Closed-Range Composition Operators on $A^2$ and the Bloch space

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Abstract. For any analytic self-map $\varphi$ of $\{z : |z| < 1\}$ we give four separate conditions, each of which is necessary and sufficient for the composition operator $C_\varphi$ to be closed-range on the Bloch space $B$. Among these conditions are some that appear in the literature, where we provide new proofs. We further show that if $C_\varphi$ is closed-range on the Bergman space $A^2$, then it is closed-range on $B$, but that the converse of this fails with a vengeance. Our analysis involves an extension of the Julia-Carathéodory Theorem.

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1. Preliminaries

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$. We let $A$ denote two-dimensional Lebesgue measure on $\mathbb{D}$. The Bergman space $A^2$ is the collection of functions $f$ that are analytic in $\mathbb{D}$ such that

$$||f||_{A^2}^2 := \int_{\mathbb{D}} |f|^2 \, dA < \infty.$$ 

As a closed subspace of $L^2(A)$, $A^2$ forms a Hilbert space with respect to the inner product $< f, g > := \int_{\mathbb{D}} fg \, dA$. The Bloch space $B$ is the collection of functions $f$ that are analytic in $\mathbb{D}$ such that

$$||f||_{B} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$ 

Now $|| \cdot ||_B$ defines a norm on $B$, and under this norm $B$ forms a Banach space. Moreover, $||f||_{A^2} \leq 3||f||_B$ for any function $f$ that is analytic in $\mathbb{D}$, and hence $B \subseteq A^2$. A function $\varphi$ that is analytic in $\mathbb{D}$ and that satisfies $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ is called an analytic self-map of $\mathbb{D}$. Analytic automorphisms of $\mathbb{D}$ are Möbius transformations of the form $z \mapsto c \frac{\alpha - z}{1 - \alpha z}$, where $c$ is some unimodular constant and $\alpha$ is some point in $\mathbb{D}$; we let $\varphi_\alpha(z) = \frac{\alpha - z}{1 - \alpha z}$. The so-called pseudohyperbolic
metric on $\mathbb{D}$ is given by $\rho(z,w) = |\varphi_w(z)|$; and is indeed a metric. For any $z$ in $\mathbb{D}$ and any $r$, $0 < r < 1$, we let $D(z,r)$ denote the pseudohyperbolic disk of radius $r$ about $z$, namely, $\{w \in \mathbb{D} : \rho(z,w) < r\}$. Now if $\varphi$ is an analytic self-map of $\mathbb{D}$, then the composition operator $C_{\varphi}$, given by $C_{\varphi}(f) := f \circ \varphi$, is a bounded operator on both $A^2$ and $B$. This result for the Bloch space is a simple consequence of the Schwarz-Pick Lemma (cf., [7], page 2), and for the Bergman space case one may consult [13], page 17. Moreover, if $\varphi$ is not constant, then $C_{\varphi}$ is one-to-one on these spaces and hence, by the Open Mapping Theorem, is closed-range if and only if it is bounded below. For any analytic self-map $\varphi$ of $\mathbb{D}$, define $\tau_{\varphi}$ on $\mathbb{D}$ by

$$\tau_{\varphi}(z) := \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}.$$ 

For $\varepsilon > 0$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_{\varphi}(z)| > \varepsilon\}$ and let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. We say that $F_\varepsilon$ satisfies the reverse Carleson condition if there exist $s$ and $c$, $0 < s$, $c < 1$, such that

$$A(F_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all $z$ in $\mathbb{D}$; cf., [10] for seminal work regarding this condition. It has been shown that $C_{\varphi}$ is closed-range on $B$ if and only if there exists $\varepsilon > 0$ such that $F_\varepsilon$ satisfies the reverse Carleson condition; cf., [9] and [3]. In fact, in [3] it is shown that, what appears to be a weaker condition than the one stated above, is indeed equivalent. To be specific, if there exists $\varepsilon > 0$ and $s$, $0 < s < 1$, such that $F_\varepsilon \cap D(z,s) \neq \emptyset$ for all $z$ in $\mathbb{D}$, then $C_{\varphi}$ is closed-range on $B$. One of the first results of this paper adds one more equivalent condition to this list, and we give a brief and rather novel proof that each of the three conditions are equivalent to $C_{\varphi}$ being closed-range on $B$; see Theorem 2.2. We then turn to connections between the Bloch and Bergman space settings. In the Bergman space setting there is an analogue of $\Lambda_\varepsilon$ that takes center stage. Indeed, if $\varphi$ is an analytic self-map of $\mathbb{D}$ and $\varepsilon > 0$, then we let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and let $G_\varepsilon = \varphi(\Omega_\varepsilon)$. In [1] it is shown that $C_{\varphi}$ is closed-range on $A^2$ if and only if there exists $\varepsilon > 0$ such that $G_\varepsilon$ satisfies the reverse Carleson condition; that is, there exist $s$ and $c$, $0 < s$, $c < 1$, such that

$$A(G_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all $z$ in $\mathbb{D}$. Here, we establish an extension of the Julia-Carathéodory Theorem (see Theorem 3.4) and use it to show that if $C_{\varphi}$ is closed-range on $A^2$, then there exist $\varepsilon$ and $s$, $0 < \varepsilon$, $s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$; see Theorem 3.5. From this we easily have the implication that if $C_{\varphi}$ is closed-range on $A^2$, then it is also closed-range on $B$; see Corollary 3.6. We also (by examples) show that the converse of Corollary 3.6 fails, without remedy. Indeed, we construct a thin Blaschke product that fixes zero and that has no angular derivative anywhere on $\mathbb{T}$, whence $C_B$ is norm preserving on $B$ and yet is compact on $A^2$; see Example 3.8. And we also construct a univalent analytic self-map $h$ of $\mathbb{D}$ that has no unimodular nontangential boundary values on $\mathbb{T}$, and thus has no angular derivative anywhere on $\mathbb{T}$ (whence, $C_h$
is compact on $\mathbb{A}^2)$, such that $C_\kappa$ is closed-range on $B$; see Example 3.10. We close the paper with a result that follows easily from work done in [1] and a remark concerning Fredholm operators; see Section 4.

2. Regarding the Bloch space

Recall that, for any analytic self-map $\varphi$ of $\mathbb{D}$ and any $\varepsilon > 0$,

$$\tau_\varphi(z) := \frac{(1-|z|^2)^2}{1-|\varphi(z)|^2},$$

and

$$\Lambda_\varepsilon := \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}.$$

Lemma 2.1. For any $\varepsilon > 0$ there exist $r$ and $s$, $0 < r, s < 1$, such that if $z \in \Lambda_\varepsilon$, then

\begin{enumerate}[(i)]
  \item $D(z, r) \subseteq \Lambda_{\frac{\varepsilon}{2}}$,
  \item $\varphi$ is univalent in $D(z, r)$ and
  \item $D(\varphi(z), s) \subseteq \varphi(D(z, r))$.
\end{enumerate}

Proof. (i) By [8], $\tau_\varphi$ is Lipschitz with respect to the pseudohyperbolic metric. Indeed, there is a positive constant $c$, independent of $\varphi$ and of $z$ and $w$ in $\mathbb{D}$, such that

$$|\tau_\varphi(z) - \tau_\varphi(w)| \leq c|z - w|.$$

Let $r = \frac{\varepsilon}{2c}$ and suppose that $|\tau_\varphi(w)| \leq \frac{\varepsilon}{2}$. Then, for $z$ in $\Lambda_\varepsilon$,

$$\frac{\varepsilon}{2} < |\tau_\varphi(z)| - |\tau_\varphi(w)| \leq |\tau_\varphi(z) - \tau_\varphi(w)| \leq c|z - w|.$$

Therefore, if $z \in \Lambda_\varepsilon$ and $\rho(z, w) < \frac{\varepsilon}{2c}$, then $w \in \Lambda_{\frac{\varepsilon}{2}}$.

(ii) Suppose that $a \in \Lambda_\varepsilon$ and $\alpha := \varphi(a)$. Notice that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is an analytic self-map of the unit disk that maps 0 to 0 and that

$$|(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)| = |\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}(0)| = |\tau_{\varphi \circ \varphi_a}(0)| = |\tau_\varphi(a)| > \varepsilon.$$

We argue that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is univalent in $\{z : |z| < r\}$; where, as in (i), $r := \frac{\varepsilon}{2c}$. Multiplying $\varphi_\alpha \circ \varphi \circ \varphi_a$ by an appropriate unimodular constant we may assume that $(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)$ is a positive real number (greater than $\varepsilon$). And using the facts that $\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}$ is Lipschitz with respect to the pseudohyperbolic metric, with the same Lipschitz constant $c$, and that $\varphi_\alpha \circ \varphi \circ \varphi_a$ maps 0 to 0, we find that

$$(2.1.1) \quad \Re((\varphi_\alpha \circ \varphi \circ \varphi_a)'(z)) > \frac{\varepsilon}{2},$$

whenever $|z| < r$. Now let $z$ and $w$ be distinct points both of which have modulus less than $r$, and define $\gamma$ on $[0, 1]$ by $\gamma(t) = (1 - t)z + tw$. Then, by (2.1.1),

$$0 \neq (w - z) \cdot \int_0^1 (\varphi_\alpha \circ \varphi \circ \varphi_a)'(\gamma(t))dt = (\varphi_\alpha \circ \varphi \circ \varphi_a)(w) - (\varphi_\alpha \circ \varphi \circ \varphi_a)(z),$$

and hence $\varphi_\alpha \circ \varphi \circ \varphi_a$ is univalent in $\{z : |z| < r\}$. It now follows that $\varphi$ is univalent in $D(a, r)$. 

(iii) Given the terminology of part (ii), \(h(z) := \frac{1}{r_\varepsilon} (\varphi_\alpha \circ \varphi \circ \varphi_\alpha)(rz)\) is analytic and univalent in \(D\), \(h(0) = 0\) and \(|h'(0)| > 1\). Therefore, by the Koebe One-Quarter Theorem (cf., [13], page 154),

\[ \{z : |z| < \frac{1}{4}\} \subseteq h(D). \]

From this it follows that

\[ \{z : |z| < \frac{r_\varepsilon}{4}\} \subseteq (\varphi_\alpha \circ \varphi \circ \varphi_\alpha)(\{z : |z| < r\}). \]

With \(s := \frac{r_\varepsilon}{4}\) we then find that \(D(\varphi(a), s) \subseteq \varphi(D(a, r))\). \(\square\)

As before, let \(\varphi\) be an analytic self-map of \(\mathbb{D}\), let \(\tau_\varphi(z) = \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}\) and for \(\varepsilon > 0\), let \(\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}\) and let \(F_\varepsilon = \varphi(\Lambda_\varepsilon)\). We now give two conditions, each of which is equivalent to \(C_\varphi\) being closed-range on \(B\); cf., [9] and [3], or Theorem 2.2 below.

(*\#) There exist \(\varepsilon > 0\) and constants \(c\) and \(s\), \(0 < c, s < 1\), such that \(A(F_\varepsilon \cap D(z, s)) \geq cA(D(z, s))\) for all \(z\) in \(\mathbb{D}\).

(*\#) There exist \(\varepsilon > 0\) and \(s\), \(0 < s < 1\), such that \(F_\varepsilon \cap D(z, s) \neq \emptyset\) for all \(z\) in \(\mathbb{D}\).

**Theorem 2.2.** Let \(\varphi\) be an analytic self-map of \(\mathbb{D}\). Then the following are equivalent.

i) \(C_\varphi\) is closed-range on \(B\).

ii) Condition (*) holds.

iii) Condition (#) holds.

iv) There are constants \(r\), \(s\) and \(c\), \(0 < r, s, c < 1\), such that, for any \(w\) in \(\mathbb{D}\), there exists \(z_w\) in \(\mathbb{D}\) with the property: \(\varphi\) is univalent on \(D(z_w, w)\), \(\varphi(D(z_w, s)) \subseteq D(w, r)\) and \(A(\varphi(D(z_w, s))) \geq c(1-|w|^2)^2\).

**Proof.** (i) \(\implies\) (iii). Since any Frostman shift of \(\varphi\) (i.e., \(\varphi_\alpha \circ \varphi\), where \(\alpha \in \mathbb{D}\)) gives rise to a closed-range composition operator on \(B\) if and only if \(\varphi\) does, we may assume that \(\varphi(0) = 0\). Now suppose that (iii) does not hold. Then we can find sequences \(\{r_n\}_{n=1}^\infty\), where \(0 < r_n < 1\) and \(\lim_{n \to \infty} r_n = 1\), and \(\{w_n\}_{n=1}^\infty\) in \(\mathbb{D}\), where \(\lim_{n \to \infty} |w_n| = 1\), such that

\[ \sup \{|\tau_\varphi(z)| : z \in \varphi^{-1}(D(w_n, r_n))\} \to 0, \]

as \(n \to \infty\). Let \(\Delta_n = \varphi^{-1}(D(w_n, r_n))\) and let \(D_n = \mathbb{D} \setminus \Delta_n\); for \(n = 1, 2, 3, \ldots\).

Now

\[ ||\varphi_{w_n} \circ \varphi||_{B/C} = \sup \{(1-|z|^2)|\varphi_{w_n} \circ \varphi'(z)| : z \in \mathbb{D}\} \]

\[ \leq \sup \{(1-\rho^2(w_n, \varphi(z)))|\tau_\varphi(z)| : z \in \mathbb{D}\} \]

\[ \leq \sup \{(1-\rho^2(w_n, \varphi(z)))|\tau_\varphi(z)| : z \in \Delta_n\} \]

\[ + \sup \{(1-\rho^2(w_n, \varphi(z)))|\tau_\varphi(z)| : z \in D_n\} \to 0, \]

as \(n \to \infty\). Yet \(||\varphi_{w_n}||_{B/C} = 1\), for all \(n\). By Theorem 0 of [9] it now
follows that $C_\varphi$ is not closed-range on $B$.

$(iii) \implies (ii)$. We assume $(iii)$, that $(\#)$ holds. Then, by Lemma 2.1, $(\ast)$ holds for $\frac{\varepsilon}{2}$.

$(ii) \implies (i)$. This follows immediately from Proposition 1 and Theorem 1 of [9].

At this point we have established the equivalence of $(i)$, $(ii)$ and $(iii)$.

$(iii) \implies (iv)$. This follows immediately from Lemma 2.1.

$(iv) \implies (iii)$. Assuming $(iv)$,

$$
\int_{D(z_w, s)} |\varphi'(z)|^2 dA(z) \geq c(1 - |w|^2)^2,
$$

and hence

$$
\int_{D(z_w, s)} \frac{|\varphi'(z)|^2}{(1 - |w|^2)^2} dA(z) \geq c.
$$

Thus we can find a positive constant $\varepsilon$, dependent only on $r$ and $s$, such that

$$
\int_{D(z_w, s)} |\tau_\varphi(z)|^2 dA(z) \geq \varepsilon^2 A(D(z_w, s)).
$$

Therefore, $|\tau_\varphi(z)| \geq \varepsilon$ for some $z$ in $D(z_w, s)$, and hence $F_\varepsilon \cap D(w, r) \neq \emptyset$ for each $w$ in $\mathbb{D}$; which gives us $(iii)$. The proof is now complete. \qed

A special case of our next result is given by Theorem 2 of [9]; namely, the case that $\varphi$ is a univalent self-map of $\mathbb{D}$. As is indicated in the proof of Theorem 2.2, if $f \in B$, then $||f||_{B/C} := \sup_{z \in \mathbb{D}} \frac{1}{1 - |z|^2} (1 - |w|^2)|f'(z)|$.

**Corollary 2.3.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi$ is closed-range on $B$ if and only if there exists $\delta > 0$ such that, for all $\alpha$ in $\mathbb{D}$, $||\varphi_\alpha \circ \varphi||_{B/C} \geq \delta$.

**Proof.** We may assume that $\varphi(0) = 0$ here since any Frostman shift of $\varphi$ gives rise to a closed-range composition operator on $B$ if and only if $\varphi$ does, and since the collection of analytic automorphisms of $\mathbb{D}$ forms a group under the operation of composition. Moreover, notice that $||\varphi_\alpha||_{B/C} = 1$ for all $\alpha$ in $\mathbb{D}$.

So, if $C_\varphi$ is closed-range on $B$, then, by Theorem 0 of [9], there exists $\delta > 0$ such that $||\varphi_\alpha \circ \varphi||_{B/C} \geq \delta$ for all $\alpha$ in $\mathbb{D}$. Conversely, suppose that there exists $\delta > 0$ such that $||\varphi_\alpha \circ \varphi||_{B/C} \geq \delta$ for all $\alpha$ in $\mathbb{D}$. Then, by Proposition 2 of [9], $(iii)$ of Theorem 2.2 holds and hence $C_\varphi$ is closed-range on $B$. \qed

### 3. The context of $A^2$ versus that of $B$

Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and, for $\varepsilon > 0$, let $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} > \varepsilon\}$, let $G_\varepsilon = \varphi(\Omega_\varepsilon)$ and let $K = \{z \in \mathbb{D} : |z| > 1\}$. By the Julia-Carathéodory Theorem (cf., [13], page 57), $\varphi$ has an angular derivative at each point $\xi$ in
$K$, which we denote by $\varphi'(\xi)$. Indeed, $\varphi'(\xi) = \zeta \bar{\xi} d$, where $\zeta := \varphi(\xi) := \angle \lim_{z \to \xi} \varphi(z)$ and $d$ is given by

$$d := \liminf_{z \to \xi} \frac{1 - |\varphi(z)|}{1 - |z|} = \liminf_{z \to \xi} \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$  

The Julia-Caratheodory Theorem tells us that $d > 0$. And since $\xi \in K$, $d \leq \frac{1}{\varepsilon}$.

**Proposition 3.1.** Given the terminology of the above discussion, $\varphi$ is continuous on $\overline{\Omega}_\varepsilon$ and $\varphi'$ is continuous on $K$.

**Proof.** The continuity of $\varphi$ on $\overline{\Omega}_\varepsilon$ was established in [1]; see Remark 2.6 in this reference. Now let $\{\xi_n\}_{n=1}^\infty$ be a sequence in $K$ that converges to $\xi_0$ in $K$, and let $d_n = |\varphi'(\xi_n)|$, for $n = 0, 1, 2, \ldots$. Since $\varphi$ is continuous on $K$, the continuity of $\varphi'$ on $K$ will follow if we show that $d_n \to d_0$, as $n \to \infty$. Now by the discussion just prior to this proposition, $\{d_n\}_{n=1}^\infty$ is bounded. And so, passing to a subsequence if necessary, we may assume that $d_n \to d$, as $n \to \infty$. Thus our goal here is to show that $d = d_0$. To this end, by the Julia-Caratheodory Theorem we can find a sequence $\{r_n\}_{n=1}^\infty$ in $(0, 1)$, such that $\lim_{n \to \infty} r_n = 1$ and $|d_n - \frac{1 - |\varphi(r_n \xi_n)|}{1 - r_n}| < \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Hence, $\{r_n \xi_n\}_{n=1}^\infty$ is a sequence in $\mathbb{D}$ that converges to $\xi_0$ and $\{\frac{1 - |\varphi(r_n \xi_n)|}{1 - r_n}\}_{n=1}^\infty$ converges to $d$. Julia’s Theorem (cf., [13], page 63) now tells us that $d = d_0$. \qed

We now set the stage for two subsequent results.

**Discussion 3.2.** For any point $\xi$ in $\mathbb{T}$ and any $\theta$, $0 < \theta < \pi$, we let $S(\xi, \theta)$ denote the interior of closed convex hull of $\{\xi\} \cup \{z : |z| \leq \sin(\theta/2)\}$. We call $S(\xi, \theta)$ the *Stolz region* based at $\xi$ with vertex angle $\theta$. For our purposes here it is sufficient that we keep the vertex angles of our Stolz regions fixed at $\frac{\pi}{2}$, though our arguments carry through for any fixed $\theta$ in the aforementioned range. Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and, for $\varepsilon > 0$, let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} > \varepsilon\}$ and let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. Define $W_\varepsilon$ by

$$W_\varepsilon = \bigcup_{\xi \in K} S(\xi, \frac{\pi}{2}).$$

Suppose that $\{z_n\}_{n=1}^\infty$ is a sequence in $W_\varepsilon$ that converges to a point $\xi_0$ in $K$. So, we can find a sequence $\{\xi_n\}_{n=1}^\infty$ in $K$ such that $z_n \in S(\xi_n, \frac{\pi}{2})$ (for $n = 1, 2, 3, \ldots$) and $\lim_{n \to \infty} \xi_n = \xi_0$. Now

- $\zeta_n := \varphi(\xi_n) := \angle \lim_{z \to \xi_n} \varphi(z)$, and
- $\angle \lim_{z \to \xi_n} \varphi'(z) := \varphi'(\xi_n) = \zeta_n \bar{\xi}_n d_n$ - the angular derivative of $\varphi$ at $\xi_n$, where $d_n := |\varphi'(\xi_n)|$.

By Proposition 3.1, $\varphi'(\xi_n) \to \varphi'(\xi_0) = \zeta_0 \bar{\xi}_0 d_0$, as $n \to \infty$, where $\zeta_0 := \varphi(\xi_0) := \angle \lim_{z \to \xi_0} \varphi(z)$ and $d_0 := |\varphi'(\xi_0)|$. Since $0 < d_0 < \infty$, we can find $M > 1$ such that $\frac{1}{M} \leq d_n \leq M$ for all $n$. 

Lemma 3.3. Assuming the terminology of Discussion 3.2, for any $\varepsilon > 0$, there exist $s$, $0 < s < 1$, and $N$ (in $\mathbb{N}$) such that

$$\left| d_n - \frac{1-|\varphi(z)|}{1-|z|} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Proof. If not, then we can find $d \neq d_0$, a subsequence $\{\xi_{n_k}\}_{k=1}^{\infty}$ of $\{\xi_n\}_{n=1}^{\infty}$ and a sequence $\{z_k\}_{k=1}^{\infty}$ such that

- $z_k' \in S(\xi_{n_k}, \frac{\pi}{2})$ for all $k$,
- $|z_k' - \xi_{n_k}| \rightarrow 0$ and hence $|z_k' - \xi_0| \rightarrow 0$ (as $k \rightarrow \infty$), and
- $(1 - |\varphi(z_k')|)/(1 - |z_k'|) \rightarrow d$, as $k \rightarrow \infty$.

By Julia’s Theorem this would then tell us that

$$d = |\varphi'(\xi_0)| = d_0;$$
a contradiction. \qed

Theorem 3.4. Assuming the terminology of Discussion 3.2, $\varphi'$ is continuous on $\overline{W}_\varepsilon$.

Proof. Our proof here is based on Lemma 3.3 and some observations concerning the proof of the Julia-Carathéodory theorem in [13]. By Proposition 3.1, all we need to show is that, given the hypothesis of Discussion 3.2, $\varphi'(z_n) \rightarrow \varphi'(\xi_0)$, as $n \rightarrow \infty$.

Claim A. For any $\varepsilon > 0$ there exist $s$, $0 < s < 1$, and $N$ (in $\mathbb{N}$) such that

$$\left| \zeta_n \xi_n d_n - \frac{\xi_n \varphi(z)}{\xi_n - z} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

To justify this claim we first observe that, by Lemma 3.3, for any $\eta > 0$, there exist $\sigma$, $0 < \sigma < 1$, and $\nu$ (in $\mathbb{N}$) such that

$$(3.4.1) \quad \left| d_n - \frac{1-|\varphi(z_n)|}{1-r} \right| < \eta \quad \text{and} \quad \left| d_n - \frac{1-|\varphi(z_n)|^2}{1-r^2} \right| < \eta$$

provided $\sigma \leq r < 1$ and $n \geq \nu$. Mimicking the proof of JC (1) $\Longrightarrow$ JC (2) (in Section 4.5 of [13]), for $n \geq \nu$ we carry the discussion to the right half-plane \{ $w : \text{Re}(w) > 0$\}. Let $\varphi_n$ and $\psi_n$ be the Möbius transformations given by $\varphi_n(z) := \frac{z + \xi_n}{\xi_n - z}$ and $\psi_n(z) := \frac{z + \xi_n}{\xi_n - z}$. Define $\Phi_n$ and $\gamma_n$ on \{ $w : \text{Re}(w) > 0$\} by $\Phi_n(w) := (\psi_n \circ \varphi \circ \varphi^{-1})(w)$ and $\gamma_n(w) := \Phi_n(w) - c_n w$, where $c_n := \frac{1}{d_n}$.

Now by (3.4.1), if $n \geq \nu$ and $\sigma \leq r < 1$, then

$$d_n - \eta < \frac{1-|\varphi(z_n)|}{1-r}, \quad \frac{1-|\varphi(z_n)|^2}{1-r^2} < d_n + \eta$$

and hence, by Julia’s Theorem and with $w_{n,r} := \varphi_n(r\xi_n)$ ($= \frac{1+r}{r}$),

$$\frac{1}{d_n} \leq \frac{\text{Re}(\Phi_n(w_{n,r}))}{\text{Re}(w_{n,r})} = \left( \frac{1-|\varphi(z_n)|^2}{1-r^2} \right) \frac{1-|r\xi_n|^2}{|\xi_n - \varphi(z_n)|^2} < \frac{d_n + \eta}{(d_n - \eta)^2},$$
Therefore, if $n$ is sufficiently large (allowing $\eta$ to be sufficiently small), one can force
\[
\frac{\text{Re}(\gamma_n(w_n, \sigma))}{\text{Re}(w_n, \sigma)}
\]
to be less than any prescribed positive real number; and $w_{n, \sigma} = \frac{1 + \sigma}{1 - \sigma}$, which clearly does not vary with $n$. We let $w_{n, \sigma}$ play the role of $w_0$ in the proof of JC (1) $\implies$ JC (2) (in Section 4.5 of [13]). And since the image under $\varphi$ of any compact subset of $\mathbb{D}$ is a compact subset of $\mathbb{D}$, $\{|\gamma_n(w_n, \sigma)|\}_{n=1}^{\infty}$ is bounded. Thus, following through with the argument in [13], we find that, for any $\tau > 0$, there is a positive real number $R$ such that if $w \in \varphi_n(S(\xi_n, \frac{\pi}{2}))$ and $|w| > R$, then
\[
(3.4.2) \quad \left| \frac{\gamma_n(w)}{w} \right| < \tau,
\]
provided $n$ is sufficiently large. Now, via the correspondence $w = \varphi_n(z)$, routine calculations give that
\[
\frac{w + 1}{\Phi_n(w) + 1} = \xi_n \bar{\xi}_n \left( \frac{\xi_n - \varphi(z)}{\xi_n - z} \right),
\]
and hence,
\[
\left| \frac{\gamma_n(w) + 1}{w + 1} \right| = \left| \frac{\xi_n - z}{\xi_n - \varphi(z)} - \frac{\xi_n \bar{\xi}_n w}{w + 1} \right|.
\]
We now find that Claim (A) follows from (3.4.2).

Claim B. For any $\varepsilon > 0$ there exist $s$, $0 < s < 1$, and $N$ (in $\mathbb{N}$) such that
\[
|\xi_n \xi_n d_n - \varphi'(z)| < \varepsilon,
\]
whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Now Claim (B) follows directly from Claim (A) and the proof of JC (2) $\implies$ JC (3) (in Section 4.6 of [13]). And by Claim (B) and the fact that $\varphi'(\xi_n) \to \varphi'(\xi_0)$, as $n \to \infty$, we find that
\[
\varphi'(z_n) \to \varphi'(\xi_0),
\]
as $n \to \infty$; which completes our proof. \hfill \Box

**Theorem 3.5.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. If $C_\varphi$ is closed-range on $\mathbb{A}^2$, then there exist $\varepsilon$ and $s$, $0 < \varepsilon$, $s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$.

**Proof.** Suppose that $C_\varphi$ is closed-range on $\mathbb{A}^2$. Then there exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies the reverse Carleson condition; cf., [1]. In particular, $\mathbb{T} \subseteq G_\varepsilon$. So, for each point $v_0$ in $\mathbb{T}$, we can find a sequence $\{w_n\}_{n=1}^{\infty}$ in $\Omega_\varepsilon$ such that $\{\varphi(w_n)\}_{n=1}^{\infty}$ converges to $v_0$. Passing to a subsequence if necessary, we may assume that $\{w_n\}_{n=1}^{\infty}$ converges to some point $\omega_0$ in $K := \mathbb{T} \cap \overline{\Omega}_\varepsilon$. Therefore, by Julia’s Theorem, $v_0 = \varphi(\omega_0) := \angle \lim_{w \to \omega_0} \varphi(w)$. Thus, $\varphi(K) = \mathbb{T}$. We proceed indirectly and suppose that the conclusion of this theorem fails. Then we can find a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D} \setminus \{0\}$, such that $\{|z_n|\}_{n=1}^{\infty}$ converges to 1 and
\[
(3.5.1) \quad \sup\{|\tau_\varphi(w) : \varphi(w) = z_n\} \to 0,
\]
as \( n \to \infty \). Since \( \varphi(K) = \mathbb{T} \), there exists \( \{\xi_n\}_{n=1}^{\infty} \) in \( K \) such that \( \varphi(\xi_n) = \zeta_n := \frac{z_n}{|z_n|} \), for \( n = 1, 2, 3, \ldots \). Passing to a subsequence if need be, we may assume that \( \{\xi_n\}_{n=1}^{\infty} \) converges to some point \( \xi_0 \) in \( K \). Since, by Proposition 3.1, \( \varphi \) is continuous on \( K \), indeed, continuous on \( \overline{\Omega}_\varepsilon \), we find that \( \{\zeta_n\}_{n=1}^{\infty} \) converges to \( \zeta_0 := \varphi(\xi_0) \). Now, by Theorem 3.4 and its proof, there exist \( \delta \) and \( s \), \( 0 < \delta, s < 1 \), and \( N \) in \( \mathbb{N} \) such that

\[ |\tau_{\varphi}(z)| \geq \delta, \]

whenever \( z \in S(\xi_n, \frac{\pi}{2}) \), \( |z| > s \) and \( n \geq N \). Moreover, by Claim (A) in the proof of Theorem 3.4 (that speaks to the conformality of \( \varphi \) at \( \xi_n \)), we can find \( \sigma, 0 < \sigma < 1 \), and \( \nu \) in \( \mathbb{N} \) such that

\[ \{r\zeta_n : \sigma \leq r < 1\} \subseteq \varphi(\{z \in S(\xi_n, \frac{\pi}{2}) : |z| > s\}), \]

whenever \( n \geq \nu \). Since \( z_n \in \{r\zeta_n : \sigma \leq r < 1\} \), if \( n \) is sufficiently large, we find that (3.5.1) above cannot occur; and our proof is complete. \( \square \)

Our next result is an immediate consequence of Theorem 3.5 and Theorem 2.2; and so we state it without proof.

**Corollary 3.6.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). If \( C_\varphi \) is closed-range on \( \mathbb{A}^2 \), then it is also closed-range on \( \mathbb{B} \).

A slight modification of the proof of Theorem 3.5 gives us the following rather surprising result. It also can be viewed as a byproduct of the nice behavior of \( \varphi \) on \( \overline{W}_\varepsilon \), as indicated by Theorem 3.4.

**Theorem 3.7.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then the following are equivalent.

i) \( C_\varphi \) is closed-range on \( \mathbb{A}^2 \).

ii) There exist \( \varepsilon, s \) and \( c, 0 < \varepsilon, s, c < 1 \), such that

\[ A(G_\varepsilon \cap D(z, s)) \geq cA(D(z, s)), \]

for all \( z \) in \( \mathbb{D} \).

iii) There exist \( \varepsilon \) and \( s \), \( 0 < \varepsilon, s < 1 \), such that \( \{z : s \leq |z| < 1\} \subseteq G_\varepsilon \).

**Proof.** The equivalence between (i) and (ii) was established in [1]. And clearly (iii) implies (ii). So we need only establish that (i) implies (iii). To this end, assume that \( C_\varphi \) is closed-range on \( \mathbb{A}^2 \) and mimic the proof of Theorem 3.5, replacing \( |\tau_{\varphi}(z)| \) by \( \frac{1-|z|^2}{1-|\varphi(z)|^2} \), throughout. The argument carries over with this modification to gives us (iii). \( \square \)

By Theorem 2.5 of [1], the only univalent analytic self-maps of \( \mathbb{D} \) that give rise to closed-range composition operators on \( \mathbb{A}^2 \) are the analytic automorphisms of \( \mathbb{D} \). This is in contrast with the Bloch space setting. Indeed, if \( \psi \) is any
conformal mapping from $\mathbb{D}$ one-to-one and onto $\mathbb{D} \setminus [0, 1)$, then $C_\psi$ is closed-range on $B$; cf., Example 2 of [9]. So, the converse of Corollary 3.6 fails. Our next two examples show that the converse fails with a vengeance. Our first is an example of a thin Blaschke product $B$ that fixes zero and has no angular derivative at any point of the unit circle $\mathbb{T}$; and by thin we mean that $(1 - |a_n|^2)|B'(a_n)| \to 1$, as $n \to \infty$, where $\{a_n\}_{n=1}^\infty$ are the zeros of $B$. Therefore, $C_B$ is norm preserving on $B$ (cf., [5], or [11]) and yet is compact on $A^2$ (cf., [13], pages 52 and 195). And since $C_B$ is compact and not of finite rank on $A^2$, it is not closed-range on $A^2$. This first example is a factor of the one produced by J. Shapiro on page 185 of [13].

**Example 3.8.** Let $B^*$ be the Blaschke product constructed by J. Shapiro on page 185 of [13] and let $\{a_n\}_{n=1}^\infty$ be the zeros of $B^*$. Associated with each $a_n$ is an arc $I_n$ of length $1/n$ of the form $I_n = \{e^{i\theta} : \theta_n \leq \theta \leq \theta_{n+1}\}$. The zeros $a_n$ are given by: $a_n := r_ne^{i\omega_n}$, where $r_n := 1 - \frac{1}{n^2}$ and $\omega_n := \frac{1}{2}(\theta_n + \theta_{n+1})$. A theorem of O. Frostman (cf., [13], page 183) is then used to show that $B^*$ has no angular derivative anywhere on $\mathbb{T}$. For each positive integer $\nu$, we define the $\nu^\text{th}$ “layer” of zeros of $B^*$ as $[a_{\nu}] := \{a_\nu, a_{\nu+1}, ..., a_{N_\nu}\}$, where $N_\nu$ is the unique positive integer that satisfies:

$$\mathbb{T} \subseteq \bigcup_{n=\nu}^{N_\nu} I_n, \text{ yet } \mathbb{T} \not\subseteq \bigcup_{n=\nu}^{N_\nu-1} I_n.$$

Since $\sum_{n=\nu}^{N_\nu-1} \frac{1}{n} < 2\pi$, it follows that $N_\nu < 540\nu$. For any positive integer $\nu$, let $B_\nu$ be the Blaschke product with (simple) zeros $[a_\nu]$. For any $a_k$ in $[a_\nu]$, let $B_\nu^k$ denote $B_\nu$ with the Blaschke factor involving $a_k$ deleted. And choose $a_k$ in $[a_\nu] \setminus \{a_k\}$ such that $\rho(a_k, a_{k^*}) \leq \rho(a_k, a_l)$, whenever $a_l \in [a_\nu] \setminus \{a_k\}$. Then, for such $l$,

$$\left| \frac{a_k - a_l}{1 - a_la_k} \right|^2 - 1 \geq -\frac{(1 - r_k^2)(1 - r_{k^*}^2)}{1 - 2r_kr_{k^*} \cos(\theta_k - \theta_{k^*}) + r_{k^*}^2r_k^2}.$$

Now, $|\theta_k - \theta_{k^*}| \geq \frac{1}{4k}$ and so, for $\nu$ sufficiently large,

$$1 - 2r_kr_{k^*} \cos(\theta_k - \theta_{k^*}) + r_{k^*}^2r_k^2 \geq \frac{1}{20k^2}.$$

Hence,

$$\left| \frac{a_k - a_l}{1 - a_la_k} \right|^2 - 1 \geq -\frac{80}{(k^*)^2} \geq -\frac{80}{\nu^2},$$

independent of $k$ and $l$ in our range here. Therefore,

$$0 > \sum_{k \neq l = \nu}^{N_\nu} \left( \left| \frac{a_k - a_l}{1 - a_la_k} \right|^2 - 1 \right) \geq (540\nu)(-\frac{80}{\nu^2}) = -\frac{43,200}{\nu} \to 0,$$

as $\nu \to \infty$; uniformly in $k, \nu \leq k \leq N_\nu$. From this it follows that

$$(3.7.1) \quad |B_\nu^k(a_k)| \to 1,$$
as \( \nu \to \infty \); uniformly in \( k, \nu \leq k \leq N_\nu \). Now since \( B^* \) is a Blaschke product,

\[
(3.7.2) \quad |B_\nu| \to 1
\]

uniformly on compact subsets of \( \mathbb{D} \), as \( \nu \to \infty \). And since, for any fixed \( \nu \), \( B_\nu \) is a finite Blaschke product,

\[
(3.7.3) \quad |B_\nu(z)| \to 1
\]

uniformly in \( z \), as \( |z| \to 1^- \). Using (3.7.1) – (3.7.3), one can find a (rapidly) increasing sequence \( \{\nu_j\}_{j=1}^{\infty} \) of positive integers such that \([a_{\nu_k}] \cap [a_{\nu_l}] = \emptyset\) if \( k \neq l \), and such that

\[
B := \prod_{j=1}^{\infty} B_{\nu_j},
\]

whose (simple) zeros we enumerate as \( \{\alpha_n\}_{n=1}^{\infty} \), satisfies

\[
|B^n(\alpha_n)| \to 1,
\]

as \( n \to \infty \); where \( B^n \) denotes \( B \) with the Blaschke factor involving \( \alpha_n \) deleted. And we may assume that \( \nu_1 = 1 \). Hence, \( B \) is a thin Blaschke product that fixes zero. Since the zeros of \( B \) consist of infinitely many disjoint layers of the zeros of \( B^* \), one can argue as in [13], page 185, and find that

\[
\sum_{n=1}^{\infty} \frac{1 - |\alpha_n|}{|\zeta - \alpha_n|^2} = \infty,
\]

for each \( \zeta \) in \( \mathbb{T} \). Thus, by a theorem of O. Frostman (cf., [13], page 183), we conclude that \( B \) has no angular derivative at any point in \( \mathbb{T} \).

**Remark 3.9.** The converse of Theorem 3.5 does not hold. Indeed, by Theorem 2.7 of [4], if \( B \) is the Blaschke product that we produced in Example 3.8, then

\[
\mathbb{D} \subseteq F_{1/2}^*;
\]

and yet \( C_B \) is far from closed-range on \( \mathbb{A}^2 \).

We now produce a univalent analytic self-map \( h \) of \( \mathbb{D} \) that has no angular derivative at any point of \( \mathbb{T} \) (whence, \( C_h \) is compact on \( \mathbb{A}^2 \)) such that \( C_h \) is closed-range on \( \mathcal{B} \). This dramatically improves upon our understanding of what is possible in the univalent case; cf., Example 2 of [9]. And since \( h(\mathbb{D}) \) contains no annulus with outer boundary equal to \( \mathbb{T} \) (and similarly for Example 2 of [9]), there is no analogue of Theorem 3.7 in the context of the Bloch space.

**Example 3.10.** Here we construct a conformal mapping \( h \) from \( \mathbb{D} \) one-to-one and onto an infinite ribbon \( G \) that spirals out to \( \mathbb{T} \) such that \( C_h \) is closed-range on \( \mathcal{B} \). So \( h \) will have no unimodular nontangential boundary values on \( \mathbb{T} \), and thus no angular derivative anywhere on \( \mathbb{T} \). We write \( h \) as the composition of three conformal mappings:
Thus, \( \zeta = i \left( \frac{1+i}{1-i} + e \right) \), which maps \( \mathbb{D} \) univalently onto \( G_1 := \{ \zeta : \text{Im}(\zeta) > e \} \),

\( \xi = \log(\zeta) \), which maps \( G_1 \) univalently onto a smoothly bounded subregion \( G_2 \) of the swath \( \{ \xi : \text{Re}(\xi) > 1 \) and \( 0 < \text{Im}(\xi) < \pi \} \) that asymptotically approximates this swath, and

\( w = \xi^i \), which maps \( G_2 \) univalently onto an infinite ribbon \( G \) that spirals out to \( T \).

Thus, \( h(z) = \left[ \log \left( i \left( \frac{1+i}{1-i} + e \right) \right) \right]^i \). Clearly \( h \) has no unimodular nontangential boundary values on \( \mathbb{T} \) and thus has no angular derivative anywhere on \( \mathbb{T} \). As we noted just prior to Example 3.8, this tells us that \( C_h \) is compact and hence not closed-range on \( \mathbb{A}^2 \). One may also refer to Theorem 2.5 of [1] to obtain that \( C_h \) is not closed-range on \( \mathbb{A}^2 \). Now let \( \Gamma = h([0,1]) \), which is an arc of infinite length that spirals out to \( T \). Our strategy in showing that \( C_h \) is closed-range on \( B \) is to first establish that there exists \( \varepsilon > 0 \) such that \( \Gamma \subseteq F_\varepsilon \) and then establish that there exists \( s, 0 < s < 1 \), such that \( \Gamma \cap D(z,s) \neq \emptyset \) for all \( z \) in \( \mathbb{D} \). Theorem 2.2 then gives us the conclusion. In what follows we use the symbol \( \sim \) between real-valued functions \( f \) and \( g \) defined on \([0,1]\) (viz., \( f \sim g \)) to indicate that there is a constant \( M > 1 \) such that \( \frac{1}{M} f(x) \leq g(x) \leq M f(x) \) for all \( x \) in \([0,1]\). Now, for \( x \) in \([0,1]\),

\[
\begin{align*}
h(x) &= \left[ \log \left( i \left( \frac{1+x}{1-x} + e \right) \right) \right]^i \\
&= \left[ \log \left( \frac{1+x}{1-x} + e \right) + \frac{i\pi}{2} \right]^i.
\end{align*}
\]

Denoting \( \log \left( \frac{1+x}{1-x} + e \right) + \frac{i\pi}{2} \) by \( \xi_x \), we have:

\[
h(x) = e^{i\log(\xi_x)} = e^{-\arg(\xi_x)} \cdot e^{i\log|\xi_x|}.
\]

Hence,

\[
1 - |h(x)| \sim \arg(\xi_x) \sim \frac{1}{\log \left( \frac{1+x}{1-x} + e \right)}.
\]

Thus, for \( x \) in \([0,1]\),

\[
\frac{1-x}{1-|h(x)|} \sim (1-x) \log \left( \frac{1+x}{1-x} + e \right).
\]

And, for such \( x \), \( h'(x) = \frac{e^{i\log(\xi_x)}}{\log \left( \frac{1+x}{1-x} + e \right) + \frac{i\pi}{2}} \cdot \frac{2}{(1-x^2) + e(1-x)^2} \); whence

\[
|h'(x)| \sim \frac{1}{(1-x) \log \left( \frac{1+x}{1-x} + e \right)}.
\]

Evidently, \( |\tau_h(x)| \sim 1 \), and so there exists \( \varepsilon > 0 \) such that \( \Gamma \subseteq F_\varepsilon \). Now, as \( x \) increases to 1 in \([0,1]\), \( h(x) \) traverses \( \Gamma \) through infinitely many counterclockwise rotations about 0 as it works its way toward \( T \). To complete our argument here it is important that we obtain a good estimate on the ratio between \( 1 - |h(x')| \) and \( 1 - |h(x)| \), if \([x, x']\) is a subinterval of \([0,1]\) over which \( h \) makes...
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precisely one rotation about 0. Recalling that $h(x) = e^{-\arg(x) \cdot e^i \log |x|}$, we find that this reduces to an examination of

$$h^*(y) := e^{-\frac{1}{y} \cdot e^i \log(y)},$$

as $y$ in $[1, \infty)$ increases to $\infty$. Notice that $h^*$ winds through $2\pi$ radians on any subinterval of $[1, \infty)$ of the form $[y, e^{2\pi}y]$. And, independent of $y$, $1 - \frac{|h^*(y)|}{1 - |h^*(e^{2\pi}y)|}$ is boundedly equivalent to $\frac{1}{e^{2\pi}}$. This then tells us that $\mathbb{D} \setminus \Gamma$ does not contain pseudohyperbolic disks of radius arbitrarily near 1. Hence, there exists $s$, $0 < s < 1$, such that $\Gamma \cap D(z, s) \neq \emptyset$, for all $z$ in $\mathbb{D}$. Since, as we have shown, $\Gamma \subset F_{\varepsilon}$, for some $\varepsilon > 0$, we can now refer to Theorem 2.2 and conclude that $C_h$ is closed-range on $B$.

4. Closing Remarks

In this final section we give a result in the context of $A^2$ for singular inner functions and we point out some implications of our work here to the theory of Fredholm operators. In our discussion we let $m$ denote normalized Lebesgue measure on $\mathbb{T}$. Recall that a compact subset $E$ of $\mathbb{T}$ is said to be porous if there exists $\varepsilon$, $0 < \varepsilon < 1$, such that whenever $I$ is a arc of $\mathbb{T}$ with $I \cap E \neq \emptyset$, then there is a subarc $J$ of $I$ where $m(J) > \varepsilon m(I)$ and $J \cap E = \emptyset$. In [12] it is shown that $E$ is a porous subset of $\mathbb{T}$ if and only if $E$ has the property: For any singular measure $\mu$ supported on $E$, every nontrivial Frostman shift of the singular inner function $S_\mu$ is a Carleson-Newman Blaschke product; that is, a finite product of interpolating Blaschke products. The proof of Corollary 3.11 in [1] also establishes our next result.

**Proposition 4.1.** Let $E$ be a porous subset of $\mathbb{T}$. If $\mu$ is any singular measure with support in $E$, then $C_{S_\mu}$ is closed-range on $A^2$.

**Remark 4.2.** We close the paper with some thoughts concerning Fredholm operators. We first recall that the little Bloch space $B_0$ is the collection of functions $f$ in $B$ for which

$$\lim_{r \to 1^-} \sup_{r < |z| < 1} (1 - |z|^2)|f'(z)| = 0.$$ 

And the Dirichlet space $D$ is the collection of functions $f(z) = \sum_{n=0}^\infty a_n z^n$, analytic in $\mathbb{D}$, such that

$$||f||_D^2 := \sum_{n=0}^\infty (n + 1) |a_n|^2 < \infty.$$ 

An operator between two Banach spaces is called a Fredholm operator if its range is closed and both the operator and its adjoint have finite dimensional kernel. If $\varphi$ is an analytic self-map of $\mathbb{D}$ and $C_\varphi$ is a Fredholm operator on a
Hilbert space of analytic functions that contains $\mathcal{D}$, then $\varphi$ is a disk automorphism; cf., [6], page 153. Now $\mathcal{D} \subseteq \mathcal{B}_0$, but we will show that the situation is different for $\mathcal{B}_0$. Indeed, there exists Fredholm composition operators on $\mathcal{B}_0$ whose symbols are not disk automorphisms. The \textit{minimal Besov space} $\mathcal{B}_1$ is the collection of all functions $f$ that are analytic in $\mathbb{D}$ of the form

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \varphi w_n(z),$$

where $\{w_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$, and $\{a_n\}_{n=1}^{\infty} \in l^1$. The norm on $\mathcal{B}_1$ is given by

$$||f||_{\mathcal{B}_1} := \inf \{ \sum_{n=0}^{\infty} |a_n| : (4.2.1) \text{ holds} \}.$$

Now $\mathcal{B}_1$ is a Banach space with respect to this norm and is invariant under disk automorphisms. Under the pairing $(f,g) = \int_{\mathbb{D}} f'(z) \overline{g'(z)} dA(z)$, the dual of $\mathcal{B}_0$ is $\mathcal{B}_1$ and the dual of $\mathcal{B}_1$ is $\mathcal{B}$; cf., [2]. Notice that, for $g$ in $\mathcal{B}_0$ and $w$ in $\mathbb{D}$,

$$(g, \varphi_w) = - \int_{\mathbb{D}} g'(z) \frac{1-|w|^2}{(1-wz)^2} dA(z) = -(1-|w|^2)g'(w),$$

and therefore,

$$(g, C_{\varphi}^*(\varphi_w)) = <C_{\varphi}(g), \varphi_w> = -(1-|w|^2) (g \circ \varphi)'(w) = -\tau_\varphi(w)(g, \varphi_{\varphi(w)}).$$

If $w \in \mathbb{D}$, then

$$C_{\varphi}^*(\varphi_w) = -\tau_\varphi(w) \varphi_{\varphi(w)},$$

and if $|w| = 1$, then $\varphi_w = w$ and

$$C_{\varphi}^* \varphi_w = 0.$$

By (4.2.2) and (4.2.3) it is easy to see that the kernel of $C_{\varphi}^*: B_1 \to B_1$ consists of the constant functions. Also, a non-constant composition operator is always one-to-one, and therefore $C_{\varphi}: \mathcal{B}_0 \to \mathcal{B}_0$ will be a Fredholm operator if it is closed-range. It is shown in [9] that if $\psi$ is a conformal mapping from $\mathbb{D}$ onto $\mathbb{D} \setminus [0,1)$, then $C_\psi$ is bounded below on $\mathcal{B}$. Any univalent self-map of $\mathbb{D}$ is in $\mathcal{B}_0$, and thus $\psi \in \mathcal{B}_0$ and $C_\psi$ is a Fredholm operator on $\mathcal{B}_0$.

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