

# Tutorial 3

In Tutorial 1 the program Macaulay 2 was introduced. In this tutorial we will create functions in Macaulay 2 to implement the algorithms in Lecture 3. A function in Macaulay 2 has the syntax

*function-name = (names for input(s) separated by commas) ->  
(computations to execute separated by semi-colons)*

Below, for example, is a function `toBinomial` which creates a binomial from an integer vector. The second function `HSAlg` encodes the Hoşten-Sturmfels Algorithm described in Lecture 3.

```
load "LLL.m2" -- load a package for doing LLL reduction
```

```
toBinomial = (b,R) -> (  
  -- take a vector b and a ring R having #b variables  
  -- return binomial x^(b-) - x^(b+)  
  pos := 1_R;  
  neg := 1_R;  
  scan(#b, i-> if b_i > 0 then pos = pos * R_i^(b_i)  
               else if b_i < 0 then neg = neg * R_i^(-b_i));  
  pos - neg);
```

```
HSAlg = (A,w) -> (  
  -- take a list of rows of A  
  -- and take a list of weights for the term order w  
  -- return a reduced GB for I_A wrt w  
  n := #(A_0);  
  R = QQ[x_1..x_n,Degrees=>transpose A,MonomialSize=>16,  
        Weights=>w];  
  B := transpose LLL syz matrix A;  
  J := ideal apply(entries B, b -> toBinomial(b,R));  
  scan(gens ring J, f -> J = saturate(J,f));  
  gens gb J)
```

Notice that the prefix ‘--’ is used to denote the start of a comment; the entire rest of the line is ignored by the program. A list in Macaulay 2 is denoted by curly braces and its elements are separated by commas. In the function `HSAlg` instead of using one of the predefined orders and the command `MonomialOrder=>name-of-order`, we use `Weights=>w` which tells Macaulay 2 to use a weight order determined by the vector  $\mathbf{w}$ . The command `saturate(J,f)` computes  $(J : f^\infty)$ . The command `#L` returns the size of the list  $L$ . More help with commands is available by typing `help name-of-command` in Macaulay 2, or better, by using the html-based help either locally or at <http://www.math.uiuc.edu/Macaulay2/Manual>.

In the program above, we used a monomial order defined by a weight vector. Another useful order is an elimination order. For an ideal  $I \subset k[\mathbf{t}, \mathbf{x}]$ , generators for the ideal  $I \cap k[\mathbf{x}]$  are found by calculating a Gröbner basis with respect to an elimination order for the  $t_i$  variables and then throwing out any generators which contain  $t_i$ 's. In Macaulay 2 including the setting `MonomialOrder => Eliminate n` in the definition of a ring will force the use of an elimination order eliminating the first  $n$  variables and refined by the graded reverse lexicographic order.

1. (a) Use the Conti-Traverso algorithm to find a reduced Gröbner basis  $\mathcal{G}_{\{3,2,1\}}$  of  $I_A$  for  $A = [1 \ 2 \ 3]$ .
- (b) Use Macaulay 2 to implement the Conti-Traverso algorithm by filling in the gaps in the following function.

```
CTAlg = (A,w) -> (
  -- take a list of rows of matrix A
  --   and a weight vector w
  -- return a reduced GB of I_A
  n := #(A_0);
  d := #A;
  R := QQ[t_0..t_d,x_1..x_n, MonomialSize=>16,
        MonomialOrder=>???];
  J := ideal (???);
  I := selectInSubring(1, gens gb J); -- select entries
  -- without vars from 1st 'part' of mon. order
  S := ???; -- the ring I should end up in
  gens gb substitute(I, S))
```

2. (a) Prove that a finite set  $H \subset \{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \text{ such that } \pi(\mathbf{u}) = \pi(\mathbf{v})\}$  is a generating set of  $I_A$  if and only if the vectors  $\mathbf{u} - \mathbf{v}$  from the

binomials in  $H$  form a connected undirected graph on the lattice points of each fiber  $\pi^{-1}(\mathbf{b})$  as  $\mathbf{b}$  varies in  $\mathbb{N}\mathcal{A}$ .

- (b) Prove that  $\mathcal{G}_\succ$  is a reduced Gröbner basis of  $I_{\mathcal{A}}$  if and only if the vectors  $\mathbf{u} - \mathbf{v}$  from the binomials of  $\mathcal{G}_\succ$  form a connected directed graph in each fiber of  $\pi$  with a unique sink at the unique monomial in the fiber with lowest degree under the ordering by  $\succ$ .
3. (a) Write an algorithm using toric ideals to solve the integer program  $\min\{\mathbf{c} \cdot \mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n, \mathbf{x} \geq 0\}$  where  $\mathbf{c}$  is the cost (weight) vector and  $\mathbf{b} \in \mathbb{N}\mathcal{A} = \{A\mathbf{u} \mid \mathbf{u} \in \mathbb{Z}^n, \mathbf{u} \geq 0\}$ .
- (b) Use your algorithm to solve the integer program about currency from the introduction to Lecture 3.
4. The programs 4ti2 and CoCoA both have specialized packages for toric computations.

To use the program 4ti2, create a file *matrix-file* for each  $A$  matrix with the following lines in the file:

line 1: *number-of-rows-in-A number-of-columns-in-A*  
 remaining lines: *entries of row of A, each separated by spaces*

A Gröbner basis may be calculated by running `groebner matrix-file`. This will return a file *matrix-file.gro* in the same format as the input. The output file can be formatted with the command `output`. To list the output as binomials, for example, type `output bin matrix-file.gro`. Caution: you must include the extension *.gro*, or else you will get binomials from the input matrix  $A$ . For further information, see the website <http://www.4ti2.de>.

In CoCoA, `Toric(matrix)` returns the toric ideal corresponding to the matrix. CoCoA also has an integer programming package. Details on using CoCoA may be found at <http://cocoa.dima.unige.it>.

Use one or both of these programs to find a Gröbner basis for the lattice ideal generated by the matrix

$$\begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

For more practice, check your previous calculations using these programs.

5. (a) Show that the result in exercise 2b can be used to enumerate all the elements of a fiber.
- (b) The program CaTS will enumerate all lattice points in a fiber using the command `cats_fiber`. The input file is a list of the columns of  $A$  in the form  $\{(\text{col } 1)(\text{col } 2) \cdots (\text{col } n)\}$  followed by an exponent vector of an element of the fiber in the form  $(d_1, d_2, \dots, d_n)$ . For more information see <http://www.soopadoopa.dk/anders/cats/cats.html>. Use CaTS to find all the lattice points in the fiber necessary to solve the currency problem in the introduction to Lecture 3.
6. Suppose a small local trucking firm has two customers, A and B, that generate shipments to the same location. Each shipment from A is a pallet weighing exactly 400 kilos and taking up 2 cubic meters of volume. Each pallet from B weighs 500 kilos and takes up 3 cubic meters of volume. The shipping firm uses small trucks that can carry any load up to 3700 kilos, and up to 20 cubic meters. B's product is more perishable, though, and they are willing to pay a higher price for on-time delivery: \$15 per pallet versus \$11 per pallet from A. How many pallets from each of the two companies should be included in each truckload to maximize the revenues generated? (*Borrowed from [?].*)
7. As we saw in exercise 4, the program 4ti2 can compute a generating set and Gröbner basis for  $I_A$ . It can also calculate Graver bases by running `graver matrix-file`.

- (a) For each of the following matrices, calculate, by hand, a Hilbert basis of the cone spanned by the columns of  $A$ . Then calculate, using 4ti2, the Graver basis for the lattice ideal defined by  $A$ .

i.  $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

ii.  $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

- (b) For the above matrices, check, by hand or using Macaulay 2, that the conjectured bound (Conjecture **reference to 8.4.11** on the total degree of any element of the Graver basis holds, that is, the total degree is always less than  $(d + 1)D(A)$  where

$$D(A) = \max\{|\det[\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_d}]| : 1 \leq i_1 \leq i_2 \leq \dots \leq i_d \leq n\}.$$

# Chapter 1

## Solutions to Tutorial 3

- (a) A reduced Gröbner basis of  $I_{\mathcal{A}}$  is  $\{x_1^2 - x_2, x_1x_2 - x_3, x_2^2 - x_1x_3\}$ .  
(b) This is one way to fill in the gaps to create a program which implements the Conti-Traverso algorithm.

```
CTAlg = (A,w) -> (  
  -- take a list of rows of matrix A  
  --   and a weight vector w  
  -- return a reduced GB of I_A  
  n := # (A_0);  
  d := #A;  
  R := QQ[t_0..t_d,x_1..x_n, MonomialSize=>16,  
    MonomialOrder=>Eliminate(d+1)];  
  J := ideal (append(  
    apply(n, j -> (  
      firstmon = x_(j+1);  
      secondmon = 1_R;  
      scan(d, i-> (  
        if A_i_j < 0  
        then firstmon = firstmon*(t_(i+1))^(A_i_j)  
        else secondmon = secondmon*(t_(i+1))^(A_i_j)));  
      firstmon - secondmon)),  
    product(toList (t_0..t_d))-1));  
  I := selectInSubring(1, gens gb J);  
  S := QQ[x_1..x_n, Degrees => transpose A, Weights=>w];  
  gens gb substitute(I, S))
```

- (a) Let  $H \subset \{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \text{ such that } \pi(\mathbf{u}) = \pi(\mathbf{v})\}$ .

Assume  $H$  generates  $I_{\mathcal{A}}$ . It forms a graph on the each fiber, so we

just need to check that these graphs are connected. Take any two vectors  $\mathbf{p}, \mathbf{q} \in \pi^{-1}(\mathbf{b})$ . Then  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$ . Since  $H$  generates  $I_{\mathcal{A}}$ , we have  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} = c_1(\mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}) + c_2(\mathbf{x}^{\mathbf{u}_2} - \mathbf{x}^{\mathbf{v}_2}) + \dots + c_n(\mathbf{x}^{\mathbf{u}_n} - \mathbf{x}^{\mathbf{v}_n})$  where each binomial in the sum is an element of  $H$ . We may also assume  $c_1\mathbf{x}^{\mathbf{u}_1} = \mathbf{x}^{\mathbf{p}}$  and  $c_n\mathbf{x}^{\mathbf{v}_n} = \mathbf{x}^{\mathbf{q}}$ . There is an edge from  $\mathbf{p}$  to  $\mathbf{v}_1$ . Since  $c_1\mathbf{x}^{\mathbf{v}_1}$  has to cancel with some other term in the sum, we can assume that it cancels with  $c_2\mathbf{x}^{\mathbf{u}_2}$ , hence there is an edge from  $\mathbf{v}_1 = \mathbf{u}_2$  to  $\mathbf{v}_2$ . Repeating this process, we find a path from  $\mathbf{p}$  to  $\mathbf{q}$ . Therefore the graph on the fiber given by  $H$  is connected.

Now assume that  $H$  forms a connected graph on  $\pi^{-1}(\mathbf{b})$ . Take  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$ . The nodes  $\mathbf{p}$  and  $\mathbf{q}$  must have a path  $\mathbf{p}, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{q}$  between them in the graph. So the binomials (or their negatives)  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_2}, \mathbf{x}^{\mathbf{v}_2} - \mathbf{x}^{\mathbf{v}_3}, \dots, \mathbf{x}^{\mathbf{v}_n} - \mathbf{x}^{\mathbf{q}}$  are elements of  $H$ . Finally we can write  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} = (\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_2}) + (\mathbf{x}^{\mathbf{v}_2} - \mathbf{x}^{\mathbf{v}_3}) + \dots + (\mathbf{x}^{\mathbf{v}_n} - \mathbf{x}^{\mathbf{q}})$ , hence  $H$  generates  $I_{\mathcal{A}}$ .

- (b) Assume that  $\mathcal{G}_{\succ}$  is a reduced Gröbner basis of  $I_{\mathcal{A}}$ . By the first part of this exercise, we know that  $\mathcal{G}_{\succ}$  gives a connected graph on each fiber. We just need to show that this graph may be directed and has a unique sink at the minimal monomial in the fiber. Since  $\mathcal{G}_{\succ}$  is reduced, if  $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\succ}$  then  $\mathbf{x}^{\mathbf{v}} - \mathbf{x}^{\mathbf{u}} \notin \mathcal{G}_{\succ}$ , so the graph is directed. Now suppose there are two sinks  $\mathbf{p}$  and  $\mathbf{q}$ . Since they are in the same fiber, the binomial  $\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}} \in I_{\mathcal{A}}$ . We may assume  $\text{in}(\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{q}}) = \mathbf{x}^{\mathbf{p}}$ . Since  $\mathcal{G}_{\succ}$  is a reduced Gröbner basis of  $I_{\mathcal{A}}$ ,  $\mathbf{x}^{\mathbf{p}} = \text{in}(g)$  for some  $g = \mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\succ}$ . This is a contradiction because then there is an edge in the graph from  $\mathbf{p}$  to  $\mathbf{v}$ .

Assume that  $\{\mathbf{u} - \mathbf{v} \mid \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\succ}\}$  forms a connected directed graph on each fiber with a unique sink that is the minimal monomial in that fiber. We want to show that  $\mathcal{G}_{\succ}$  is a reduced Gröbner basis of  $I_{\mathcal{A}}$ . By the first part of the exercise,  $\mathcal{G}_{\succ}$  generates  $I_{\mathcal{A}}$ , so  $\{\text{in}(g) \mid g \in \mathcal{G}_{\succ}\} \subset \text{in}(I_{\mathcal{A}})$ . To verify equality, take a monomial  $\mathbf{x}^{\mathbf{p}} \in \text{in}(I_{\mathcal{A}})$ . Either  $\mathbf{p}$  is a node with an edge leaving it or it is the unique minimal sink of its connected component. So either  $\mathbf{p} = \text{in}(g)$  for some  $g \in \mathcal{G}_{\succ}$  or it is minimal in the term order and hence cannot have been the initial term of some polynomial in  $I_{\mathcal{A}}$ .

3. (a) First calculate  $I_{\mathcal{A}}$  in the polynomial ring with  $\deg(x_i) = c_i$ . Then find a monomial of degree  $b$ . Finally reduce it modulo  $I_{\mathcal{A}}$  and take the exponent vector to get the  $x$  which minimizes the integer program.

- (b) For this problem  $c = \langle 1, 1, 1, 1 \rangle$ ,  $A = [5 \ 10 \ 25 \ 50]$ , and  $b = 100$ . A monomial of degree 100 in  $I_{\mathcal{A}}$  is  $x_1^{20}$ . Reducing this monomial by  $I_{\mathcal{A}}$  gives  $x_4^2$ , so the solution to the integer program is  $c \cdot x = \langle 1, 1, 1, 1 \rangle \cdot \langle 0, 0, 0, 2 \rangle = 2$ .
4. Using `4ti2`, the output of the command `output bin exercise.gro`, where `exercise` is the file containing the matrix and `exercise.gro` is the result of running `groebner exercise`, is

```
[
x[5]*x[6]-x[4]*x[8],
x[7]^2-x[6]*x[8],
x[3]*x[6]-x[2]*x[7],
x[4]^2-x[2]*x[6],
x[8]^2-x[7]*x[9],
x[2]*x[4]-x[1]*x[6],
x[2]*x[5]-x[1]*x[8],
x[3]*x[7]-x[2]*x[8],
x[3]*x[4]-x[1]*x[7],
x[2]^2-x[1]*x[4],
x[5]^2-x[3]*x[9],
x[5]*x[7]-x[4]*x[9],
x[3]*x[5]-x[1]*x[9],
x[4]*x[5]-x[2]*x[8],
x[7]*x[8]-x[6]*x[9],
x[3]*x[8]-x[2]*x[9],
x[3]^2-x[1]*x[5],
x[2]*x[3]*x[9]-x[1]*x[5]*x[8]
]
```

5. (a) Given a reduced Gröbner basis  $\mathcal{G}_{\succ}$  of  $I_{\mathcal{A}}$ , Exercise 2b shows that the exponent vectors form connected directed graphs on the elements of each fiber. The idea is to start with some element of the fiber and then wander around the graph to find all the other elements of the fiber. Computationally, start with a fiber element  $u \in \pi^{-1}(\mathbf{b})$ . Reduce  $x^u$  modulo  $\mathcal{G}_{\succ}$  to get a monomial whose exponent is the unique minimal element of the fiber. Now the elements of  $\mathcal{G}_{\succ}$  can be used to move backward from the unique sink to find all nodes of the graph.

(b) Create a file containing the following line:

```
{(5) (10) (25) (50)} (20,0,0,0)
```

Then run the program `cats_fiber` on it to get the elements of the fiber. The output should be the same as the fiber elements listed in the chapter.

6. Let  $a$  be the number of pallets of A's product shipped and  $b$  be the number of pallets of B's product shipped. Then we want to maximize  $15b + 11a$  subject to the constraints  $400a + 500b \leq 3700$  and  $2a + 3b \leq 20$ . Solving this integer program using the method of problem 4 or 5, you should find the revenues will be maximized with 4 pallets of each.
7. (a) To find the Hilbert basis for a set of vectors  $\mathcal{P}$ , we draw the cone  $pos(\mathcal{P})$  and the lattice  $\mathbb{Z}\mathcal{P}$ . Then we find a minimal set of vectors whose  $\mathbb{N}$ -linear combinations give the vectors of  $pos(\mathcal{P}) \cap \mathbb{Z}\mathcal{P}$ .
- i.  $\mathcal{P} = \{(1, 0), (1, 1), (1, 3), (1, 4)\}$  can be drawn as the black dots in the figure below. The cone  $pos(\mathcal{P})$  is the gray shaded region and  $\mathbb{Z}\mathcal{P}$  is the set of all shaded points along with the original black points.

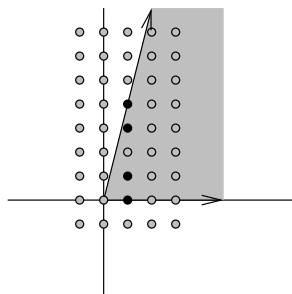


Figure 1.1:

Clearly the intersection is the set of all integer lattice points within the cone, so the Hilbert basis is  $\{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4)\}$ . Now running `graverfilename` from 4ti2 on this matrix and using output `binfilename.gra` to format the output file, we find that the Graver basis is  $\{x_2^3 - x_1^2x_3, x_2^4 - x_1^3x_4, x_2x_3 - x_1x_4, x_2^2x_4 - x_1x_3^2, x_2x_4^2 - x_3^3, x_4^4 - x_1x_3^3\}$ .

- ii.  $\mathcal{P} = \{(1, 0), (2, 1), (2, 3), (3, 4)\}$  can be drawn as the black dots in the figure below. The cone  $pos(\mathcal{P})$  is the gray shaded region

and  $\mathbb{Z}\mathcal{P}$  is the set of all shaded points along with the original black points.

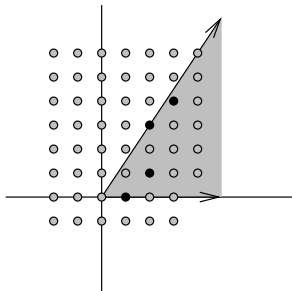


Figure 1.2:

Again the intersection is the set of all integer lattice points within the cone, so the Hilbert basis is  $\{(1, 0), (1, 1), (2, 3)\}$ .

Again using 4ti2, we find that the Graver basis is  $\{x_2^3 - x_1^4 x_3, x_2^4 - x_1^5 x_4, x_2^5 x_4, x_2 x_3 - x_1 x_4, x_2^2 x_4 - x_1^3 x_3^2, x_2 x_4^2 - x_1^2 x_3^3, x_1 x_3^4 - x_4^3, x_2 x_3^5 - x_4^4\}$ .

- (b) i. First we calculate the total degrees of the elements of the Graver basis.

element	total degree
$x_2^3 - x_1^4 x_3$	3
$x_2^4 - x_1^5 x_4$	4
$x_2 x_3 - x_1 x_4$	2
$x_2^2 x_4 - x_1^3 x_3^2$	3
$x_2 x_4^2 - x_1^2 x_3^3$	3
$x_3^4 - x_1 x_4^3$	4

Now we calculate the determinants of the two by two minors of  $A$ . This can be done by hand, or using Macaulay 2 by the command `minors(2,matrix{1,1,1,1,0,1,3,4})`. Either way, we find that  $D(A) = 4$ . So now we check that indeed  $4 \leq (3)(4)$ .

- ii. First we calculate the total degrees of the elements of the Graver

	element	total degree
	$x_2^3 - x_1^4 x_3$	5
	$x_2^4 - x_1^5 x_4$	6
basis.	$x_2 x_3 - x_1 x_4$	2
	$x_2^2 x_4 - x_1^3 x_3^2$	5
	$x_2 x_4^2 - x_1^2 x_3^3$	5
	$x_1 x_3^4 - x_4^3$	5
	$x_2 x_3^5 - x_4^4$	6

Now we calculate the determinants of the two by two minors of  $A$  and find that  $D(A) = 5$ . So now we check that indeed  $6 \leq (3)(5)$ .

# Chapter 2

## Tutorial 5

Recall from Tutorial 1 that Macaulay 2 can calculate minimal free resolutions of  $R/I$  using the command `res I`. Also, if the resolution is named  $C$ , one may either view the maps in detail using the command `C.dd` or view the Betti diagram by the command `betti C`. If your ideal is defined by calling a function, its ring may not have a short name and hence the resolution will be hard to view. You may remedy this by using `S = ring I` to set the name of the ring to  $S$ .

1. Using Macaulay 2 find a minimal free resolution and Betti diagram of  $I_A$  for each of the following matrices.

(a)  $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(b)  $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

(c)  $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

2. Let  $R = k[x]/\langle x^2 \rangle$  and let  $I = \langle x \rangle \subset R$ . What should a minimal free resolution look like? Now use Macaulay 2 to calculate a minimal resolution of  $I$ . Is the answer what you expected?
3. Let  $R = k[x_1, x_2, \dots, x_n]$  and let  $I \subset R$  be an ideal. Let  $\mathcal{G}_\succ = \{g_1, g_2, \dots, g_s\}$  be a Gröbner basis of  $I$  with respect to any monomial order  $\succ$ . Define

$$s_{ij} = \frac{LCM(\text{in}_\succ(g_i), \text{in}_\succ(g_j))}{\text{in}_\succ(g_i)} \mathbf{e}_i - \frac{LCM(\text{in}_\succ(g_i), \text{in}_\succ(g_j))}{\text{in}_\succ(g_j)} \mathbf{e}_j - \mathbf{a}_{ij}$$

where  $\mathbf{a}_{ij}$  is the vector of coefficients of the  $g_k$ 's in the expansion of the  $S$ -polynomial  $S\text{-pair}(g_i, g_j)$  in terms of  $\mathcal{G}_\prec$  and  $\mathbf{e}_i$  is the standard basis vector.

**Schreyer's Theorem** With  $R$ ,  $I$ , and  $\mathcal{G}_\prec$  as above, the set  $\{s_{ij} : 1 \leq i, k \leq s\}$  generates  $Syz(I)$ , the syzygies on  $I$ , as an  $R$ -module.

Consider the following matrices:

$$M = \begin{pmatrix} x^2 - x & xy & y^2 - y \end{pmatrix} \quad N = \begin{pmatrix} y & 0 \\ -x + 1 & y - 1 \\ 0 & -x \end{pmatrix}$$

Let  $I$  be the ideal generated by the entries of  $M$ .

- (a) Verify that the matrix product  $MN$  equals the  $1 \times 2$  matrix of all zeros. Explain why this shows that the module generated by the columns of the matrix  $N$  is contained in  $Syz(I)$ .

To show that  $Syz(I)$  is generated by the columns of  $N$ , we can use Schreyer's Theorem.

- (b) Check that the generators for  $I$  form a Gröbner basis for  $I$  with respect to lexicographic order.
- (c) Compute the syzygies  $s_{12}, s_{13}, s_{23}$  obtained from the  $S$ -polynomials on the generators of  $I$ . By Schreyer's Theorem, these three syzygies generate  $Syz(I)$ .
- (d) How are the the columns of  $N$  related to the generators  $s_{12}, s_{13}, s_{23}$  of  $S$ ? Why does  $N$  only have two columns?  
(Borrowed from [?].)

4. The ideal of the twisted cubic is the toric ideal of  $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ .

Compute the minimal graded free resolution of  $I_A$ . For each degree in which there is a nonzero Betti number, calculate the fiber and then the simplicial complex  $\Delta_{\mathbf{b}}$ . Are all the fibers basic? Do they have the homology of a simplicial sphere? (A helpful Macaulay 2 command is `basis(b,R)` which gives the fiber  $\pi^{-1}(b)$  in the ring  $R$ .)

Corollary ?? says that the toric ideal  $I_A$  has a minimal generator in degree  $\mathbf{b}$  iff  $\Delta_{\mathbf{b}}$  is disconnected. Verify that this is true for the three generators of the twisted cubic.

# Chapter 3

## Solutions to Tutorial 5

- Using the Conti-Traverso algorithm written in Tutorial 3 and a weight vector of all ones, we get the following resolutions and Betti diagrams. The Hoşten-Sturmfels algorithm may also be used.

- We display the differentials in the resolution below. The command `complete` forces Macaulay 2 to finish its calculations of the matrices in order to display them.

```
i2 : Ia = ideal CTAlg({{1,2,3}},{1,1,1});
```

```
i3 : S = ring Ia;
```

```
i4 : (complete res Ia).dd
```

```
o4 = 0 : S 1 <----- S 2 : 1
      | x_1^2-x_2 x_1x_2-x_3 |
```

```
1 : S 2 <----- S 1 : 2
      {2} | -x_1x_2+x_3 |
      {3} | x_1^2-x_2 |
```

```
2 : S 1 <----- 0 : 3
      0
```

```
o4 : ChainComplexMap
```

```
i5 : betti res Ia
```

```
o5 = total: 1 2 1
      0: 1 . .
      1: . 1 .
      2: . 1 .
      3: . . 1
```

- (b) In this example the differential matrices are larger than will fit nicely on the page, so they have not been displayed.

```
i6 : Ib = ideal CTA1g({{1,1,1,1},{0,1,3,4}},{1,1,1,1});
```

```
i7 : S = ring Ib;
```

```
i8 : res Ib
```

```
      1      4      4      1
o8 = S <-- S <-- S <-- S <-- 0
      0      1      2      3      4
```

```
o8 : ChainComplex
```

```
i9 : betti res Ib
```

```
o9 = total: 1 4 4 1
      0: 1 . . .
      1: . 1 . .
      2: . 3 4 1
```

- (c) Again the differentials have not been displayed.

```
i10 : Ic=ideal CTA1g({{1,2,2,3},{0,1,3,4}},{1,1,1,1});
```

```
i11 : S = ring Ic;
```

```
i12 : res Ic
```

```

      1      5      6      2
o12 = S <-- S <-- S <-- S <-- 0
      0      1      2      3      4

```

```
o12 : ChainComplex
```

```
i13 : betti res Ic
```

```

o13 = total: 1 5 6 2
      0: 1 . . .
      1: . . . .
      2: . . . .
      3: . 1 . .
      4: . . . .
      5: . 1 . .
      6: . 1 1 .
      7: . 1 2 .
      8: . 1 2 1
      9: . . 1 1

```

2. In a polynomial ring an ideal with one generator has no syzygies, however, over  $R = k[x]/\langle x^2 \rangle$ , multiplication by  $x$  sends the generator to  $x^2$  which is zero in  $R$ . Now for the syzygies on the syzygies, we again want syzygies on  $x$ , so again we get  $x$ . Repeating this process we should get an infinite repeating resolution

$$0 \longleftarrow R/I \longleftarrow R \xleftarrow{(x)} R \xleftarrow{(x)} R \xleftarrow{(x)} R \longleftarrow \dots$$

Putting this ring and ideal into Macaulay 2, we find that the resolution stops after two steps. The computer cannot handle the infinite resolution.

```
i14 : S=(ZZ/101[x])/ideal(x^2);
```

```
i15 : res ideal(x)
```

```

      1      1      1
o15 = S <-- S <-- S
      0      1      2

```

o15 : ChainComplex

3. (a) Since the generators of  $I$  are the three entries of the matrix  $M$ , any column vector  $v$  such that  $Mv = 0$  is a syzygy on the generators of  $I$ . The matrix  $MN$  contains only zeros, so each column of  $N$  is a syzygy on the generators of  $I$ , and hence the module the columns generate must be contained in the syzygy module  $Syz(I)$ .
- (b) To check these polynomials form a Gröbner basis, we calculate the S-pairs. Let  $g_1 = x^2 - x$ ,  $g_2 = xy$  and  $g_3 = y^2 - y$  and  $G = \{g_1, g_2, g_3\}$ .

$$\begin{aligned} S\text{-pair}(g_1, g_2) &= \frac{x^2y}{x^2}(x^2 - x) - \frac{x^2y}{xy}(xy) = -xy = -g_2 \\ &\equiv 0 \pmod{G} \\ S\text{-pair}(g_1, g_3) &= \frac{x^2y^2}{x^2}(x^2 - x) - \frac{x^2y^2}{y^2}(y^2 - y) = -xy^2 + x^2y \\ &= (xy)(x - y) = g_2(x - y) \equiv 0 \pmod{G} \\ S\text{-pair}(g_2, g_3) &= \frac{xy^2}{xy}(xy) - \frac{xy^2}{y^2}(y^2 - y) = xy = g_2 \\ &\equiv 0 \pmod{G} \end{aligned}$$

Since all three S-pairs reduce to 0,  $G$  is a Gröbner basis for  $I$ .

- (c) From our calculations above, we see

$$\begin{aligned} s_{12} &= \frac{x^2y}{x^2}\mathbf{e}_1 - \frac{x^2y}{xy}\mathbf{e}_2 - \langle 0, -1, 0 \rangle = \langle y, -x + 1, 0 \rangle \\ s_{13} &= \frac{x^2y^2}{x^2}\mathbf{e}_1 - \frac{x^2y^2}{y^2}\mathbf{e}_3 - \langle 0, x - y, 0 \rangle = \langle y^2, -(x - y), -x^2 \rangle \\ s_{23} &= \frac{xy^2}{xy}\mathbf{e}_2 - \frac{xy^2}{y^2}\mathbf{e}_3 - \langle 0, 1, 0 \rangle = \langle 0, y - 1, -x \rangle \end{aligned}$$

- (d) The syzygies  $s_{12}$  and  $s_{23}$  are the columns of  $N$ . The remaining syzygy  $s_{13} = \langle y^2, -(x - y), -x^2 \rangle = ys_{12} + xs_{23}$  so it is in the module generated by  $s_{12}$  and  $s_{23}$ . Hence it is unnecessary to include it in a minimal generating set for  $Syz(I)$ .
4. First we use Macaulay 2 and the Conti-Traverso algorithm to find the ideal and its resolution.

```

i16 : I = ideal CTA1g({{1,1,1,1},{0,1,2,3}},{1,1,1,1});
i17 : S = ring I;
i18 : (complete res I).dd

o18 = 0 : S  $\begin{array}{c} 1 \\ \hline | \ x_2^2-x_1x_3 \ x_2x_3-x_1x_4 \ x_3^2-x_2x_4 \ | \\ \hline \end{array}$  S : 1

1 : S  $\begin{array}{c} 3 \\ \hline \{2,2\} \ | \ -x_3 \ x_4 \ | \\ \{2,3\} \ | \ x_2 \ -x_3 \ | \\ \{2,4\} \ | \ -x_1 \ x_2 \ | \\ \hline \end{array}$  S : 2

2 : S  $\begin{array}{c} 2 \\ \hline 0 \\ \hline \end{array}$  0 : 3

o18 : ChainComplexMap

```

Looking at the resolution, we see that there are generators in multidegrees  $\{2, 2\}$ ,  $\{2, 3\}$ , and  $\{2, 4\}$ . We add the multidegree of any monomial from the first column of the differential map from  $S^2$  to  $S^3$  to the multidegree of its row, and we find  $\{3, 4\}$  is the multidegree of one of the first syzygies. Similarly, using the second column, we find the other first syzygy has multidegree  $\{3, 5\}$ .

The fiber in multidegree  $\mathbf{b}$  can be calculated in Macaulay 2 by find a basis for  $S_{\mathbf{b}}$  using the command `basis`. For example if  $\mathbf{b} = \{2, 2\}$ :

```

i1 : basis({2,2},S)

o19 = | x_1x_3 x_2^2 |

o19 : Matrix S  $\begin{array}{c} 1 \quad 2 \\ \hline \end{array}$  S

```

So the maximal faces of the simplicial complex  $\Delta_{\{2,2\}}$  are  $\{1, 3\}$  and  $\{2\}$ , which means  $\Delta_{\{2,2\}}$  consists of a three vertices and one edge connecting

two of the vertices. In other words,  $\Delta_{\{2,2\}}$  is contractible to a 0-sphere (see figure 3.1), and hence has the same homology as a simplicial sphere. Also clearly this fiber is basic since the greatest common divisor of  $x_1x_3$  and  $x_2^2$  is 1.

For  $\mathbf{b} = \{2, 3\}$ , the fiber is  $\{x_1x_4, x_2x_3\}$ , and for  $\mathbf{b} = \{2, 4\}$ , the fiber is  $\{x_2x_4x_3^2\}$ . For both of these it is clear that the fiber is basic since the gcd of each pair of generators is 1. The simplicial complexes  $\Delta_{\{2,3\}}$  and  $\Delta_{\{2,4\}}$  are both contractible to 0-spheres (see figure 3.1) as we expected.

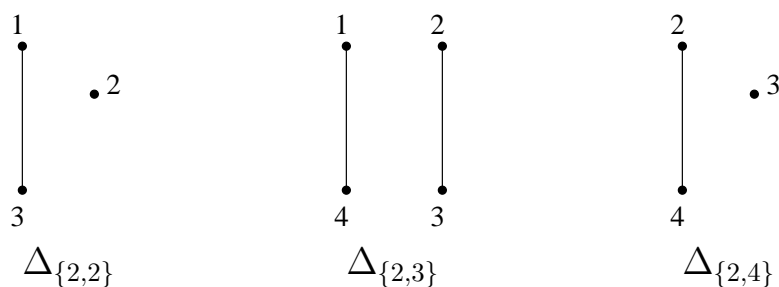


Figure 3.1:

For  $\mathbf{b} = \{3, 4\}$ , the fiber is  $\{x_1x_2x_4x_1x_3^2x_2^2x_3\}$ . Clearly  $\gcd(x_1x_2x_4x_1x_3^2x_2^2x_3) = 1$ . Removing one element at a time, we see that  $\gcd(x_1x_3^2x_2^2x_3) = x_3$ ,  $\gcd(x_1x_2x_4x_3^2) = x_2$ , and  $\gcd(x_1x_2x_4x_1x_3^2) = x_1$ . Hence this fiber is basic. The maximal faces of the simplicial complex are  $\{1, 2, 4\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$  which contracts to a simplicial 1-sphere (see figure 3.2).

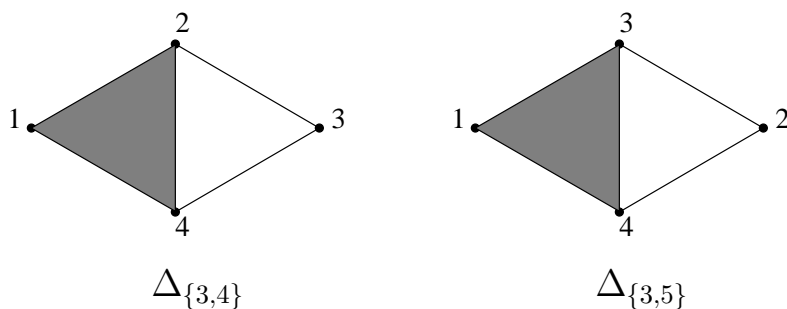


Figure 3.2:

Similarly for  $\mathbf{b} = \{3, 5\}$ , the fiber is  $\{x_1x_3x_4x_2^2x_4x_2x_3^2\}$ . We can check this fiber is basic and the associated simplicial complex contracts to a simplicial 1-sphere (see figure 3.2).

Finally, looking back at the simplicial complexes associated to the minimal generators, it is clear that they are indeed disconnected.