_j \, da_i$$

WE NOW FORMALLY DEFINE THE TENSOR

$$E_{ij} = \frac{1}{2}\left\{ \frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \left(\frac{\partial u_\ell}{\partial a_j}\right)\left(\frac{\partial u_\ell}{\partial a_i}\right) \right\}$$

AS THE NON-LINEAR FORM OF THE LAGRANGIAN STRAIN TENSOR. THIS TENSOR WAS INTRODUCED BY GREEN AND ST. VENANT AND IS OFTEN TIMES REFERRED TO AS GREEN'S STRAIN TENSOR. HOWEVER WE WILL ADOPT THE TERMINOLOGY FROM HYDRODYNAMICS AND REFER TO THIS TENSOR AS THE LAGRANGIAN STRAIN TENSOR. NOTE THAT

$$ds^2 - ds_0^2 = 2E_{ij} \, da_j \, da_i$$

SIMILARLY, IF WE ADOPT THE EULERIAN DESCRIPTION OF DEFORMATION

$$u_i = u_i(x_1, x_2, x_3)$$

$$a_i = a_i(x_1, x_2, x_3)$$

WHERE

$$da_i = \left(\frac{\partial a_i}{\partial x_j}\right) dx_j$$

AND WE CONSIDER THE DIFFERENCE BETWEEN THE SQUARES OF THE LENGTHS ds AND ds_0

$$\begin{aligned} ds^2 - ds_0^2 &= \delta_{ij} \, dx_j \, dx_i - \delta_{ij} \, da_j \, da_i \\ &= \delta_{ij} \, dx_j \, dx_i - \delta_{k\ell}\left(\frac{\partial a_k}{\partial x_j}\right) dx_j \left(\frac{\partial a_\ell}{\partial x_i}\right) dx_i \\ &= \left\{ \delta_{ij} - \delta_{k\ell}\left(\frac{\partial a_k}{\partial x_j}\right)\left(\frac{\partial a_\ell}{\partial x_i}\right) \right\} dx_j \, dx_i \end{aligned}$$

ONCE AGAIN, INTRODUCING THE DISPLACEMENT VECTOR

$$a_k = x_k - u_k$$

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THEN

$$ds^2 - ds_o^2 = \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_e}{\partial x_j} \frac{\partial u_e}{\partial x_i} \right\} dx_j \, dx_i$$

AND WE FORMALLY DEFINE THE TENSOR

$$e_{ij} = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \left(\frac{\partial u_e}{\partial x_j} \right) \left(\frac{\partial u_e}{\partial x_i} \right) \right\}$$

AS THE NONLINEAR FORM OF THE EULERIAN STRAIN TENSOR. THIS TENSOR WAS INTRODUCED BY CAUCHY FOR INFINITESIMAL STRAINS AND BY ALMANSI AND HAMMEL FOR FINITE STRAINS. THIS TENSOR IS OFTEN TIMES REFERRED TO AS ALMANSI'S STRAIN TENSOR, BUT ONCE AGAIN WE ADOPT THE TERMINOLOGY FROM HYDRODYNAMICS. NOTE THAT

$$ds^2 - ds_o^2 = 2 e_{ij} \, dx_j \, dx_i$$

UP TO THIS POINT WE HAVE BEEN REFERRING TO E_{ij} AND e_{ij} AS TENSORS. TO BE ABLE TO DO THIS WE MUST SHOW FORMALLY THAT THEY OBEY THE TRANSFORMATION LAW FOR SECOND ORDER TENSORS. WE WILL DEMONSTRATE THIS FOR THE EULERIAN STRAIN TENSOR. CONSIDER AN ARBITRARY ROTATION OF COORDINATES AXES SUCH THAT

$$u_i = \alpha_{mi} u_m'$$

$$u_m' = u_m'(x_n')$$

THUS IN TERMS OF THE PRIMED COORDINATE SYSTEM

$$e_{ij} = \left(\frac{1}{2}\right) \left\{ \left[\frac{\partial (\alpha_{mi} u_m')}{\partial x_n'} \right] \left(\frac{dx_n'}{dx_j} \right) + \left[\frac{\partial (\alpha_{nj} u_n')}{\partial x_m'} \right] \left(\frac{dx_m'}{dx_i} \right) - \left[\frac{\partial (\alpha_{kr} u_k')}{\partial x_m'} \right] \left(\frac{dx_m'}{dx_i} \right) \left[\frac{\partial (\alpha_{er} u_e')}{\partial x_n'} \right] \left(\frac{dx_n'}{dx_j} \right) \right\}$$

BUT UNDER THE ARBITRARY ROTATION OF COORDINATE AXES

$$x_n' = \alpha_{jn} x_j$$

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TAKING THE DIFFERENTIAL OF THIS EXPRESSION YIELDS

$$
\begin{aligned}
dx'_n &= d(\alpha_{nj} x_j) \\
&= [d\alpha_{nj}]x_j + \alpha_{nj}\, dx_j \\
&= [0]x_j + \alpha_{nj}\, dx_j \\
&= \alpha_{nj}\, dx_j
\end{aligned}
$$

THUS

$$\frac{dx'_n}{dx_j} = \alpha_{nj} \qquad \left[\frac{dx_j}{dx'_n} = \alpha_{jn} \right]$$

AND

$$e_{ij} = \left(\frac{1}{2}\right)\left\{ \alpha_{mi}\alpha_{nj}\left(\frac{\partial u'_m}{\partial x'_n}\right) + \alpha_{nj}\alpha_{mi}\left(\frac{\partial u'_n}{\partial x'_m}\right) - \alpha_{kr}\alpha_{mi}\alpha_{\ell r}\alpha_{nj}\left(\frac{\partial u'_k}{\partial x'_m}\right)\left(\frac{\partial u'_\ell}{\partial x'_n}\right)\right\}$$

$$= \left(\frac{1}{2}\right)\alpha_{mi}\alpha_{nj}\left\{\left(\frac{\partial u'_m}{\partial x'_n}\right) + \left(\frac{\partial u'_n}{\partial x'_m}\right) - \alpha_{kr}\alpha_{\ell r}\left(\frac{\partial u'_k}{\partial x'_m}\right)\left(\frac{\partial u'_\ell}{\partial x'_n}\right)\right\}$$

WITH

$$\alpha_{kr}\alpha_{\ell r} = \delta_{k\ell}$$

THEN THE ABOVE BECOMES

$$
\begin{aligned}
e_{ij} &= \left(\frac{1}{2}\right)\alpha_{mi}\alpha_{nj}\left\{\left(\frac{\partial u'_m}{\partial x'_n}\right) + \left(\frac{\partial u'_n}{\partial x'_m}\right) - \delta_{k\ell}\left(\frac{\partial u'_k}{\partial x'_m}\right)\left(\frac{\partial u'_\ell}{\partial x'_n}\right)\right\} \\
&= \left(\frac{1}{2}\right)\alpha_{mi}\alpha_{nj}\left\{\left(\frac{\partial u'_m}{\partial x'_n}\right) + \left(\frac{\partial u'_n}{\partial x'_m}\right) - \left(\frac{\partial u'_\ell}{\partial x'_m}\right)\left(\frac{\partial u'_\ell}{\partial x'_n}\right)\right\} \\
&= \left(\frac{1}{2}\right)\alpha_{mi}\alpha_{nj}\,(2\, e'_{mn}) \\
&= \alpha_{mi}\alpha_{nj}\, e'_{mn} \qquad \underline{\underline{Q.E.D.}}
\end{aligned}
$$

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THUS THE EULERIAN STRAIN TENSOR OBEYS THE TRANSFORMATION LAW FOR SECOND ORDER TENSORS. IN A SIMILAR FASHION IT CAN BE DEMONSTRATED THAT

$$E_{ij} = \alpha_{mi} \alpha_{nj} E'_{mn}$$

FINALLY, WE HAVE DEFINED THE LAGRANGIAN AND EULERIAN STRAIN TENSORS USING THE DIFFERENCE BETWEEN THE SQUARES OF A LENGTH IN THE REFERENCE STATE (PQ) AND THE LENGTH IN THE DEFORMED STATE ($\bar{P}\bar{Q}$), THAT IS

$$ds^2 - ds_o^2$$

IF THE NEIGHBORHOOD IN THE VICINITY OF PQ DOES NOT EXPERIENCE ANY STRAIN IN GOING FROM THE REFERENCE STATE TO THE DEFORMED STATE, THEN

$$ds^2 - ds_o^2 \equiv 0$$

THUS IN THE ABSENCE OF STRAIN

$$0 \equiv 2E_{ij}\, da_j\, da_i$$

$$0 \equiv 2e_{ij}\, dx_j\, dx_i$$

THIS IMPLIES

$$[0] = \frac{1}{2}\left\{ \frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_e}{\partial a_j} \frac{\partial u_e}{\partial a_i} \right\}$$

$$[0] = \frac{1}{2}\left\{ \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_e}{\partial x_j} \frac{\partial u_e}{\partial x_i} \right\}$$

IN THE ABSENCE OF STRAIN ALL THE COMPONENTS OF THE EULERIAN OR LAGRANGIAN STRAIN TENSOR MUST VANISH. CONVERSELY, IF THE COMPONENTS OF THESE STRAIN TENSORS VANISH, THEN THERE MUST BE A TOTAL ABSENCE OF STRAIN. WE CALL ANY DISPLACEMENT THAT TAKES PLACE IN THE ABSENCE OF STRAIN "<u>RIGID</u> <u>BODY</u> <u>MOTION</u>".

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GEOMETRICAL INTERPRETATION OF THE LINEAR STRAIN TENSOR

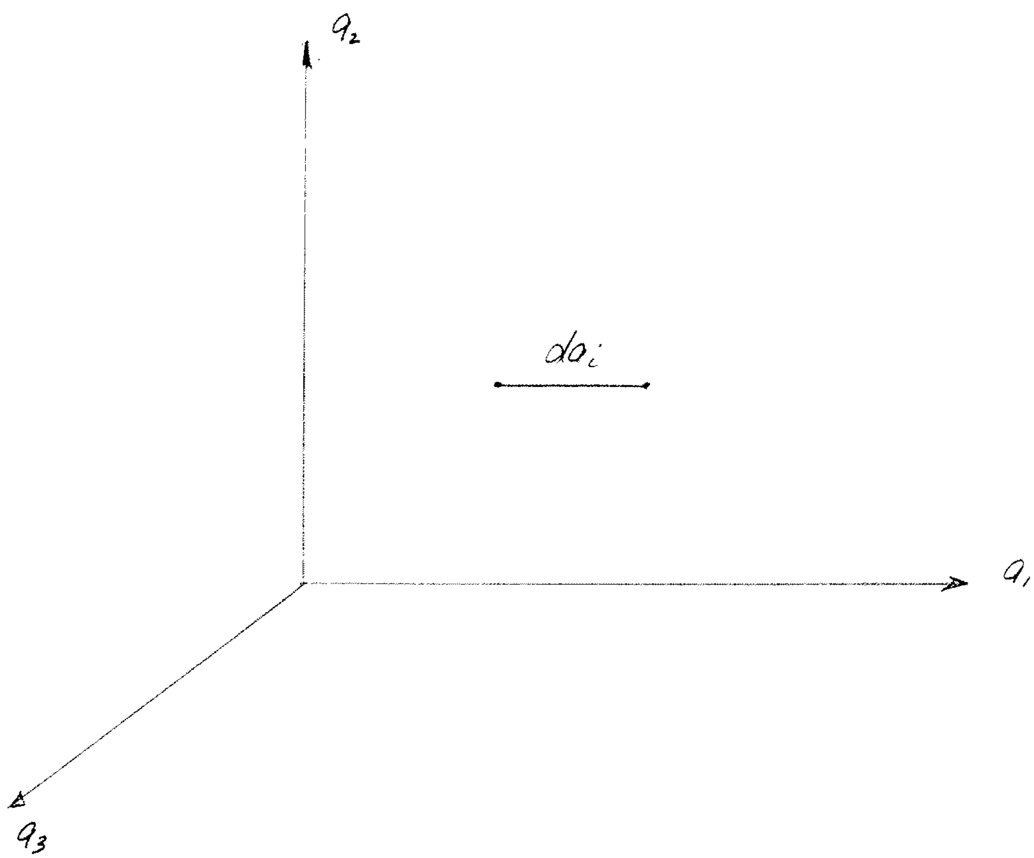
IN A PRECEEDING SECTION, THE LINEAR LAGRANGIAN STRAIN TENSOR WAS GIVEN AS

$$E_{ij} = \frac{1}{2}\left\{ \frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} \right\}$$

FIRST WE WILL OBTAIN A PHYSICAL MEANING OF THE DIAGONAL COMPONENTS OF THIS TENSOR, I.E., E_{11}, E_{22} AND E_{33}. CONSIDER A DIFFERENTIAL LINE SEGMENT da_i PARALLEL TO THE a_1-AXIS. HERE

$$da_i = (da_1, 0, 0)$$

GRAPHICALLY



WHERE

$$|da_i| = ds_0$$
$$= da_1$$

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IN GENERAL, THE CHANGE OF THE SQUARE LENGTH OF AN ARBITRARY LINE SEGMENT DUE TO A DEFORMATION IS

$$ds^2 - ds_o^2 = 2E_{ij}\, da_i\, da_j$$

$$(ds - ds_o)(ds + ds_o) = 2E_{ij}\, da_i\, da_j$$

FOR THE CASE OF INFINITESIMAL STRAINS

$$ds \cong ds_o$$

NOTE: da_i & da_j ARE COINCIDENT VECTORS, I.E., BOTH VECTORS ARE PARALLEL TO THE a_1 - AXIS

HENCE

$$ds + ds_o = 2ds_o$$

THUS

$$(ds - ds_o)\, 2ds_o = 2E_{ij}\, da_i\, da_j$$

$$(ds - ds_o) = \frac{E_{ij}\, da_i\, da_j}{ds_o}$$

DIVIDING BOTH SIDES OF THIS EXPRESSION BY ds_o

$$\frac{(ds - ds_o)}{ds_o} = E_{ij}\left(\frac{da_i}{ds_o}\right)\left(\frac{da_j}{ds_o}\right)$$

BUT

$\dfrac{da_i}{ds_o}$ = UNIT VECTOR OF DIRECTION COSINES FOR da_i

$= n_i$

$\dfrac{da_j}{ds_o}$ = UNIT VECTOR OF DIRECTION COSINES FOR da_j

$= n_j$

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ALSO DEFINE

$$E = \text{CHANGE IN LENGTH PER UNIT LENGTH OF } da_i$$
$$= \frac{ds - ds_o}{ds_o}$$

THEN

$$E = E_{ij}\, n_i\, n_j$$

RETURNING TO THE FACT THAT

$$da_i = (da_1, 0, 0)$$

THE VECTOR OF DIRECTION COSINES IS GIVEN BY

$$n_i = (1, 0, 0)$$

THUS

$$\begin{aligned} E &= E_{ij}\, n_i\, n_j \\ &= E_{1j}\, n_1\, n_j + E_{2j}\, n_2\, n_j + E_{3j}\, n_3\, n_j \\ &= E_{1j}\, n_1\, n_j \\ &= E_{11}\, n_1\, n_1 + E_{12}\, n_1\, n_2 + E_{13}\, n_1\, n_3 \\ &= E_{11} \end{aligned}$$

THUS E_{11} IS THE CHANGE IN LENGTH PER UNIT LENGTH OF THE DIFFERENTIAL LINE SEGMENT da_i ORIENTED IN THE a_1-DIRECTION. SIMILARLY IT CAN BE SHOWN THAT E_{22} AND E_{33} ARE CHANGES IN LENGTH PER UNIT LENGTH OF LINE SEGMENTS PARALLEL TO THE a_2 AND a_3 AXES, RESPECTIVELY. THESE ARE TERMED "NORMAL STRAINS" IN AN INTRODUCTORY MECHANICS COURSE.

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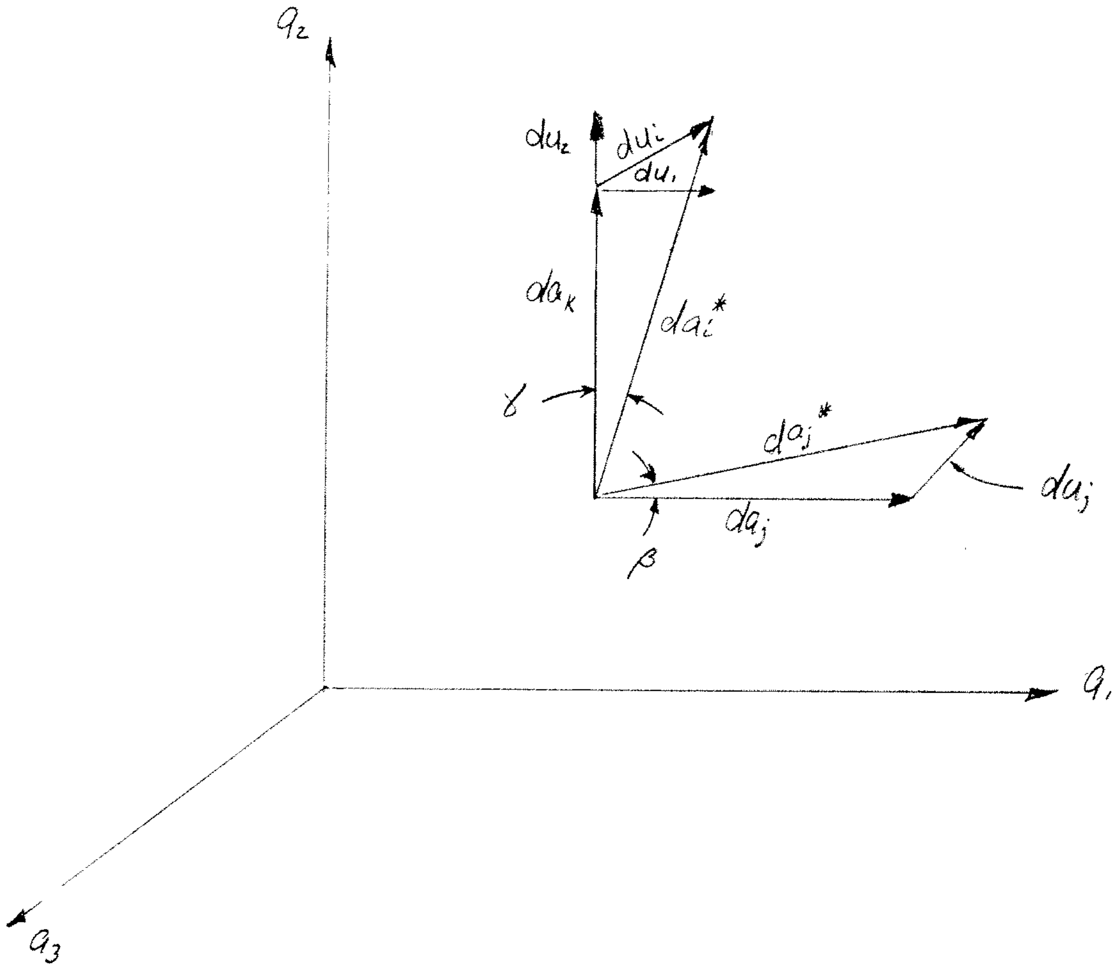
THE REMAINING COMPONENTS OF E_{ij} FOR $i \neq j$ REQUIRE A CONSIDERATION OF TWO PERPENDICULAR LINE SEGMENTS IN THE UNDEFORMED STATE THAT ARE SUBJECTED TO AN ARBITRARY DIFFERENTIAL DEFORMATION



HERE

$$da_k = (0, da_2, 0)$$

$$da_j = (da_1, 0, 0)$$

AND WE RESTRICT THE DEFORMATION TO THE $a_1 - a_2$ PLANE SUCH THAT

$$du_i = (du_1, du_2, 0)$$

$$du_j = (du_1^*, du_2^*, 0)$$

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WITH

$$du_i = \left(\frac{\partial u_i}{\partial a_k}\right) da_k$$

WHICH HAS COMPONENTS

$$du_1 = \left(\frac{\partial u_1}{\partial a_k}\right) da_k$$

$$= \left(\frac{\partial u_1}{\partial a_1}\right) da_1 + \left(\frac{\partial u_1}{\partial a_2}\right) da_2 + \left(\frac{\partial u_1}{\partial a_3}\right) da_3$$

$$= \left(\frac{\partial u_1}{\partial a_2}\right) da_2$$

SIMILARLY

$$du_2 = \left(\frac{\partial u_2}{\partial a_2}\right) da_2$$

$$du_3 = 0$$

THE COMPONENTS OF du_j CAN BE FOUND IN A SIMILAR MANNER

$$du_1^* = \left(\frac{\partial u_1}{\partial a_j}\right) da_j$$

$$= \left(\frac{\partial u_1}{\partial a_1}\right) da_1 + \left(\frac{\partial u_1}{\partial a_2}\right) da_2 + \left(\frac{\partial u_1}{\partial a_3}\right) da_3$$

$$= \left(\frac{\partial u_1}{\partial a_1}\right) da_1$$

$$du_2^* = \left(\frac{\partial u_2}{\partial a_1}\right) da_1$$

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$$du_3^* = 0$$

THE ANGLE γ CAN BE DEFINED FROM THE GEOMETRY AS

$$\tan \gamma = \frac{du_1}{da_2 + du_2}$$

FOR SMALL DISPLACEMENTS

$$da_2 \gg du_2$$

THUS

$$da_2 + du_2 \cong da_2$$

AND

$$\tan \gamma \cong \frac{du_1}{da_2} = \frac{\partial u_1}{\partial a_2}$$

SIMILARLY

$$\tan \beta \cong \frac{\partial u_2}{\partial a_1}$$

FOR INFINITESIMAL STRAINS THE ANGLES γ AND β ARE SMALL THUS

$$\gamma \cong \tan \gamma$$

$$\beta \cong \tan \beta$$

AND

$\gamma + \beta =$ DECREASE IN THE RIGHT ANGLE BETWEEN da_i AND da_j

$$= \frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1} = 2E_{12}$$

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E_{13} AND E_{23} HAVE SIMILAR INTERPRETATIONS. THUS THE THE THREE TENSOR COMPONENTS ARE EQUAL TO ONE HALF THE CHANGE OF ANGLE BETWEEN TWO LINE SEGMENTS ORIGINALLY 90° APART ALONG THE DIRECTIONS INDICATED BY THE NUMERICAL SUBSCRIPTS. THESE ARE ONE HALF THE FAMILIAR "SHEARING STRAINS" DEFINED IN A STRENGTH OF MATERIALS CLASS.

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THE CAUCHY-GREEN STRAIN TENSORS

THE LAGRANGIAN STRAIN TENSOR E_{ij} AND THE EULERIAN STRAIN TENSOR e_{ij} ARE DEFINED BASED ON AN ANALYSIS OF DEFORMATION OF INITIAL AND FINAL SQUARED LENGTHS, I.E.

$$\text{CLASSICAL} \begin{cases} ds^2 - ds_o^2 = 2 E_{ij}\, da_i\, da_j \\ ds^2 - ds_o^2 = 2 e_{ij}\, dx_i\, dx_j \end{cases}$$

THE CAUCHY-GREEN STRAIN TENSORS ARE CLOSELY RELATED TO THE CLASSICAL STRAIN TENSORS GIVEN ABOVE. INSTEAD OF GIVING THE CHANGE IN THE SQUARED LENGTHS, THE CAUCHY-GREEN TENSORS RELATE THE ORIGINAL SQUARED LENGTHS TO THE DEFORMED SQUARED LENGTH IN THE FOLLOWING MANNER

$$ds^2 = C_{ij}\, da_i\, da_j$$

$$ds_o^2 = B_{ij}\, dx_i\, dx_j$$

WE CAN DEVELOP RELATIONSHIPS BETWEEN THE CLASSICAL STRAIN TENSORS AND THE CAUCHY-GREEN STRAIN TENSORS IN THE FOLLOWING MANNER

$$ds^2 - ds_o^2 = 2 E_{ij}\, da_i\, da_j$$

$$C_{ij}\, da_i\, da_j - ds_o^2 = 2 E_{ij}\, da_i\, da_j$$

$$ds_o^2 = C_{ij}\, da_i\, da_j - 2E_{ij}\, da_i\, da_j$$

$$0 = (C_{ij} - 2E_{ij})\, da_i\, da_j - ds_o^2$$

$$0 = (C_{ij} - 2E_{ij})\, da_i\, da_j - da_k\, da_k$$

$$0 = (C_{ij} - 2E_{ij})\, da_i\, da_j - \delta_{ik}\, da_i\, \delta_{jk}\, da_j$$

$$0 = (C_{ij} - 2E_{ij} - \delta_{ik}\delta_{jk})\, da_i\, da_j$$

$$0 = (C_{ij} - 2E_{ij} - \delta_{ij})\, da_i\, da_j$$

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THUS FOR ARBITRARY LINE SEGMENT da_i (da_j)

$$[0] = C_{ij} - 2E_{ij} - \delta_{ij}$$

$$C_{ij} = \delta_{ij} + 2E_{ij}$$

SIMILARLY

$$ds^2 - ds_0^2 = 2e_{ij}\, dx_i\, dx_j$$

$$ds^2 - B_{ij}\, dx_i\, dx_j = 2e_{ij}\, dx_i\, dx_j$$

$$ds^2 = (B_{ij} + 2e_{ij})\, dx_i\, dx_j$$

HOWEVER

$$ds^2 = dx_k\, dx_k$$

AND

$$dx_k\, dx_k = (B_{ij} + 2e_{ij})\, dx_i\, dx_j$$

$$\delta_{ik}\, dx_i\, \delta_{jk}\, dx_j = (B_{ij} + 2e_{ij})\, dx_i\, dx_j$$

$$0 = (B_{ij} + 2e_{ij} - \delta_{ij})\, dx_i\, dx_j$$

THUS

$$B_{ij} + 2e_{ij} = \delta_{ij}$$

$$B_{ij} = \delta_{ij} - 2e_{ij}$$

NOTE THAT IN THE ABSENCE OF STRAIN THE CLASSICAL STRAIN TENSORS VANISH, I.E.,

$$E_{ij} \equiv [0]$$

$$e_{ij} \equiv [0]$$

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HOWEVER IN THE ABSENCE OF STRAIN THE CAUCHY-GREEN TENSORS DO NOT VANISH, IN FACT

$$B_{ij} + [0] = \delta_{ij} \qquad\qquad C_{ij} - 2[0] = \delta_{ij}$$

$$B_{ij} = \delta_{ij} \qquad\qquad C_{ij} = \delta_{ij}$$

THEREFORE IN THE ABSENCE OF STRAIN THE CAUCHY-GREEN TENSORS REDUCE TO THE KRONECKER DELTA.

ONE FINAL COMMENT ON THE NOMENCLATURE. THE CAUCHY-GREEN TENSOR C_{ij} IS OFTEN REFERRED TO AS

1. THE CAUCHY STRAIN TENSOR, OR
2. THE RIGHT CAUCHY-GREEN STRAIN TENSOR

SIMILARLY B_{ij} IS REFERRED TO AS

1. FINGERS TENSOR, OR
2. THE LEFT CAUCHY-GREEN STRAIN TENSOR

STRETCH TENSOR AND THE JACOBIAN

DEFINE STRETCH IN THE FOLLOWING MANNER

$$\text{STRETCH} = \frac{ds}{dS_0}$$

FOR A LINE SEGMENT ORIGINALLY IN THE x_1 - DIRECTION

$$da_i = (da_1, 0, 0)$$

AND

$$da_1 = dS_0$$

UTILIZING THE RIGHT CAUCHY GREEN TENSOR

$$ds^2 = C_{ij}\, da_i\, da_j$$

FOR $i = j = 1$

$$ds^2 = C_{11}\, da_1\, da_1$$

$$= C_{11}\, dS_0\, dS_0$$

THUS

$$C_{11} = \left(\frac{ds}{dS_0}\right)^2$$

AND THE STRETCH OF A LINE SEGMENT ORIGINALLY PARALLEL TO THE x_1 - AXIS IS

$$^1\left(\frac{ds}{dS_0}\right) = \sqrt{C_{11}}$$

$$= (1 + 2E_{11})^{1/2}$$

SIMILARLY, THE STRETCH OF A LINE SEGMENT ORIGINALLY PARALLEL TO THE x_2 - AXIS IS

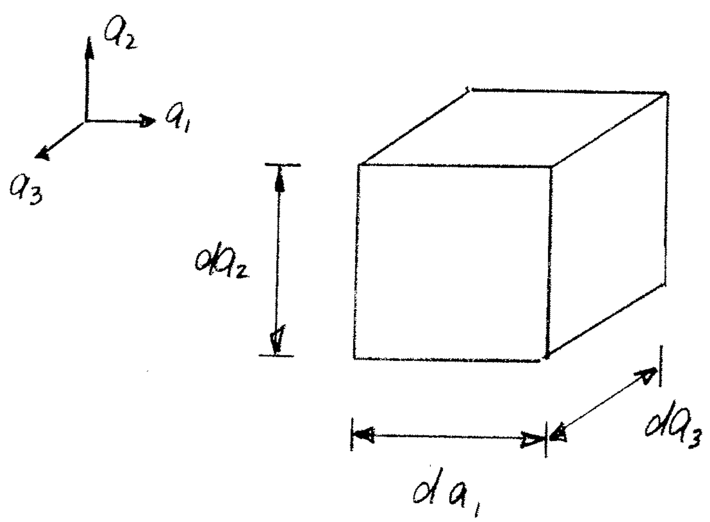
$$^2\left(\frac{ds}{dS_0}\right) = \sqrt{C_{22}} = (1 + 2E_{22})^{1/2}$$

FINALLY, THE STRETCH OF A LINE SEGMENT ORIGINALLY PARALLEL TO THE a_3' - AXIS IS

$$
{}^{3}\left(\frac{ds}{dS_0}\right) = \sqrt{C_{33}}
$$

$$
= (1 + 2E_{33})^{1/2}
$$

WITH THE FOLLOWING DIFFERENTIAL VOLUME ELEMENT



$$
dV_0 = da_1\, da_2\, da_3
$$

AND THE FOLLOWING STATE OF PRINCIPLE STRAIN APPLIED TO THE ELEMENT

$$
E_{ij} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}
$$

THEN

$$
C_{ij} = \begin{bmatrix} 1+2E_1 & 0 & 0 \\ 0 & 1+2E_2 & 0 \\ 0 & 0 & 1+2E_3 \end{bmatrix}
$$

THE NEW DIMENSIONS OF THE DIFFERENTIAL VOLUME ARE

$$da_1^* = {}^1\left(\frac{ds}{ds_0}\right) da_1 = \sqrt{C_1}\, da_1$$

$$da_2^* = {}^2\left(\frac{ds}{ds_0}\right) da_2 = \sqrt{C_2}\, da_2$$

$$da_3^* = {}^3\left(\frac{ds}{ds_0}\right) da_2 = \sqrt{C_3}\, da_3$$

THUS THE NEW DIFFERENTIAL VOLUME CAN BE EXPRESSED AS

$$\begin{aligned} dV &= da_1^*\, da_2^*\, da_3^* \\ &= (\sqrt{C_1}\, da_1)(\sqrt{C_2}\, da_2)(\sqrt{C_3}\, da_3) \\ &= (C_1 C_2 C_3)^{1/2}\, da_1\, da_2\, da_3 \\ &= (C_1 C_2 C_3)^{1/2}\, dV_0 \end{aligned}$$

AND

$$\frac{dV}{dV_0} = (C_1 C_2 C_3)^{1/2}$$

HOWEVER, IN TERMS OF THE PRINCIPLE VALUES OF C_{ij}, THE THIRD INVARIANT OF THE RIGHT GREEN TENSOR IS

$$\tilde{C}_3^{\gamma} = C_1 C_2 C_3$$

BY THE SAME TOKEN

$$\tilde{C}_1^{\gamma} = C_1 + C_2 + C_3$$

$$\tilde{C}_2^{\gamma} = C_1 C_2 + C_2 C_3 + C_1 C_3$$

SO WE STATE THE RATIO OF DIFFERENTIAL VOLUMES AS

$$\frac{dV}{dV_o} = \left(\tilde{C}_3\right)^{1/2}$$

THUS THE JACOBIAN IS EQUAL TO THE SQUARE ROOT OF THE THIRD INVARIANT OF THE RIGHT HAND CAUCHY GREEN STRAIN TENSOR, I.E.,

$$J = \frac{dV}{dV_o}$$

$$= \left(\tilde{C}_3\right)^{1/2}$$

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PRINCIPAL STRAINS AND INVARIANTS OF THE LINEAR STRAIN TENSOR

EVEN THOUGH THE NON-LINEAR STRAIN TENSORS POSSESS SIMILAR PROPERTIES THAT ARE DISCUSSED HERE, THE PROPERTIES OF THE LINEAR STRAIN TENSORS ARE MORE USEFUL.

JUST AS σ_{ij} POSSESSES ONE PRINCIPAL SET OF AXES AND PRINCIPAL VALUES, THE LINEAR STRAIN TENSORS E_{ij} AND e_{ij} ALSO POSSESS A SET OF PRINCIPAL AXES AND PRINCIPAL STRAINS. IF THESE PRINCIPAL AXES ARE TAKEN AS THE COORDINATE AXES, THE MATRIX OF THE STRAIN TENSOR ASSUMES DIAGONAL FORM.

FOLLOWING THE SAME PROCEDURE USED IN DEFINING PRINCIPAL DIRECTIONS FOR THE CAUCHY STRESS TENSOR, LET US DEFINE A PRINCIPAL DIRECTION FOR THE STRAIN TENSOR AS AN ORIENTATION OF A LINE SEGMENT FOR WHICH THE LINE SEGMENT RETAINS THE PROPERTY OF PERPENDICULARITY TO THE ORIGINAL PERPENDICULAR PLANE. BY DEFINITION, FOR SUCH AN ORIENTATION, THE FOLLOWING EQUATION MUST BE SATISFIED

$$E_i = E n_i$$

$$= E \delta_{ij} n_j$$

WHERE

$$E_i = \frac{du_i}{dS_o}$$

$$E_i = \left(\frac{1}{dS_o}\right) E_{ij}\, da_j$$ } EQUIVALENT TO THE CAUCHY RELATIONSHIP FOR STRESSES, I.E., $T_i = \sigma_{ij} n_j$

BUT

$$\frac{da_j}{dS_o} = n_j$$

SINCE dS_o IS THE MAGNITUDE OF da_j.

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THUS

$$E\,\delta_{ij}\,n_j = E_{ij}\,n_j$$

$$(E_{ij} - E\,\delta_{ij})\,n_j = \tilde{0}$$

FOR A NON-TRIVIAL SOLUTION

$$\mathrm{DET}\,(E_{ij} - E\,\delta_{ij}) = 0$$

WHICH YIELDS THE FOLLOWING CHARACTERISTIC POLYNOMIAL

$$-E^3 + \tilde{I}_1 E^2 - \tilde{I}_2 E + \tilde{I}_3 = 0$$

WHERE

$$\tilde{I}_1 = E_{ii}$$

$$\tilde{I}_2 = (1/2)(E_{ii}E_{jj} - E_{ij}E_{ij})$$

$$\tilde{I}_3 = (1/6)(E_{ii}E_{jj}E_{kk} - 3E_{ii}E_{jk}E_{jk} + 2E_{ij}E_{jk}E_{ki})$$

AND THE ROOTS OF THE CHARACTERISTIC POLYNOMIAL ARE THE PRINCIPAL STRAINS E_1, E_2 AND E_3.

A SIMILAR DEVELOPMENT CAN BE CONSTRUCTED FOR THE LINEAR EULERIAN STRAIN TENSOR.

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LINEAR DEVIATORIC STRAIN AND CUBICAL DILATATION

IT IS POSSIBLE AND DESIRABLE TO SEPARATE THE LINEAR STRAIN TENSORS INTO DEVIATORIC AND SPHERICAL PARTS AS FOLLOWS

$$E_{ij} = \Sigma_{ij} + (1/3)\,\delta_{ij} E_{KK}$$

WHERE Σ_{ij} IS THE LINEAR LAGRANGIAN DEVIATORIC STRAIN TENSOR. NOTE THAT BY DEFINITION

$$\Sigma_{ii} \equiv 0$$

NOW DEFINE CUBICAL DILATATION IN THE LAGRANGIAN DESCRIPTION AS

$$\begin{aligned} \tilde{D} &= \text{CHANGE IN VOLUME PER UNIT VOLUME} \\ &= \frac{\Delta V}{V} \end{aligned}$$

IN TERMS OF THE LINEAR LAGRANGIAN STRAIN TENSOR

$$\begin{aligned}
\tilde{D} &= \frac{da_1 [1 + E_{11}]\, da_2 [1 + E_{22}]\, da_3 [1 + E_{33}] - da_1\, da_2\, da_3}{da_1\, da_2\, da_3} \\
&= (1+E_{11})(1+E_{22})(1+E_{33}) - 1 \\
&= (1 + E_{22} + E_{11} + E_{11}E_{22})(1 + E_{33}) - 1 \\
&= 1 + E_{11} + E_{22} + E_{33} + E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} + E_{11}E_{22}E_{33} - 1 \\
&= E_{11} + E_{22} + E_{33} + E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} + E_{11}E_{22}E_{33}
\end{aligned}$$

UNDER THE ASSUMPTION OF SMALL STRAIN THEORY THE PRODUCT TERMS ARE NOW DROPPED. THUS

$$\tilde{D} = E_{11} + E_{22} + E_{33} = E_{KK}$$

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SUBSTITUTION OF $\tilde{D}$ INTO E_{ij} YIELDS

$$E_{ij} = \Sigma_{ij} + (1/3)\,\delta_{ij}\,\tilde{D}$$

THUS THE PORTION OF THE STRAIN TENSOR WE EARLIER IDENTIFIED AS THE SPHERICAL PART REALLY REPRESENTS THE VOLUME CHANGE OF A DIFFERENTIAL CUBE. THIS IMPLIES THE DEVIATORIC STRAIN TENSOR REPRESENTS DISTORTION OR THE SHEARING COMPONENT OF THE STRAIN TENSOR, I.E.,

$$E_{ij} = (\text{DISTORTION COMPONENT}) + (\text{VOLUMETRIC CHANGE COMPONENT})$$

USING THE EULERIAN DESCRIPTION, THE CUBICAL DILATATION WOULD BE GIVEN AS

$$D = e_{ii}$$

AS THE DISPLACEMENT GRADIENTS ARE ASSUMED TO BE SMALL THE LINEAR CUBICAL DILATATION GIVEN BY BOTH THE LAGRANGIAN AND EULERIAN DESCRIPTION ARE IDENTICAL

$$\begin{aligned}\tilde{D} &= E_{ii}\\ &= e_{ii}\\ &= D\end{aligned}$$

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INTRODUCTION TO THE COMPATIBILITY EQUATIONS

RETURNING TO THE LINEAR STRAIN-DISPLACEMENT RELATIONSHIP WHERE

$$E_{ij} = \left(\frac{1}{2}\right)\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$

OR USING COMMAS TO REPRESENT PARTIAL DERIVATIVES WITH RESPECT TO SPATIAL COORDINATES

$$E_{ij} = 1/2\,(u_{i,j} + u_{j,i})$$

IF THE DISPLACEMENT IS SPECIFIED, WE CAN READILY COMPUTE THE STRAIN TENSOR BY SUBSTITUTING u_i IN THE ABOVE EXPRESSION. THE INVERSE PROBLEM OF FINDING THE DISPLACEMENT FROM THE STRAIN TENSOR IS NOT SO SIMPLE. HERE THE DISPLACEMENT u_i, CONSISTING OF THREE FUNCTIONAL COMPONENTS, MUST BE DETERMINED BY INTEGRATION OF SIX PARTIAL DIFFERENTIAL EQUATIONS. IN ORDER TO INSURE SINGLE-VALUED, CONTINUOUS SOLUTIONS FOR THE COMPONENTS OF u_i, WE MUST IMPOSE CERTAIN RESTRICTIONS ON E_{ij}. THE RESTRICTIONS WE WILL REACH IN RENDERING u_i SINGLE VALUED AND CONTINUOUS ARE KNOWN AS THE COMPATABILITY EQUATIONS.

WE SHALL FIRST SET FORTH THE NECESSARY CONDITIONS ON E_{ij} FOR SINGLE-VALUEDNESS AND CONTINUITY OF u_i. NEXT WE SHOW THAT THESE NECESSARY CONDITIONS ARE ALSO SUFFICIENT CONDITIONS. TO DO THIS, FORM THE FOLLOWING SPATIAL DERIVATIVES OF E_{ij}

$$E_{ij,kl} = 1/2\,(u_{i,jkl} + u_{j,ikl})$$

$$E_{kl,ij} = 1/2\,(u_{k,lij} + u_{l,kij})$$

$$E_{lj,ki} = 1/2\,(u_{l,jki} + u_{j,lki})$$

$$E_{ki,lj} = 1/2\,(u_{k,ijl} + u_{i,kjl})$$

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BY ADDING THE FIRST TWO EQUATIONS AND THEN SUBTRACTING THE LAST TWO EQUATIONS YIELDS

$$E_{ij,kl} + E_{kl,ij} - E_{lj,ki} - E_{ki,lj}$$

$$= \tfrac{1}{2}\left(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij} - u_{l,jki} - u_{j,lki} - u_{k,ijl} - u_{i,kjl} \right)$$

NOTE THAT

$$u_{i,jkl} = \frac{\partial^3 u_i}{\partial a_j\, \partial a_k\, \partial a_l} = \frac{\partial^3 u_i}{\partial a_k\, \partial a_j\, \partial a_l} = u_{i,kjl}$$

THUS

$$E_{ij,kl} + E_{kl,ij} - E_{lk,ki} - E_{ki,lj}$$

$$= \tfrac{1}{2}\left[\left(u_{j,ikl} - u_{j,lki}\right) + \left(u_{k,lij} - u_{k,ijl} \right) + \left(u_{i,jkl} - u_{i,kjl}\right) + \left(u_{l,kij} - u_{l,jki}\right) \right]$$

AND

$$E_{ij,kl} + E_{kl,ij} - E_{lj,ki} - E_{ki,lj} = [0]$$

EACH TERM OF THIS TENSOR EXPRESSION REPRESENTS A FOURTH ORDER TENSOR. AS A RESULT THIS TENSOR EQUATION ALSO REPRESENTS

$$(3)^4 = 81$$

SCALAR EQUATIONS KNOWN AS THE COMPATABILITY EQUATIONS WHICH THE COMPONENTS OF OF A STRAIN TENSOR MUST SATISFY IF THEY ARE TO BE RELATED TO THE COMPONENTS OF THE DISPLACEMENT VECTOR.

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ACTUALLY ONLY SIX OF THE EIGHTY-ONE EQUATIONS OF COMPATABILITY ARE INDEPENDENT. THE REST ARE EITHER IDENTITIES OR REPETITIONS DUE TO THE SYMMETRY OF E_{ij}. THE SIX INDEPENDENT EQUATIONS ARE AS FOLLOWS

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2}$$

$$\frac{\partial^2 E_{11}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_1^2} = 2\frac{\partial^2 E_{13}}{\partial X_1 \partial X_3}$$

$$\frac{\partial^2 E_{22}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 2\frac{\partial^2 E_{23}}{\partial X_2 \partial X_3}$$

$$\frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = -\frac{\partial^2 E_{23}}{\partial X_1^2} + \frac{\partial^2 E_{12}}{\partial X_1 \partial X_3} + \frac{\partial^2 E_{13}}{\partial X_1 \partial X_2}$$

$$\frac{\partial^2 E_{22}}{\partial X_1 \partial X_3} = -\frac{\partial^2 E_{13}}{\partial X_2^2} + \frac{\partial^2 E_{12}}{\partial X_2 \partial X_3} + \frac{\partial^2 E_{23}}{\partial X_2 \partial X_1}$$

$$\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2} = -\frac{\partial^2 E_{12}}{\partial X_3^2} + \frac{\partial^2 E_{13}}{\partial X_3 \partial X_2} + \frac{\partial^2 E_{23}}{\partial X_3 \partial X_1}$$

THESE EQUATIONS REPRESENT THE NECESSARY CONDITIONS. WE SHALL NOW CONSIDER WHETHER THESE CONDITIONS ARE SUFFICIENT FOR GENERATING SINGLE-VALUED CONTINUOUS COMPONENTS FOR THE DISPLACEMENT VECTOR u_i.

SUFFICIENT CONDITIONS FOR COMPATIBILITY

LET $P^\circ(X_i)$ BE A POINT AT WHICH DISPLACEMENT u_i° AND ROTATION TENSOR Ω_{ij} ARE KNOWN. THE DISPLACEMENT OF ANY OTHER POINT P^* IN THE CONTINUUM CAN BE REPRESENTED BY A LINE INTEGRAL ALONG A CONTINUOUS CURVE C FROM P° TO P^* IN THE FOLLOWING MANNER

$$u_i^* = u_i^\circ + \int_{P^\circ}^{P^*} du_i$$

$$= u_i^\circ + \int_{a_j^\circ}^{a_j^*} \left(\frac{\partial u_i}{\partial a_j}\right) da_j$$

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AT THIS POINT WE STIPULATE THAT THE REGION OCCUPIED BY THE CONTINUUM IS A SIMPLY CONNECTED REGION, I.E., A REGION FOR WHICH EACH AND EVERY CURVE CAN BE SHRUNK TO A POINT WITHOUT CUTTING A BOUNDARY.

NOW INTRODUCE THE ROTATION TENSOR Ω_{ij}

$$u_i^* = u_i^\circ + \int_{a_j^\circ}^{a_j^*} (E_{ij} + \Omega_{ij})\, da_j$$

$$= u_i^\circ + \int_{a_j^\circ}^{a_j^*} E_{ij}\, da_j + \int_{a_j^\circ}^{a_j^*} \Omega_{ij}\, da_j$$

WITH

$$P_j = a_j - a_j^* \qquad a_j^* \rightarrow \text{FIXED REFERENCE VECTOR}$$

$$dP_j = d(a_j - a_j^*)$$

$$= da_j - d(a_j^*)$$

$$= da_j - \bar{0}$$

NEXT, CHANGE THE INTEGRATION VARIABLE SUCH THAT

$$\int_{a_j^\circ}^{a_j^*} \Omega_{ij}\, da_j = \int_{a_j^\circ - a_j^*}^{0} \Omega_{ij}\, d(a_j - a_j^*)$$

NOW INTEGRATE THIS EXRESSION BY PARTS

$$\int_{a_j^\circ - a_j^*}^{0} \Omega_{ij}\, d(a_j - a_j^*) = \Omega_{ij}\,(a_j - a_j^*)\Big|_{a_j^\circ - a_j^*}^{0} - \int_{a_j^\circ - a_j^*}^{0} (a_j - a_j^*)\, d\Omega_{ij}$$

$$= \Omega_{ij}\left\{ -a_j^* - (a_j^\circ - a_j^* - a_j^*)\right\} - \int_{a^\circ - a^*}^{0} (a_j - a_j^*)\left(\frac{\partial \Omega_{ij}}{\partial a_k}\right) da_k$$

$$= \Omega_{ij}\,(a_j^* - a_j^\circ) - \int_{a_k^\circ - a_k^*}^{0} (a_j - a_j^*)\left(\frac{\partial \Omega_{ij}}{\partial a_k}\right) da_k$$

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BUT

$$\frac{\partial \Omega_{ij}}{\partial a_k} = \frac{1}{2}\left[\frac{\partial^2 u_i}{\partial a_k \partial a_j} - \frac{\partial^2 u_j}{\partial a_k \partial a_i}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 u_i}{\partial a_k \partial a_j} + \frac{\partial^2 u_k}{\partial a_i \partial a_j}\right] - \frac{1}{2}\left[\frac{\partial^2 u_j}{\partial a_k \partial a_i} + \frac{\partial^2 u_k}{\partial a_i \partial a_j}\right]$$

$$= \frac{\partial E_{ik}}{\partial a_j} - \frac{\partial E_{jk}}{\partial a_i}$$

THUS

$$u_i^* = u_i^\circ + \int_{a_j^\circ}^{a_j^*} E_{ij}\, da_j + \Omega_{ij}(a_j^* - a_j^\circ)$$

$$- \int_{a_k^\circ - a_k^*}^{0} (a_j - a_j^*)\left[\frac{\partial E_{ik}}{\partial a_j} - \frac{\partial E_{jk}}{\partial a_i}\right] da_k$$

BUT

$$\int_{a_j^\circ}^{a_j^*} E_{ij}\, da_k = \int_{a_j^\circ - a_j^*}^{0} E_{ij}\, d(a_j - a_j^*)$$

$$= \int_{a_j^\circ - a_j^*}^{0} E_{ij}\, da_j$$

$$= \int_{a_k^\circ - a_k^*}^{0} E_{ik}\, da_k$$

THUS

$$u_i^* = u_i^\circ + \Omega_{ij}(a_j^* - a_j^\circ)$$

$$+ \int_{a_k^\circ - a_k^*}^{0} \left\{ E_{ik} - (a_j - a_j^*)\left[\frac{\partial E_{ik}}{\partial a_j} - \frac{\partial E_{jk}}{\partial a_i}\right]\right\} da_k$$

$$= u_i^\circ + \Omega_{ij}(a_j^* - a_j^\circ) + \int_{a_k^\circ - a_k^*}^{0} Q_{ik}\, da_k$$

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WHERE

$$Q_{ik} = E_{ik} - (a_j - a_j^*)\left[\frac{\partial E_{ik}}{\partial a_j} - \frac{\partial E_{jk}}{\partial a_i}\right]$$

IF u_i^* IS TO BE SINGLE-VALUED AND CONTINUOUS, THE INTEGRAL INVOLVING Q_{ik} MUST BE PATH INDEPENDENT. THIS IN TURN STIPULATES THAT $(Q_{ik}\, da_k)$ MUST BE AN EXACT DIFFERENTIAL. IN A SIMPLY CONNECTED DOMAIN, THE NECESSARY AND SUFFICIENT CONDITION FOR $(Q_{ik}\, da_k)$ TO BE AN EXACT DIFFERENTIAL IS THAT

$$\frac{\partial Q_{ik}}{\partial a_l} - \frac{\partial Q_{il}}{\partial a_k} = 0$$

THUS

$$E_{ik,l} - a_{j,l}(E_{ik,j} - E_{jk,i}) - (a_j - a_j^*)(E_{ik,jl} - E_{jk,il})$$
$$- E_{il,k} + a_{j,k}(E_{il,j} - E_{jl,i}) + (a_j - a_j^*)(E_{il,jk} - E_{jl,ik}) = [0]$$

$$E_{ik,l} - E_{il,k} - \delta_{jl}(E_{ik,j} - E_{jk,i}) + \delta_{jk}(E_{il,j} - E_{jl,i})$$
$$+ (a_j - a_j^*)(E_{il,jk} - E_{jl,ik} - E_{ik,jl} + E_{jk,il}) = [0]$$

$$E_{ik,l} - E_{il,k} - E_{ik,l} + E_{lk,i} + E_{il,k} - E_{kl,i}$$
$$+ (a_j - a_j^*)(E_{il,jk} - E_{jl,ik} - E_{ik,jl} + E_{jk,il}) = [0]$$

THE FIRST SIX TERMS CANCEL WITH EACH OTHER, AND FOR FINITE $(a_j - a_j^*)$

$$E_{il,jk} + E_{jk,il} - E_{jl,ik} - E_{ik,jl} = [0]$$

THUS THE COMPATABILITY EQUATIONS ARE NOT ONLY NECESSARY BUT ALSO SUFFICIENT.

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PRAGER OFFERED THE FOLLOWING EXAMPLE OF HOW VELOCITIES ARE REPORTED WITH RESPECT TO LAGRANGIAN AND EULERIAN DESCRIPTIONS

> THE ONE-DIMENSIONAL EXAMPLE OF THE FLOW OF VEHICLES ON A ONE-WAY STREET, ON WHICH PASSING IS FORBIDDEN, MAY SERVE TO ILLUSTRATE THE DIFFERENCE BETWEEN THESE TWO WAYS OF DESCRIBING THE SAME MOTION. THE EULERIAN DESCRIPTION CORRESPONDS TO THE OBSERVATIONS OF TRAFFIC POLICEMEN, WHO REPORT ON THE VELOCITIES WITH WHICH VEHICLES PASS THEIR (I.E, THE POLICEMEN) FIXED OBSERVATION STATIONS. THE LAGRANGIAN DESCRIPTION CORRESPONDS TO THE OBSERVATIONS OF DRIVERS, WHO REPORT ON THEIR PROGRESS ALONG THE STREET.

THUS THE EULERIAN OBSERVOR REPORTS INFORMATION AS PARTICLES PASS A SPECIFIC REGION IN SPACE. THE LAGRANGIAN OBSERVOR IS ATTACHED TO INDIVIDUAL PARTICLES AND REPORTS INFORMATION CONSTANTLY AT ANY ARBITRARY REGION OF SPACE.

REMOVED FROM NOTES