GENERIC SUBIDEALS OF GRAPH IDEALS AND FREE RESOLUTIONS

LEAH GOLD

Abstract. For a graph of an n-cycle Δ with Alexander dual Δ^*, we study the free resolution of a subideal $G(\mathrm{n})$ of the Stanley-Reisner ideal I_{Δ^*}. We prove that if $G(\mathrm{n})$ is generated by 3 generic elements of I_{Δ^*}, then the second syzygy module of $G(\mathrm{n})$ is isomorphic to the second syzygy module of $(x_1, x_2, \ldots, x_n)$. A result of Bruns shows that there is always a 3-generated ideal with this property. We show that it can be chosen to have a particularly nice form.

1. Introduction and Background

Let Δ be a cycle and Δ^* be its Alexander dual. The Stanley-Reisner ideals of such graphs and their free resolutions have been studied by many people, such as in [1], [2], [3], [9], [10], [15]. In this paper we study the free resolution of a subideal $G(\mathrm{n})$ of I_{Δ^*} consisting of three generic elements of I_{Δ^*}. The study of these ideals led to the following observation, which is our main theorem.

Theorem 1. *Let $G(n)$ be as above and let $\mathrm{Syz}_2(G(n))$ be the module of second syzygies. Then the resolution of $\mathrm{Syz}_2(G(n))$ is the same as that of $\mathrm{Syz}_2((x_1, x_2, \ldots, x_n))$.*

That is to say, the tails of the resolutions, i.e. the modules and maps in the later part of the complexes, of the ideals $G(\mathrm{n})$ and $(x_1, x_2, \ldots, x_n)$ are identical. For example, in 5 variables the three generators of $G(5)$ are $\alpha = r_1cde + r_2ade + r_3abe + r_4bcd + r_5abc$, $\beta = s_1cde + s_2ade + s_3abe + s_4bcd + s_5abc$, and $\gamma = t_1cde + t_2ade + t_3abe + t_4bcd + t_5abc$. The minimal free resolution of $G(5)$ looks like

$$0 \to R \xrightarrow{d_5} R^5 \xrightarrow{d_4} R^{10} \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where the maps d_4 and d_5 are exactly the same as the ones for the resolution of (a, b, c, d, e).

A result of Bruns [4] shows that for any ideal I and any integer m, there exists a 3-generated ideal I' such that the resolutions of $\mathrm{Syz}_m I$ and $\mathrm{Syz}_m(I')$ are identical. A consequence of our study of these graph ideals is that we have found a particularly simple 3-generated ideal related to the Koszul complex on the ideal of variables.

1.1. Criterion for Exactness.

If $\mathcal{F}$ is a complex of finitely generated free modules over a Noetherian ring, the necessary and sufficient conditions for exactness are given by the following result due to Buchsbaum and Eisenbud [7].

Let φ be a matrix of rank r. Define $I(\varphi) = I_r(\varphi)$ to be the ideal generated by the $r \times r$ minors of φ.

Gold is partially supported by an NSF-VIGRE postdoctoral fellowship.

Theorem 2 (Buchsbaum, Eisenbud)**.** *Let R be a Noetherian ring. A complex*

$$\mathcal{F}\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules is exact if and only if for all $k = 1, 2, \ldots, n$,

(1) $\operatorname{rank}(\varphi_k) + \operatorname{rank}(\varphi_{k+1}) = \operatorname{rank}(F_k)$, *and*
(2) $\operatorname{depth}(I(\varphi_k)) \geq k$.

Note, for any complex $\operatorname{rank}(\varphi_k) + \operatorname{rank}(\varphi_{k+1}) \leq \operatorname{rank}(F_k)$. Hence condition 1 asserts equality.

Recall that $\operatorname{depth}(I)$ is the length of a maximal R-sequence contained in I. In general, the depth of an ideal is less than or equal to its codimension. In the case of a polynomial ring, the depth of an ideal is equal to its codimension, and the codimension of an ideal is equal to the codimension of its radical. Hence, we may restate condition 2 as $\operatorname{codim}(\operatorname{rad}(I(\varphi_k))) \geq k$.

1.2. **Buchsbaum-Eisenbud Structure Theorem.**

In 1974 Buchsbaum and Eisenbud published a paper containing some structure theorems for finite free resolutions [8]. The structure theorems were further explained and slightly generalized in papers by Eagon and Northcott [11], and Bruns [5]. The assumptions they made may be relaxed in the case when R is an integral domain. Since we are working over an integral domain, we state the structure theorem for this case.

Definition 3. Let $A = (a_{i,j})$ be a $p \times q$ matrix and let ν be a non-negative integer. We say that A *factorizes completely* if there exists elements $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ of R such that $a_{i,j} = u_i v_j$ for all i and j. When A is a row matrix, that is, when $p = 1$, we say that the complete factorization $u_1 = 1$ and $v_j = a_{1,j}$ is the *canonical complete factorization.*

The entries of $\bigwedge^\nu A$ are the $\nu \times \nu$ minors of the matrix A. If $J = \{j_1, \ldots, j_\nu\}$ with $1 \leq j_1 < j_2 < \cdots < j_\nu \leq p$ and $K = \{k_1, \ldots, k_\nu\}$ with $1 \leq k_1 < k_2 < \cdots < k_\nu \leq q$, then

$$(\bigwedge\nolimits^\nu A)_{J,K} = \det \begin{pmatrix} a_{j_1,k_1} & a_{j_1,k_2} & \cdots & a_{j_1,k_\nu} \\ a_{j_2,k_1} & a_{j_2,k_2} & \cdots & a_{j_2,k_\nu} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_\nu,k_1} & a_{j_\nu,k_2} & \cdots & a_{j_\nu,k_\nu} \end{pmatrix}.$$

Now let B be a $q \times t$ matrix. Let μ and ν be non-negative integers such that $\mu + \nu = q$. Assume that $\bigwedge^\mu A$ and $\bigwedge^\nu B$ factorize completely. Thus, $(\bigwedge^\mu A)_{J,K} = u_J v_K$ and $(\bigwedge^\nu B)_{M,N} = w_M z_N$.

Definition 4. The two factorizations above are said to be *complementary* if, for every M, $w_M = \operatorname{sgn}(M, M')v_{M'}$ where M' denotes the complement of M in $\{1, 2, \ldots, q\}$.

Proposition 5 (Eagon, Northcott)**.** *Let $\mu = \operatorname{rank}(A)$ and $\nu = \operatorname{rank}(B)$. Suppose that $AB = 0$ with $\mu + \nu = q$. Assume there is given a complete factorization of $\bigwedge^\mu A$ and that $\operatorname{codim}(I_\mu(A)) \geq 2$. Then there is a unique complete factorization of $\bigwedge^\nu B$ that is complementary to the given factorization of $\bigwedge^\mu A$.*

Using this proposition, Eagon and Northcott reproved the first structure theorem of Buchsbaum and Eisenbud.

Corollary 6 (Buchsbaum, Eisenbud)**.** *Let*

$$\mathcal{C}\colon 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R$$

be a complex of finitely generated free R-modules. Choose a basis for each F_i and let A_i be the matrix with respect to these bases for $1 \leq i \leq n$. Suppose $\operatorname{codim}(I(\varphi_i)) \geq 2$ *for all $i \geq 2$. Then for $1 \leq i \leq n$ there exist ideals B_i such that $I(\varphi_n) = B_n$ and $I(\varphi_i) = B_{i+1}B_i$ for $1 \leq i \leq n-1$.*

1.3. **Bruns's Construction.**

Suppose we restrict our discussion to ideals with a given number of generators. An ideal with a single generator has no syzygies and a trivial resolution. An ideal with two generators has a single first syzygy and a simple resolution. When we consider an ideal with three generators, however, the resolutions are more complicated. In 1976, Bruns published a result in [4] which proved, in more generality, a conjecture of Buchsbaum and Eisenbud from section 11 of their paper [8]. This result showed that every finite free resolution has the same tail as the finite free resolution of a 3-generated ideal. The following theorem is a special case of a Bruns's result.

Theorem 7 (Bruns)**.** *Let R be a polynomial ring and let I be an ideal of R. Suppose a projective resolution of R/I has the form*

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R.$$

Let $r := \operatorname{rank}(f_3)$. Then there exist homomorphisms $c\colon F_2 \to R^{r+2}$, $f_2'\colon R^{r+2} \to R^3$, $f_1'\colon R^3 \to R$ with $f_3' = c \circ f_3$, such that the sequence

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \to \cdots \to F_4 \xrightarrow{f_4} F_3 \xrightarrow{f_3'} R^{r+2} \xrightarrow{f_2'} R^3 \xrightarrow{f_1'} R$$

is exact.

Note that c is a projection. Also notice that there are many homomorphisms c, f_1, and f_2 that satisfy the theorem.

Definition 8. Let I and J be ideals, and let the minimal free resolutions of R/I and R/J, respectively, be of the form

$$\mathcal{F}\colon 0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R$$

and

$$\mathcal{G}\colon 0 \to G_n \xrightarrow{g_n} G_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_3} G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} R.$$

We say that I and J are *tail resolution equivalent* if the modules $F_i = G_i$ for $3 \leq i \leq n$, and the maps $f_i = g_i$ for $4 \leq i \leq n$.

This definition is an equivalence relation on ideals. With this definition, we can restate the result of Theorem 7: For any ideal I, there is a 3-generated ideal J that is tail resolution equivalent to I.

In the remainder of this paper, we will develop our main result, namely, a method of constructing simple ideals that are tail resolution equivalent to $(x_1, x_2, \ldots, x_n)$.

2. A Special Family of 3-generated Ideals

We will now describe a family of 3-generated ideals. Fix an integer $n \geq 4$. Let K be the complete graph on n vertices. Let $R = k[x_1, x_2, \ldots, x_n]$ where $k = k'(r_1, r_2, \ldots, r_n, s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n)$ and k' is a field. Label the vertices of K by the x_i's. Let the graph L be the complement of the cycle $\Delta = \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\}$ in K. Let the ideal I be

$$\bigcap_{\{x_i, x_j\} \in L} (x_i, x_j).$$

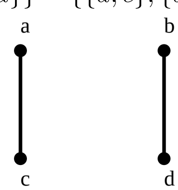
This ideal I is I_{Δ^*}, the Alexander dual of the Stanley-Reisner ideal of Δ. For convenience, let $\mathbf{m}_{i_1, i_2, \ldots, i_k} = \prod_{p \neq i_1, i_2, \ldots, i_k} x_p$. Wherever subscripts appear, consider them as being modulo n, so, for example, $\mathbf{m}_{n,n+1} = x_2 \cdots x_{n-1}$. Then the n generators of I are $\mathbf{m}_{i,i+1}$, for $i = 1, \ldots, n$.

Now we want to take a generic linear combination of these generators. So let M be the $3 \times n$ matrix

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}.$$

Define the ideal G(n) to be the 3-generated ideal whose generators are the entries of $(\mathbf{m}_{1,2}\, \mathbf{m}_{2,3} \, \cdots \, \mathbf{m}_{n,1})M^t$.

Example 9. Consider the graph on four vertices, labeled a, b, c, and d, with edge set $K - \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}\} = \{\{a,c\}, \{b,d\}\}$.



The monomial ideal I is $(a,c) \cap (b,d) = (ab, bc, ad, cd)$. We find the 3-generated ideal $G(4)$ by taking generic combinations of ab, bc, ad, and cd. So, $G(4) = (r_1cd + r_2ad + r_3bc + r_4ab, s_1cd + s_2ad + s_3bc + s_4ab, t_1cd + t_2ad + t_3bc + t_4ab)$.

Due to the construction, it is clear that $G(\text{n}) \subset (x_i, x_j)$ for all i and j that are nonadjacent integers mod n. Each ideal (x_i, x_j) where i and j are not adjacent mod n, therefore, is a codimension two component of $G(\text{n})$. The following proposition shows that there are no other codimension two components.

Lemma 10. *If P is a codimension two associated prime of the ideal $G(n)$ and if $x_i \in P$, then $P = (x_i, x_j)$ for some j that is not adjacent mod n to i.*

Proof. Let $G(\text{n}) = (\alpha, \beta, \gamma)$, so α, β, γ are contained in P. Separate the terms of the generators of $G(\text{n})$ into those that involve x_i and those that do not.

$$\alpha = fx_i + r_{i-1}\mathbf{m}_{i-1,i} + r_i\mathbf{m}_{i,i+1} = fx_i + (r_{i-1}x_{i+1} + r_ix_{i-1})\mathbf{m}_{i-1,i,i+1}$$
$$\beta = gx_i + s_{i-1}\mathbf{m}_{i-1,i} + s_i\mathbf{m}_{i,i+1} = gx_i + (s_{i-1}x_{i+1} + s_ix_{i-1})\mathbf{m}_{i-1,i,i+1}$$
$$\gamma = hx_i + t_{i-1}\mathbf{m}_{i-1,i} + t_i\mathbf{m}_{i,i+1} = hx_i + (t_{i-1}x_{i+1} + t_ix_{i-1})\mathbf{m}_{i-1,i,i+1}$$

Since $x_i \in P$, we also have that $\alpha - fx_i$, $\beta - gx_i$, and $\gamma - hx_i$ are in P. Therefore, either the terms $r_{i-1}x_{i+1} + r_ix_{i-1}$, $s_{i-1}x_{i+1} + s_ix_{i-1}$, and $t_{i-1}x_{i+1} + t_ix_{i-1}$ are in P, or x_j is in P for some $j \neq i-1, i, i+1$. In the former case, x_{i-1} and x_{i+1} are in P along with x_i, and this contradicts $\operatorname{codim}(P) = 2$. In the latter case, we get the desired result. □

Proposition 11. *The codimension two associated primes of the ideal $G(n)$ are exactly the ideals (x_i, x_j) where $1 \leq i < j \leq n$ and i, j are not adjacent integers mod n.*

Proof. Let $G(\mathrm{n}) = (\alpha, \beta, \gamma)$. The ideals (x_i, x_j) where i and j are not adjacent mod n are certainly codimension 2 components of $G(\mathrm{n})$.

Now, suppose P is some other prime ideal of codimension two containing $G(\mathrm{n})$. We will show that no such P exists.

If P contains a variable, then by Lemma 10 it is of the form (x_i, x_j) where i and j are not adjacent mod n. Hence, we may assume that P does not contain any variables.

Write the generators of $G(\mathrm{n})$ by splitting them into those terms that involve x_1 and those that do not:

$$\begin{aligned} \alpha &= fx_1 + r_1\mathbf{m}_{1,2} + r_n\mathbf{m}_{1,n} &&= fx_1 + \mathbf{m}_{1,2,n}l_1 \\ \beta &= gx_1 + s_1\mathbf{m}_{1,2} + s_n\mathbf{m}_{1,n} &&= gx_1 + \mathbf{m}_{1,2,n}l_2 \\ \gamma &= hx_1 + t_1\mathbf{m}_{1,2} + t_n\mathbf{m}_{1,n} &&= hx_1 + \mathbf{m}_{1,2,n}l_3 \end{aligned}$$

where

$$\begin{aligned} f &= r_2\mathbf{m}_{1,2,3} + r_3\mathbf{m}_{1,3,4} + \cdots + r_{n-1}\mathbf{m}_{1,n-1,n} \\ g &= s_2\mathbf{m}_{1,2,3} + s_3\mathbf{m}_{1,3,4} + \cdots + s_{n-1}\mathbf{m}_{1,n-1,n} \\ h &= t_2\mathbf{m}_{1,2,3} + t_3\mathbf{m}_{1,3,4} + \cdots + t_{n-1}\mathbf{m}_{1,n-1,n} \end{aligned}$$

$$l_1 = (r_1x_n + r_nx_2) \qquad l_2 = (s_1x_n + s_nx_2) \qquad l_3 = (t_1x_n + t_nx_2)$$

There are two cases: either P contains f, g, and h, or it does not contain at least one of them.

Case 1: $f, g, h \in P$.

Since α, β, and γ are in P, we also know that P contains $\mathbf{m}_{1,2,n}(r_1x_n + r_nx_2)$, $\mathbf{m}_{1,2,n}(s_1x_n + s_nx_2)$, and $\mathbf{m}_{1,2,n}(t_1x_n + t_nx_2)$. Since P does not contain any variables, $(r_1x_n + r_nx_2)$, $(s_1x_n + s_nx_2)$, and $(t_1x_n + t_nx_2)$ are in P. Containing these elements also forces x_2 and x_n to be in P, and this contradicts our assumption that there are no variables in P.

Case 2: One of f, g, h is not in P.

Without loss of generality, suppose $f \notin P$.

Let $\widehat{P} = P \cap k[x_2, \ldots, x_n]$. Since $\alpha, \beta, \gamma \in P$, we have the elements

$$\begin{aligned} f\beta - g\alpha &= \mathbf{m}_{1,2,n}(fl_2 - gl_1), \\ f\gamma - h\alpha &= \mathbf{m}_{1,2,n}(fl_3 - hl_1), \text{ and} \\ g\gamma - h\beta &= \mathbf{m}_{1,2,n}(gl_3 - hl_2). \end{aligned}$$

are in $\widehat{P}$. The prime ideal $\widehat{P}$ does not contain any variables, so $fl_2 - gl_1$, $fl_3 - hl_1$, and $gl_3 - hl_2$ are in $\widehat{P}$.

Since P is codimension two, $V(P)$ is dimension $n - 2$. The projection, p, from n variables to $n - 1$ variables given by dropping x_1 gives a birational map between $V(P)$ and its image $p(V(P))$. So $p(V(P))$ is also dimension $n - 2$, hence $\widehat{P}$ is codimension $(n - 1) - (n - 2) = 1$.

Since $\widehat{P}$ is codimension one, these elements must have a common factor. We claim, however, that they are irreducible. So, we have a contradiction and such a P cannot exist.

It is now sufficient to show that $fl_2 - gl_1$ is irreducible.

$$\begin{aligned} gl_1 - fl_2 &= [s_2\mathbf{m}_{1,2,3} + s_3\mathbf{m}_{1,3,4} + \cdots + s_{n-1}\mathbf{m}_{1,n-1,n}](r_1x_n + r_nx_2) \\ &\quad -[r_2\mathbf{m}_{1,2,3} + r_3\mathbf{m}_{1,3,4} + \cdots + r_{n-1}\mathbf{m}_{1,n-1,n}](s_1x_n + s_nx_2) \end{aligned}$$

$$\begin{aligned} = \quad & [(r_1s_2 - r_2s_1)x_4\cdots x_n^2 + (r_1s_3 - r_3s_1)x_2x_5\cdots x_n^2 + \cdots \\ & +(r_1s_{n-2} - r_{n-2}s_1)x_2\cdots x_{n-3}x_n^2 + (r_1s_{n-1} - r_{n-1}s_1)x_2\cdots x_{n-2}x_n] \\ + \quad & [(r_ns_2 - r_2s_n)x_2x_4\cdots x_n + (r_ns_3 - r_3s_n)x_2^2x_5\cdots x_n + \cdots \\ & +(r_ns_{n-2} - r_{n-2}s_n)x_2^2x_3\cdots x_{n-3}x_n + (r_ns_{n-1} - r_{n-1}s_n)x_2^2x_3\cdots x_{n-2}] \end{aligned}$$

Letting $z_i = (r_1s_i - r_is_1)x_n + (r_ns_i - r_is_n)x_2$, we can rewrite the above expression.

$$\begin{aligned} & z_2\mathbf{m}_{1,2,3} + z_3\mathbf{m}_{1,3,4} + z_4\mathbf{m}_{1,4,5} + \cdots + z_{n-1}\mathbf{m}_{1,n-1,n} \\ &= (z_2x_4 + z_3x_2)x_5\cdots x_n + x_2x_3(z_4x_6\cdots x_n \\ &\quad +z_5x_4x_7\cdots x_n + \cdots + z_{n-1}x_4\cdots x_{n-2}) \\ &= \delta + x_3\epsilon. \end{aligned}$$

In order for this expression to factor, δ and ϵ must have a common factor. None of the variables divide all terms of both δ and ϵ, so the factor cannot be divisible by a variable. So, the only possible common factor is $z_2x_4 + z_3x_2$. In order for $z_2x_4 + z_3x_2$ to divide the sum of the latter terms, any specialization of the variables that make this expression zero must also make the sum of the latter terms zero. Consider the specialization where $x_4 = 0$, $x_2 = r_3s_1 - r_1s_3$, $x_n = r_ns_3 - r_3s_n$ and all the other variables are non-zero. Then $z_2x_4 + z_3x_2$ becomes 0, but the remaining term of the sum $z_4x_6\cdots x_{n-1}$ is non-zero. Hence this expression is irreducible. □

3. The Tail Resolution Equivalence

With the help of a computer and Macaulay 2, it is easy to construct examples of the ideals discussed in section 2. When we do so for rings with between 4 and 12 variables and look at their resolutions, the results are rather striking.

Example 12. Consider the ideal $(a,b,c,d) \subset R = k[a,b,c,d]$. A resolution of $R/(a,b,c,d)$ is the Koszul resolution, namely,

$$0 \to R \xrightarrow{d_4} R^4 \xrightarrow{d_3} R^6 \xrightarrow{d_2} R^4 \xrightarrow{d_1} R$$

with the maps described below.

$$d_1 = \begin{pmatrix} a & b & c & d \end{pmatrix}$$

$$d_2 = \begin{pmatrix} -b & 0 & 0 & -d & -c & 0 \\ a & -c & 0 & 0 & 0 & -d \\ 0 & b & -d & 0 & a & 0 \\ 0 & 0 & c & a & 0 & b \end{pmatrix}$$

$$d_3 = \begin{array}{c} \\ a\wedge b \\ b\wedge c \\ c\wedge d \\ a\wedge d \\ a\wedge c \\ b\wedge d \end{array} \begin{array}{c} \begin{array}{cccc} a\wedge b\wedge c & a\wedge b\wedge d & a\wedge c\wedge d & b\wedge c\wedge d \end{array} \\ \begin{pmatrix} c & d & 0 & 0 \\ a & 0 & 0 & d \\ 0 & 0 & a & b \\ 0 & -b & -c & 0 \\ -b & 0 & d & 0 \\ 0 & a & 0 & -c \end{pmatrix} \end{array}$$

$$d_4 = \begin{pmatrix} -d \\ c \\ -b \\ a \end{pmatrix}$$

Using this resolution in the manner discussed above, we get the following candidate for the resolution of $R/\mathrm{G}(4)$.

$$0 \to R \xrightarrow{d_4} R^4 \xrightarrow{\varphi_3} R^5 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where

$$\varphi_1 = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} r_1cd+r_2ad+r_3bc+r_4ab \\ s_1cd+s_2ad+s_3bc+s_4ab \\ t_1cd+t_2ad+t_3bc+t_4ab \end{pmatrix}^t$$

$$\varphi_2 = \begin{array}{c} \begin{matrix} g_1 & g_2 & g_3 & a\wedge c & b\wedge d \end{matrix} \\ \begin{pmatrix} 0 & \gamma & \beta & * & * \\ \gamma & 0 & -\alpha & * & * \\ -\beta & -\alpha & 0 & * & * \end{pmatrix} \end{array}$$

where the column labeled $a \wedge c$ is

$$\begin{matrix} -(s_2t_4 - s_4t_2)\,a^2 + -(s_1t_4 - s_4t_1)\,ac + -(s_2t_3 - s_3t_2)\,ac + -(s_1t_3 - s_3t_1)\,c^2 \\ (r_2t_4 - r_4t_2)\,a^2 + (r_1t_4 - r_4t_1)\,ac + (r_2t_3 - r_3t_2)\,ac + (r_1t_3 - r_3t_1)\,c^2 \\ -(r_2s_4 - r_4s_2)\,a^2 + -(r_1s_4 - r_4s_1)\,ac + -(r_2s_3 - r_3s_2)\,ac + -(r_1s_3 - r_3s_1)\,c^2 \end{matrix}$$

and the column labeled $b \wedge d$ is

$$\begin{matrix} -(s_4t_3 - s_3t_4)\,b^2 + (s_1t_4 - s_4t_1)\,bd + -(s_2t_3 - s_3t_2)\,bd + (s_1t_2 - s_2t_1)\,d^2 \\ (r_4t_3 - r_3t_4)\,b^2 + -(r_1t_4 - r_4t_1)\,bd + (r_2t_3 - r_3t_2)\,bd + -(r_1t_2 - r_2t_1)\,d^2 \\ -(r_4s_3 - r_3s_4)\,b^2 + (r_1s_4 - r_4s_1)\,bd + -(r_2s_3 - r_3s_2)\,bd + (r_1s_2 - r_2s_1)\,d^2 \end{matrix}$$

$$\varphi_3 = \begin{matrix} \\ g_1 \\ g_2 \\ g_3 \\ a\wedge c \\ b\wedge d \end{matrix} \begin{array}{c} \begin{matrix} a\wedge b\wedge c & a\wedge b\wedge d & a\wedge c\wedge d & b\wedge c\wedge d \end{matrix} \\ \begin{pmatrix} r_2a+r_1c & r_3b+r_1d & r_4a+r_3c & r_4b+r_2d \\ -s_2a-s_1c & -s_3b-s_1d & -s_4a-s_3c & -s_4b-s_2d \\ t_2a+t_1c & t_3b+t_1d & t_4a+t_3c & t_4b+t_2d \\ -b & 0 & d & 0 \\ 0 & a & 0 & -c \end{pmatrix} \end{array}$$

Notice that if we specialize to $M = \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&1&0 \end{pmatrix}$, then $G(4) = (ab, cd, ad + bc)$. This is the resolution in example 2 of section 11 of Buchsbaum and Eisenbud's Structure Theorems paper [8].

Example 13. In five variables, G(5) has the following generators.

$$\begin{aligned} \alpha &= r_1cde + r_2ade + r_3abe + r_4bcd + r_5abc \\ \beta &= s_1cde + s_2ade + s_3abe + s_4bcd + s_5abc \\ \gamma &= t_1cde + t_2ade + t_3abe + t_4bcd + t_5abc \end{aligned}$$

The resolution of $R/(\alpha, \beta, \gamma)$ is as follows.

$$0 \to R \xrightarrow{d_5} R^5 \xrightarrow{d_4} R^{10} \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where the d_i's are the Koszul maps and the other maps are defined as follows.

$$\varphi_1 = \begin{pmatrix} \alpha \; \beta \; \gamma \end{pmatrix}$$

$$\varphi_2 = \begin{pmatrix} 0 & \gamma & \beta & * & * & * & * & * \\ \gamma & 0 & -\alpha & * & * & * & * & * \\ -\beta & -\alpha & 0 & * & * & * & * & * \end{pmatrix}$$

In order to condense the matrices so that they fit on the page, let $xylm = x_ly_m - x_my_l$. The missing entries from the above matrix can be written as the product of a 3×10 matrix with a 10×5 matrix.

$$\begin{pmatrix} st12 & st13 & st15 & st14 & st23 & st25 & st24 & st35 & st34 & st54 \\ -rt12 & -rt13 & -rt15 & -rt14 & -rt23 & -rt25 & -rt24 & -rt35 & -rt34 & -rt54 \\ rs12 & rs13 & rs15 & rs14 & rs23 & rs25 & rs24 & rs35 & rs34 & rs54 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & d^2e & de^2 & 0 \\ -ace & 0 & bde & be^2 & ce^2 \\ -ac^2 & -acd & 0 & bce & c^2e \\ -c^2d & -cd^2 & 0 & 0 & 0 \\ -a^2e & 0 & 0 & 0 & ae^2 \\ -a^2c & -a^2d & -abd & 0 & ace \\ -acd & -ad^2 & -bd^2 & -bde & 0 \\ 0 & -a^2b & -ab^2 & 0 & 0 \\ 0 & -abd & -b^2d & -b^2e & -bce \\ 0 & 0 & 0 & -b^2c & -bc^2 \end{pmatrix}$$

$\varphi_3 =$

$$\begin{pmatrix} r_2a+r_1c & r_1d & r_4b+r_1e & r_3a & r_4c & r_5a+r_4d & r_3b+r_2d & r_2e & r_5b & r_5c+r_3e \\ -s_2a-s_1c & -s_1d & -s_4b-s_1e & -s_3a & -s_4c & -s_5a-s_4d & -s_3b-s_2d & -s_2e & -s_5b & -s_5c-s_3e \\ t_2a+t_1c & at_1d & t_4b+t_1e & t_3a & t_4c & t_5a+t_4d & t_3b+t_2d & t_2e & t_5b & t_5c+t_3e \\ -b & 0 & 0 & d & e & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & -c & 0 & e & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & -c & 0 & e & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & -c & -d & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & b & 0 & -d \end{pmatrix}$$

Example 14. In six variables, G(6) has the following generators.

$$\begin{aligned} \alpha &= r_6abcd + r_5bcde + r_4abcf + r_3abef + r_2adef + r_1cdef \\ \beta &= s_6abcd + s_5bcde + s_4abcf + s_3abef + s_2adef + s_1cdef \\ \gamma &= t_6abcd + t_5bcde + t_4abcf + t_3abef + t_2adef + t_1cdef \end{aligned}$$

The resolution of $R/(\alpha, \beta, \gamma)$ is

$$0 \to R \xrightarrow{d_6} R^6 \xrightarrow{d_5} R^{15} \xrightarrow{d_4} R^{20} \xrightarrow{\varphi_3} R^{12} \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where the d_i's are the Koszul maps. We will describe the φ_i's later in this section.

Here it is enough to notice that except for the first several syzygy matrices and free modules, the resolution is the same as the resolution of the complete intersection $(x_1, \dots, x_n)$. This pattern leads us to the following theorem.

Theorem 15. *The ideal $G(n)$ is tail resolution equivalent to $(x_1, \dots, x_n)$.*

In order to prove this theorem, we will exhibit a resolution for G(n). In the process we will show that it has the same tail as the Koszul resolution on n variables.

Let the following complex be the Koszul resolution of $R/(x_1, \dots, x_n)$.

$$\mathcal{K}\colon 0 \to \bigwedge^n(R^n) \xrightarrow{d_n} \cdots \xrightarrow{d_4} \bigwedge^3(R^n) \xrightarrow{d_3} \bigwedge^2(R^n) \xrightarrow{d_2} R^n \xrightarrow{d_1} R$$

Let G_1 be the free submodule of $\bigwedge^2(R^n)$ generated by $\{x_i \wedge x_{i+1}, 1 \le i \le n\}$ and let G_2 be the complementary free submodule of $\bigwedge^2(R^n)$ generated by $\{x_i \wedge x_j$ such that i and j are not adjacent integers mod $n\}$. Recall that all subscripts on variables are to be considered mod n. G_1 and G_2 determine maps $\psi_1\colon \bigwedge^3(R^n) \to G_1$ and $\psi_2\colon \bigwedge^3(R^n) \to G_2$ such that $d_3 = \psi_1 \oplus \psi_2$. Hence ψ_1 is given by a $n \times \binom{n}{3}$ matrix and ψ_2 by a $(\binom{n}{2} - n) \times \binom{n}{3}$ matrix. Now let $M\colon G_1 \to R^3$ be given by the $3 \times n$ matrix

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}.$$

We define the map $\varphi_3\colon \bigwedge^3 R^n \to R^3 \oplus G_2$ to be $\begin{pmatrix} M\psi_1 \\ \psi_2 \end{pmatrix}$.

We define the map $\varphi_1\colon \bigwedge^2 R^3 \simeq R^{3^*} \to R^*$ to be the composite of $M^t\colon R^{3^*} \simeq \bigwedge^2 R^3 \to G_1^*$ and $\mu\colon G_1^* \to R$ given by the matrix $(\mathbf{m}_{1,2}\ \mathbf{m}_{2,3}\ \cdots\ \mathbf{m}_{1,n})$. Recall $\mathbf{m}_S = \prod_{p \notin S} x_p$.

Let $K\colon \bigwedge^2 R^{3*} \to \bigwedge^2 R^3$ be the matrix of Koszul syzygies on α, β, and γ, the generators of $G(\mathrm{n})$. Let $P\colon G_2 \to \bigwedge^2 G_1$ be determined by the following map on the generators of G_2:

$$\underset{i<j-1}{x_i \wedge x_j} \longmapsto \sum_{\substack{l\in\{i,\dots,j-1\}\\ m\in\{1,\dots,i-1,j,\dots,n\}}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} (x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1})$$

Note that the fraction $\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}}$ is, in fact, a ring element. Let $N\colon G_2 \to R^{3^*}$ be the composite of P and $\bigwedge^2 M\colon \bigwedge^2 G_1 \to \bigwedge^2 R^3 \simeq R^{3^*}$. Define $\varphi_2\colon R^3 \oplus G_2 \to \bigwedge^2 R^3 \simeq R^{3^*}$ to be K on R^3 and N on G_2, so $\varphi_2 = (K \mid N)$.

Then the proposed resolution for $R/G(\mathrm{n})$ is

$$\mathcal{J}\colon 0 \to \bigwedge^n R^n \xrightarrow{d_n} \cdots \xrightarrow{d_4} \bigwedge^3 R^n \xrightarrow{\varphi_3} R^3 \oplus G_2 \xrightarrow{\varphi_2} \bigwedge^2 R^3 \simeq R^{3^*} \xrightarrow{\varphi_1} R^*.$$

Theorem 16. *The sequence $\mathcal{J}$ described above is an exact complex.*

This theorem provides a proof of Theorem 15.

Proof of Theorem 15. Theorem 16 shows that the complex $\mathcal{J}$ is a resolution. By its construction it is tail resolution equivalent to the ideal $(x_1, \dots, x_n)$. □

We will prove Theorem 16 in the next section.

4. Proof of Tail Resolution Equivalence

In the following two subsections, we prove Theorem 16 by first showing $\mathcal{J}$ is a complex and then showing that it is exact.

4.1. **Proof of Complex Structure.**

Lemma 17. *The sequence $\mathcal{J}$ of Theorem 16 is a complex.*

Proof. We show that $\mathcal{J}$ is a complex by checking that the composition of every pair of adjacent maps is zero. Notice that the maps d_i, for $4 \le i \le n$, are exactly the same as those in the Koszul resolution. Hence, all the compositions $d_i d_{i+1}$ for $4 \le i < n$ are zero, and we are left with only three pairs of maps to check.

Consider $\varphi_3 d_4$. The rows of φ_3 are either rows of d_3, or linear combinations of rows of d_3. Since $d_3 d_4 = 0$, we get $\varphi_3 d_4 = 0$.

There are two compositions left to check.

Lemma 18. $\varphi_1\varphi_2 = 0$

Lemma 19. $\varphi_2\varphi_3 = 0$.

In the remainder of this section, we prove these lemmas, and hence complete the proof that $\mathcal{J}$ is a complex. □

Before we prove the lemmas, we describe the maps involved in more detail.

Let Δ_{pml} be the 3×3 minor of M using columns p, m, and l.

Let $\{g_1, g_2, g_3\}$ be a basis for R^3 and $\{h_1, h_2, h_3\}$ be a basis for R^{3^*}.

$$\begin{aligned}
\varphi_3\colon \bigwedge^3 R^n &\longrightarrow R^3 \oplus G_2 \\
\underset{i<j<k}{x_i \wedge x_j \wedge x_k} &\longmapsto \sum_{l,m \text{ s.t. } \{l,l+1,m\}=\{i,j,k\}} (r_l x_m\, g_1 - s_l x_m\, g_2 + t_l x_m\, g_3) \\
&\quad + x_i\, x_j \wedge x_k, \text{ if } k \neq j-1, j+1 \\
&\quad + x_j\, x_k \wedge x_i, \text{ if } i \neq k-1, k+1 \\
&\quad + x_k\, x_i \wedge x_j, \text{ if } j \neq i-1, i+1
\end{aligned}$$

$$\begin{aligned}
P\colon G_2 &\longrightarrow \bigwedge^2 G_1 \\
\underset{i<j-1}{x_i \wedge x_j} &\longmapsto \sum_{\substack{i\le l<j \\ 1\le m<i}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} (x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1}) \\
&\quad + \sum_{\substack{i\le l<j \\ j\le m\le n}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} (x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1})
\end{aligned}$$

$$\begin{aligned}
\bigwedge^2 M\colon \bigwedge^2 G_1 &\longrightarrow \bigwedge^2 R^3 \simeq R^{3^*} \\
\underset{m<l}{(x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1})} &\longmapsto (s_m t_l - s_l t_m)\, h_1 - (r_m t_l - r_l t_m)\, h_2 \\
&\quad +(r_m s_l - r_l s_m)\, h_3
\end{aligned}$$

$$\begin{aligned}
\mu M^t \bigwedge^2 M\colon \bigwedge^2 G_1 &\longrightarrow R^* \\
\underset{m<l}{(x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1})} &\longmapsto \sum_{p=1}^{n} \Delta_{pml}\mathbf{m}_{p,p+1}
\end{aligned}$$

$$\begin{aligned}
K\colon \bigwedge^2 R^{3^*} &\longrightarrow \bigwedge^2 R^3 \simeq R^{3^*} \\
g_1 &\longmapsto \sum_{i=1}^{n} t_i \mathbf{m}_{i,i+1}\, h_2 - \sum_{i=1}^{n} s_i \mathbf{m}_{i,i+1}\, h_3 \\
g_2 &\longmapsto \sum_{i=1}^{n} t_i \mathbf{m}_{i,i+1}\, h_1 - \sum_{i=1}^{n} r_i \mathbf{m}_{i,i+1}\, h_3 \\
g_3 &\longmapsto \sum_{i=1}^{n} s_i \mathbf{m}_{i,i+1}\, h_1 - \sum_{i=1}^{n} r_i \mathbf{m}_{i,i+1}\, h_2
\end{aligned}$$

Proof of Lemma 18. By definition of K, $\varphi_1 K$ is zero. So, it is sufficient to show that $\varphi_1 N = (\mu M^t)(\bigwedge^2 M)P = 0$.

Applying P to a general element of G_2, we get

$$\underset{i+1<j}{x_i \wedge x_j} \longmapsto \sum_{\substack{l\in\{i,\dots,j-1\} \\ m\in\{1,\dots,i-1,j,\dots,n\}}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} (x_m \wedge x_{m+1}) \wedge (x_l \wedge x_{l+1}).$$

Under the map $\mu M^t \bigwedge^2 M$, $P(x_i \wedge x_j)$ goes to

$$\sum_{m<i\leq l<j} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} \sum_{p=1}^{n} \Delta_{pml}\mathbf{m}_{p,p+1}$$

$$+ \sum_{i\leq l<j\leq m} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}} \sum_{p=1}^{n} \Delta_{pml}\mathbf{m}_{p,p+1}$$

Letting $C_{pml} = \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}\mathbf{m}_p}{\mathbf{m}_{i,j}} \Delta_{pml}$, the above expression can be rewritten as

$$= \sum_{\substack{1\leq m<i\\ i\leq l<j}} \sum_{1\leq p\leq n} C_{pml} + \sum_{\substack{i\leq l<j\\ j\leq m\leq n}} \sum_{1\leq p\leq n} C_{pml}$$

$$= \sum_{\substack{1\leq m<i\\ i\leq l<j\\ 1\leq p<i}} C_{pml} + \sum_{\substack{1\leq m<i\\ i\leq l<j\\ i\leq p<j}} C_{pml} + \sum_{\substack{1\leq m<i\\ i\leq l<j\\ j\leq p\leq n}} C_{pml}$$

$$+ \sum_{\substack{i\leq l<j\\ j\leq m\leq n\\ 1\leq p<i}} C_{pml} + \sum_{\substack{i\leq l<j\\ j\leq m\leq n\\ i\leq p<j}} C_{pml} + \sum_{\substack{i\leq l<j\\ j\leq m\leq n\\ j\leq p\leq n}} C_{pml}.$$

The first and last sums cancel with themselves as $\{p, m\}$ ranges over the specified values. The second and fifth sums cancel with themselves as $\{p, l\}$ ranges over the specified values. Since $C_{pml} = -C_{mpl}$, the third and fourth sums cancel with each other. Hence this whole expression is zero. □

Proof of Lemma 19. We want to show that $KM\psi_1 + (\bigwedge^2 M)N\psi_2 = 0$.

There are 3 possible forms for a basis element of $\bigwedge^3 R^n$. We will treat each one separately.

Case 1: $x_i \wedge x_{i+1} \wedge x_{i+2}$.

Under φ_3 this element maps to

$$r_i x_{i+2} g_1 - s_i x_{i+2} g_2 + t_i x_{i+2} g_3 + r_{i+1} x_i g_1 - s_{i+1} x_i g_2 + t_{i+1} x_i g_3 - x_{i+1} x_i \wedge x_{i+2}$$

$$= (r_i x_{i+2} + r_{i+1} x_i) g_1 - (s_i x_{i+2} + s_{i+1} x_i) g_2 + (t_i x_{i+2} + t_{i+1} x_i) g_3 - x_{i+1} x_i \wedge x_{i+2}.$$

In turn, under φ_2, $\varphi_3(x_i \wedge x_{i+1} \wedge x_{i+2}$ maps to

$$(r_i x_{i+2} + r_{i+1} x_i)(\sum_{1\leq\delta\leq n} t_\delta \mathbf{m}_{\delta,\delta+1} h_2 - \sum_{1\leq\delta\leq n} s_\delta \mathbf{m}_{\delta,\delta+1} h_3)$$

$$-(s_i x_{i+2} + s_{i+1} x_i)(\sum_{1\leq\delta\leq n} t_\delta \mathbf{m}_{\delta,\delta+1} h_1 - \sum_{1\leq\delta\leq n} r_\delta \mathbf{m}_{\delta,\delta+1} h_3)$$

$$+(t_i x_{i+2} + t_{i+1} x_i)(\sum_{1\leq\delta\leq n} s_\delta \mathbf{m}_{\delta,\delta+1} h_1 - \sum_{1\leq\delta\leq n} r_\delta \mathbf{m}_{\delta,\delta+1} h_2)$$

$$-x_{i+1} \sum_{\substack{l=i,i+1\\ 1\leq m<i}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} [(s_m t_l - s_l t_m) h_1 - (r_m t_l - r_l t_m) h_2 + (r_m s_l - r_l s_m) h_3]$$

$$-x_{i+1} \sum_{\substack{l=i,i+1\\ i+2\leq m\leq n}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} [(s_m t_l - s_l t_m) h_1 - (r_m t_l - r_l t_m) h_2 + (r_m s_l - r_l s_m) h_3].$$

The h_1 component is

$$-(s_i x_{i+2} + s_{i+1} x_i) \sum_{1\leq\delta\leq n} t_\delta \mathbf{m}_{\delta,\delta+1} \ + \ (t_i x_{i+2} + t_{i+1} x_i) \sum_{1\leq\delta\leq n} s_\delta \mathbf{m}_{\delta,\delta+1}$$

$$-x_{i+1} \sum_{\substack{l=i,i+1 \\ 1\leq m\leq i-1}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} (s_m t_l - s_l t_m)$$

$$-\ x_{i+1} \sum_{\substack{l=i,i+1 \\ i+2\leq m\leq n}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} (s_m t_l - s_l t_m).$$

Rearranging this expression it becomes

$$\sum_{1\leq\delta\leq n} [(s_\delta t_{i+1} - s_{i+1} t_\delta) x_i + (s_\delta t_i - s_i t_\delta) x_{i+2}] \mathbf{m}_{\delta,\delta+1}$$

$$- \sum_{\substack{l=i,i+1 \\ 1\leq m\leq i-1}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_l - s_l t_m)$$

$$- \sum_{\substack{l=i,i+1 \\ i+2\leq m\leq n}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_l - s_l t_m).$$

The i and $i+1$ terms of the first sum cancel with each other. So, now the first sum has two parts: those terms where $\delta \leq i-1$ and those terms where $\delta \geq i+2$. The other two sums can be broken up into terms where $l = i$ and terms where $l = i+1$. So, we now have

$$\begin{aligned}
&\sum_{1\leq\delta\leq i-1} [(s_\delta t_{i+1} - s_{i+1} t_\delta) x_i + (s_\delta t_i - s_i t_\delta) x_{i+2}] \mathbf{m}_{\delta,\delta+1} \\
&\quad + \sum_{i+2\leq\delta\leq n} [(s_\delta t_{i+1} - s_{i+1} t_\delta) x_i + (s_\delta t_i - s_i t_\delta) x_{i+2}] \mathbf{m}_{\delta,\delta+1} \\
&\quad - \sum_{1\leq m\leq i-1} \frac{\mathbf{m}_{i,i+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_i - s_i t_m) \\
&\quad - \sum_{1\leq m\leq i-1} \frac{\mathbf{m}_{i+1,i+2}\ \mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_{i+1} - s_{i+1} t_m) \\
&\quad - \sum_{i+2\leq m\leq n} \frac{\mathbf{m}_{i,i+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_i - s_i t_m) \\
&\quad - \sum_{i+2\leq m\leq n} \frac{\mathbf{m}_{i+1,i+2}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+2}} x_{i+1} (s_m t_{i+1} - s_{i+1} t_m).
\end{aligned}$$

It is easy to see that $\frac{\mathbf{m}_{i,i+1}x_{i+1}}{\mathbf{m}_{i,i+2}} = x_{i+2}$ and $\frac{\mathbf{m}_{i+1,i+2}x_{i+1}}{\mathbf{m}_{i,i+2}} = x_i$. Now we have

$$\begin{aligned}
&\sum_{1\le\delta\le i-1}[(s_\delta t_{i+1}-s_{i+1}t_\delta)x_i+(s_\delta t_i-s_it_\delta)x_{i+2}]\mathbf{m}_{\delta,\delta+1}\\
&\qquad+\sum_{i+2\le\delta\le n}[(s_\delta t_{i+1}-s_{i+1}t_\delta)x_i+(s_\delta t_i-s_it_\delta)x_{i+2}]\mathbf{m}_{\delta,\delta+1}\\
&-\sum_{1\le m\le i-1}\mathbf{m}_{m,m+1}x_{i+2}(s_mt_i-s_it_m)+\sum_{i+2\le m\le n}\mathbf{m}_{m,m+1}x_i(s_{i+1}t_m-s_mt_{i+1})\\
&-\sum_{1\le m\le i-1}\mathbf{m}_{m,m+1}x_i(s_mt_{i+1}-s_{i+1}t_m)+\sum_{i+2\le m\le n}\mathbf{m}_{m,m+1}x_{i+2}(s_it_m-s_mt_i)\\
&=0.
\end{aligned}$$

The h_2 and h_3 components can be similarly shown to be zero. So for case 1, the composition of these maps is zero.

Case 2: $x_i \wedge x_{i+1} \wedge x_k$ where $x_i \wedge x_k$ and $x_{i+1} \wedge x_k$ are not part of the basis of G_1.

We may assume $i < k$ in order to simplify the calculations. Under φ_3 $x_i \wedge x_{i+1} \wedge x_k$ maps to

$$r_ix_k\ g_1 - s_ix_k\ g_2 + t_ix_l\ g_3 + x_i\ x_{i+1}\wedge x_k - x_{i+1}\ x_i \wedge x_k.$$

In turn, under φ_2, this element goes to

$$\begin{aligned}
&r_ix_k(\sum_{1\le\delta\le n}t_\delta\mathbf{m}_{\delta,\delta+1}\ h_2-\sum_{1\le\delta\le n}s_\delta\mathbf{m}_{\delta,\delta+1}\ h_3)\\
&-s_ix_k(\sum_{1\le\delta\le n}t_\delta\mathbf{m}_{\delta,\delta+1}\ h_1-\sum_{1\le\delta\le n}r_\delta\mathbf{m}_{\delta,\delta+1}\ h_3)\\
&+t_ix_k(\sum_{1\le\delta\le n}s_\delta\mathbf{m}_{\delta,\delta+1}\ h_1-\sum_{1\le\delta\le n}r_\delta\mathbf{m}_{\delta,\delta+1}\ h_2)\\
&+x_i\sum_{\substack{i+1\le l<k\\1\le m<i+1}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i+1,k}}[(s_mt_l-s_lt_m)\ h_1-(r_mt_l-r_lt_m)\ h_2+(r_ms_l-r_ls_m)\ h_3]\\
&-x_i\sum_{\substack{i+1\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i+1,k}}[(s_mt_l-s_lt_m)\ h_1-(r_mt_l-r_lt_m)\ h_2+(r_ms_l-r_ls_m)\ h_3]\\
&-x_{i+1}\sum_{\substack{i\le l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}[(s_mt_l-s_lt_m)\ h_1-(r_mt_l-r_lt_m)\ h_2+(r_ms_l-r_ls_m)\ h_3]\\
&+x_{i+1}\sum_{\substack{i\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}[(s_mt_l-s_lt_m)\ h_1-(r_mt_l-r_lt_m)\ h_2+(r_ms_l-r_ls_m)\ h_3]
\end{aligned}$$

Taking just the coefficient of h_1, we get

$$\begin{aligned}
&\sum_{1\le\delta\le n}(s_\delta t_i-s_it_\delta)x_k\mathbf{m}_{\delta,\delta+1}\\
&+x_i\sum_{\substack{i+1\le l<k\\1\le m<i+1}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i+1,k}}(s_mt_l-s_lt_m)-x_i\sum_{\substack{i+1\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i+1,k}}(s_mt_l-s_lt_m)
\end{aligned}$$

$$-x_{i+1}\sum_{\substack{i\le l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}(s_mt_l-s_lt_m)+x_{i+1}\sum_{\substack{i\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}(s_mt_l-s_lt_m).$$

Remove the terms where $m=1$ or $l=1$ from the second, fourth, and fifth sums above. Then notice that $\frac{\mathbf{m}_{i,i+1}x_{i+1}}{\mathbf{m}_{i,k}}\frac{\mathbf{m}_{i,i+1}x_i}{\mathbf{m}_{i+1,k}}=x_k$ and rewrite the above expression as

$$\sum_{1\le\delta\le n}(s_\delta t_i-s_it_\delta)x_k\mathbf{m}_{\delta,\delta+1}$$
$$+\sum_{\substack{i+1\le l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+1,k}}(s_mt_l-s_lt_m)-\sum_{\substack{i+1\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+1,k}}(s_mt_l-s_lt_m)$$
$$-\sum_{\substack{i<l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+1,k}}(s_mt_l-s_lt_m)+\sum_{\substack{i<m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,i+1,k}}(s_mt_l-s_lt_m)$$
$$+\sum_{i+1\le l<k}\mathbf{m}_{l,l+1}x_k(s_it_l-s_lt_i)-\sum_{1\le m<i}\mathbf{m}_{m,m+1}x_k(s_mt_i-s_it_m)$$
$$+\sum_{k\le l\le n}\mathbf{m}_{l,l+1}x_k(s_it_l-s_lt_i).$$

So we see that the sum is zero, as desired. The calculations for the h_2 and h_3 components similarly yield zero.

Case 3: $x_i\wedge x_j\wedge x_k$ where no pair among x_i, x_j, and x_k form a basis element of G_1.

Again we write down the image of this element under the map φ_3 followed by φ_2 and then take the coefficient of h_1. We will assume $i<j<k$. In this case, we get

$$x_i\sum_{\substack{j\le l<k\\1\le m<j}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{j,k}}(s_mt_l-s_lt_m)-x_i\sum_{\substack{j\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{j,k}}(s_mt_l-s_lt_m)$$
$$-x_j\sum_{\substack{i\le l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}(s_mt_l-s_lt_m)+x_j\sum_{\substack{i\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,k}}(s_mt_l-s_lt_m)$$
$$+x_k\sum_{\substack{i\le l<j\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}}(s_mt_l-s_lt_m)-x_k\sum_{\substack{i\le m<j\\j\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}}(s_mt_l-s_lt_m).$$

The expression simplifies to

$$\sum_{\substack{j\le l<k\\1\le m<j}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m)-\sum_{\substack{j\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m)$$
$$-\sum_{\substack{i\le l<k\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m)+\sum_{\substack{i\le m<k\\k\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m)$$
$$+\sum_{\substack{i\le l<j\\1\le m<i}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m)-\sum_{\substack{i\le m<j\\j\le l\le n}}\frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j,k}}(s_mt_l-s_lt_m).$$

Now we can see that everything cancels. The fifth sum cancels the terms of the third sum where $i\le l<j$. The remaining terms of the third sum, those where $j\le l<k$, cancel with the terms of the terms of the first sum where $m<i$. The remaining terms of the first sum, those where $i\le m<j$, cancel with the terms

of the sixth sum where $j \le l < k$. The remaining terms of the sixth sum, those where $k \le l \le n$, cancel with the terms of the fourth sum where $i \le m < j$. The remaining terms of the fourth sum, those where $j \le m \le k$, cancel with the entire second sum.

The calculations for the h_2 and h_3 components similarly yield zero and so the composition of these maps is zero as desired. □

4.2. **Proof of Exactness.** Recall that the complex $\mathcal{K}$ is the Koszul resolution of $R/(x_1, \ldots, x_n)$. Let A_i be the matrix of the map d_i with respect to the usual bases. In particular, we denote the rows of A_3 corresponding to generators of G_1 by $y_1, \ldots, y_n$ and the rows of A_3 corresponding to generators of G_2 by $y_{n+1}, \ldots, y_{\binom{n}{2}}$.

The complex $\mathcal{K}$ satisfies the conditions of Corollary 6 so we may simultaneously factor the matrices $\bigwedge^{\operatorname{rank} A_i} A_i$. In order to calculate the minors of A_3, we describe the first three steps in the complete factorization of $\mathcal{K}$.

$(\bigwedge^{\operatorname{rank} A_1} A_1)_{1,j} = x_j$,

$(\bigwedge^{\operatorname{rank} A_2} A_2)_{I,J} = (a_1)_I(a_2)_J$ where $(a_1)_I = \operatorname{sgn}(I, I')x_{I'}$,

and $(\bigwedge^{\operatorname{rank} A_3} A_3)_{L,N} = (a_2')_L(a_3)_N$ where $(a_2')_L = \operatorname{sgn}(L, L')(a_2)_{L'}$,

Choose $i \in \{1, \ldots, n\}$. Let $L = \{i, n+1, \ldots, \binom{n}{2}\}$. So $L' = \{1, \ldots, n\} \setminus \{i\}$. If $I = \{1, \ldots, n\} \setminus \{i\}$, we have that $(\bigwedge^{\operatorname{rank} A_2} A_2)_{I,L'}$ is the determinant of the matrix

$$\begin{pmatrix} -x_2 & 0 & 0 & \ldots & x_n \\ x_1 & -x_3 & 0 & \ldots & 0 \\ 0 & x_2 & -x_4 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -x_1 \end{pmatrix}$$

with the ith row and column removed. Since $(a_1)_I = x_i$, $(\bigwedge^{\operatorname{rank} A_2} A_2)_{I,L'} = (-1)^{n-1}\mathbf{m}_{i,i+1}x_i$. So $(a_2')_L = \operatorname{sgn}(L, L')(a_2)_{L'} = \operatorname{sgn}(L, L')(-1)^{n-1}\mathbf{m}_{i,i+1}$. Then for any choice of $\binom{n}{2} - n + 1$ columns N of A_3, we have that $(\bigwedge^{\operatorname{rank} A_3} A_3)_{L,N} = \operatorname{sgn}(L, L')(-1)^{n-1}\mathbf{m}_{i,i+1}(a_3)_N$.

Let $\lambda = \binom{n}{2} - n + 1$.

Lemma 20. $\operatorname{codim}(I_\lambda(\varphi_3)) \ge 2$ *and* $\operatorname{rank}(\varphi_3) = \lambda$.

Proof. Let the rows of φ_3 be $z_1, \ldots, z_{\lambda+2}$ where the first three rows are $\sum_{1 \le i \le n} r_i y_i$ $\sum_{1 \le i \le n} s_i y_i$, $\sum_{1 \le i \le n} t_i y_i$, and the remaining $\lambda - 1$ rows are $y_{n+1}, \ldots y_{\binom{n}{2}}$.

Take K to be $\{1, 4, 5, \ldots, \lambda+2\}$. Then by multi-linearity

$$(\bigwedge^{\lambda} \varphi_3)_{K,N} = \sum_{1 \le i \le n} r_i \det \begin{pmatrix} y_i \\ y_{n+1} \\ \vdots \\ y_{\binom{n}{2}} \end{pmatrix}_N = \sum_{1 \le i \le n} r_i (\bigwedge^{\operatorname{rank} A_3} A_3)_{L,N}$$

$$= \sum_{1 \le i \le n} \operatorname{sgn}(L, L')(-1)^{n-1} r_i \mathbf{m}_{i,i+1}(a_3)_N$$

for any subset N of size λ and $L = \{i, n+1, \ldots, \binom{n}{2}\}$.

Similarly, if we take $K = \{2, 4, 5, \ldots, \lambda+2\}$ and $K = \{3, 4, 5, \ldots, \lambda+2\}$, we get

$$(\bigwedge^{\lambda} \varphi_3)_{K,N} = \sum_{1 \le i \le n} \operatorname{sgn}(L, L')(-1)^{n} s_i \mathbf{m}_{i,i+1}(a_3)_N$$

$$\text{and} \quad (\bigwedge^{\lambda} \varphi_3)_{K,N} = \sum_{1 \le i \le n} \operatorname{sgn}(L, L')(-1)^{n-1} t_i \mathbf{m}_{i,i+1}(a_3)_N$$

respectively, for any choice of N.

If $\operatorname{codim}(I_\lambda(\varphi_3)) = 1$, then the $(\bigwedge^{\lambda} \varphi_3)_{K,N}$ over all K and N must have a common factor. Hence $\{(a_3)_N\}$ over all choices for N must have a common factor. This leads to a contradiction because the ideal generated by all the $(a_3)_N$ is of codimension n.

We know there must be an N such that $(a_3)_N \neq 0$, so we have also found a nonzero $\lambda \times \lambda$ minor of φ_3. By construction, the rank of φ_3 cannot be larger than λ, therefore it is equal to λ. □

Now we are prepared to prove the following lemma.

Lemma 21. *The complex $\mathcal{J}$ of Theorem 16 is exact.*

We will show the complex $\mathcal{J}$ exact by applying the Buchsbaum-Eisenbud Exactness Theorem [Theorem 2].

Proof. We know that all the conditions of Theorem 2 are satisfied for $k \geq 4$ because the tail of the complex is the same as the tail of the Koszul resolution of n variables. It remains to be shown that the conditions hold for $k = 1,\ 2,$ and 3.

Claim: $\operatorname{rank}(\varphi_1) + \operatorname{rank}(\varphi_2) = \operatorname{rank}(R^3)$.

Since $M \neq 0$, $\operatorname{rank}(\varphi_1) = \operatorname{rank}\left(\, \alpha\ \beta\ \gamma \,\right) = 1$. So, we just need to show that $\operatorname{rank} \varphi_2 = 2$. The sum of the ranks of the maps is always less than or equal to the rank of the module. So, we know that $\operatorname{rank}(\varphi_2) \leq 3 - 1 = 2$. To show equality, we just need to find a 2×2 submatrix with non-zero determinant. Since φ_2 includes the 3×3 Koszul matrix, it also contains a 2×2 submatrix with non-zero determinant, namely the product of two generators of the 3-generated ideal.

Claim: $\operatorname{rank}(\varphi_2) + \operatorname{rank}(\varphi_3) = \operatorname{rank}(R^3 \oplus G_2)$.

From above, we know that $\operatorname{rank}(\varphi_2) = 2$. Lemma 20 shows that $\operatorname{rank}(\varphi_3) = \binom{n}{2} - n + 1$ and we know that $\operatorname{rank}(R^3 \oplus G_2) = 3 + \binom{n}{2} - n$.

Claim: $\operatorname{rank}(\varphi_3) + \operatorname{rank}(d_4) = \operatorname{rank}(\bigwedge^3 R^n)$.

The rank of d_4 is $\binom{n}{3} - \binom{n}{2} + n - 1$ because it is part of the Koszul complex, which is exact. We showed above that $\operatorname{rank} \varphi_3 = \binom{n}{2} - n + 1$. So $\operatorname{rank} \varphi_3 + \operatorname{rank}(d_4) = \binom{n}{2} - n + 1 + \binom{n}{3} - \binom{n}{2} + n - 1 = \binom{n}{3} = \operatorname{rank}(\bigwedge^3 R^n)$.

Claim: $\operatorname{codim}(I(\varphi_1)) \geq 1$.

We know $\operatorname{rank}(\varphi_1) = 1$ so $I(\varphi_1)$ is generated by the entries of φ_1. Since $M \neq 0$, this ideal is non-zero and so its codimension must be at least one.

Claim: $\operatorname{codim}(I(\varphi_2)) \geq 2$.

The rank of φ_2 is 2, so $I(\varphi_2) = I_2(\varphi_2)$. The map given by the matrix K is the Koszul relations on α, β, and γ, so the 2×2 minors of it (and hence also of φ_2) contain the ideal $(\alpha^2, \beta^2, \gamma^2)$. So $\operatorname{codim}(I(\varphi_2)) \geq \operatorname{codim}(\alpha^2, \beta^2, \gamma^2)$. Since $\operatorname{codim}(\alpha^2, \beta^2, \gamma^2) = \operatorname{codim}(\alpha, \beta, \gamma) = 2$, we have that $\operatorname{codim}(I_2(\varphi_2)) \geq 2$.

Claim: $\operatorname{codim}(I(\varphi_3)) \geq 3$.

Choose bases for F_i. Let A_i be the matrix of the map d_i for $i \geq 4$, and let A_i be the matrix of the map φ_i for $1 \leq i \leq 3$. Since the Koszul resolution is exact, by Theorem 2 $\operatorname{codim}(I(d_i)) \geq i$ for $i \geq 4$. By Lemma 20 $\operatorname{codim}(I(\varphi_3)) \geq 2$. We showed above that $\operatorname{codim}(I(\varphi_2)) \geq 2$. So by application of Corollary 6 to $\mathcal{J}$, there are ideals B_i such that $I(d_4) = B_5B_4$, $I(\varphi_3) = B_4B_3$, $I(\varphi_2) = B_3B_2$, and $I(\varphi_1) = B_2B_1$. The ideal $I(\varphi_1)$ must have trivial complete factorization, so $B_1 = (1)$ and $B_2 = J$.

The codimension of a product of ideals is the minimum of the codimension of the factors. Since $\operatorname{codim}(I(\varphi_4)) \geq 4$, we also know that $\operatorname{codim}(B_4) \geq 4$. Therefore we would be done if we could show that $\operatorname{codim}(B_3) \geq 3$, and since $\operatorname{codim}(I(\varphi_3)) \geq 2$, we just need to show that $\operatorname{codim}(B_3) \neq 2$.

Suppose that $\operatorname{codim}(B_3) = 2$. Then there is a codimension 2 prime P such that $B_3 \subset P$. By construction, B_3 contains J and the entries of N. Therefore P is a codimension 2 component of J. Hence, by Proposition 11 $P = (x_i, x_j)$ for some nonadjacent integers mod n i and j. Consider an entry of N in the $x_i \wedge x_j$ column,

$$\sum_{\substack{l \in \mathcal{I} \\ m \in \mathcal{I}^c}} \frac{\mathbf{m}_{l,l+1}\mathbf{m}_{m,m+1}}{\mathbf{m}_{i,j}}(s_m t_l - s_l t_m).$$

If $s_i t_j - s_j t_i \neq 0$, then the term of the sum where $l = i$ and $m = j$ is non-zero. In fact, this term is $\mathbf{m}_{i+1,j+1}(s_i t_j - s_j t_i)$. None of the other terms can possibly cancel with this term and so the sum is not contained in the ideal P. This is a contradiction. Therefore $\operatorname{codim}(B_3) \geq 3$ and so $\operatorname{codim}(I(\varphi_3)) \geq 3$. □

5. A Menagerie of Binomial Ideals

5.1. Specializations.

The family of ideals above have generic coefficients so for almost all specializations, the resolution is still exact. One wonders whether it is possible to specialize these coefficients to get binomial ideals tail resolution equivalent to $(x_1, \ldots, x_n)$. For projective dimension less than seven, it is possible as the following examples show.

Example 22 (Projective Dimension 4)**.** This example is the resolution for 4 variables with the specialization that $r_3 = r_4 = s_1 = s_4 = t_1 = t_2 = 0$, giving that the generators of the ideal are

$$\begin{aligned} \alpha &= r_1cd + r_2ad, \\ \beta &= s_2ad + s_3ab, \text{ and} \\ \gamma &= t_3ab + t_4bc, \end{aligned}$$

and the resolution of $R/(\alpha, \beta, \gamma)$ has the form

$$0 \to R \xrightarrow{d_4} R^4 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where

$$\varphi_1 = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \text{ and}$$

$$\varphi_2 = \begin{pmatrix} 0 & \gamma & \beta & * & * \\ \gamma & 0 & -\alpha & * & * \\ -\beta & -\alpha & 0 & * & * \end{pmatrix}.$$

The missing entries denoted by $*$ are polynomials of degree 2 in the variables and degree 2 in the coefficients.

$$\varphi_3 \quad = \quad \begin{pmatrix} r_2a{+}r_1c & r_1d & 0 & r_2d \\ -s_2a & 0 & -s_3a & -s_3b{-}s_2d \\ 0 & t_4b & t_3a{+}t_4c & t_3b \\ -b & 0 & d & 0 \\ 0 & a & 0 & -c \end{pmatrix}$$

Example 23 (Projective Dimension 5)**.** This example is the resolution for 5 variables with the specialization that $r_2 = r_3 = r_5 = s_1 = s_3 = s_4 = t_1 = t_2 = t_4 = 0$, giving that the generators of the ideal are

$$\begin{aligned} \alpha &= r_1cde + r_4abc, \\ \beta &= s_2ade + s_5bcd, \text{ and} \\ \gamma &= t_3abe + t_5bcd, \end{aligned}$$

and the resolution of $R/(\alpha, \beta, \gamma)$ has the form

$$0 \to R \xrightarrow{d_5} R^5 \xrightarrow{d_4} R^{10} \xrightarrow{\varphi_3} R^8 \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where

$$\varphi_1 \quad = \quad \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \text{ and}$$

$$\varphi_2 \quad = \quad \begin{pmatrix} 0 & \gamma & \beta & * & * & * & * & * \\ \gamma & 0 & -\alpha & * & * & * & * & * \\ -\beta & -\alpha & 0 & * & * & * & * & * \end{pmatrix}.$$

The missing entries denoted by $*$ are polynomials of degree 3 in the variables and degree 2 in the coefficients.

$$\varphi_3 \quad = \quad \begin{pmatrix} r_1c & r_1d & r_1e & 0 & 0 & r_4a & 0 & 0 & r_4b & r_4c \\ -s_2a & 0 & -s_5b & 0 & -s_5c & -s_5d & -s_2d & -s_2e & 0 & 0 \\ 0 & & t_5b & t_3a & t_5c & t_5d & t_3b & 0 & 0 & t_3e \\ -b & 0 & 0 & d & e & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & -c & 0 & e & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & -c & 0 & e & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & -c & -d & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & b & 0 & -d \end{pmatrix}$$

Example 24 (Projective Dimension 6)**.** This example is the resolution for 6 variables with the specialization that $r_2 = r_3 = r_5 = r_6 = s_1 = s_3 = s_4 = s_6 = t_1 = t_2 = t_4 = t_5 = 0$ giving that the generators of the ideal are

$$\begin{aligned} \alpha &= r_1cdef + r_4abcf, \\ \beta &= s_2adef + s_5abcd, \text{ and} \\ \gamma &= t_3abef + t_6bcde, \end{aligned}$$

and the resolution of $R/(\alpha, \beta, \gamma)$ has the form

$$0 \to R \xrightarrow{d_6} R^6 \xrightarrow{d_5} R^{15} \xrightarrow{d_4} R^{20} \xrightarrow{\varphi_3} R^{12} \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R$$

where

$$\varphi_1 \quad = \quad \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \text{ and}$$

$$\varphi_2 \quad = \quad \begin{pmatrix} 0 & \gamma & \beta & * & * & * & * & * & * & * & * & * \\ \gamma & 0 & -\alpha & * & * & * & * & * & * & * & * & * \\ -\beta & -\alpha & 0 & * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

The missing entries denoted by $*$ are polynomials of degree 4 in the variables and degree 2 in the coefficients. $\varphi_3 =$

$$\begin{pmatrix}
r_1c & r_1d & r_1e & r_1f & 0 & 0 & 0 & r_4a & 0 & 0 & 0 & 0 & 0 & r_4b & 0 & 0 & r_4c & 0 & 0 & -r_4f \\
0 & 0 & 0 & -t_6b & -t_3a & 0 & t_6c & 0 & t_6d & t_6e & -t_3b & 0 & 0 & 0 & 0 & 0 & -t_3e & t_3f & 0 & 0 \\
s_2a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_5a & s_2d & s_2e & -s_2f & 0 & 0 & -s_5b & 0 & 0 & -s_5c & -s_5d \\
-b & 0 & 0 & 0 & d & e & -f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -b & 0 & 0 & -c & 0 & 0 & e & -f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 & -c & 0 & -d & 0 & -f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & e & -f & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & -d & 0 & -f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & d & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & -d & 0 & -f & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & -d & -e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & c & 0 & -e
\end{pmatrix}$$

5.2. **Random Examples.**

For projective dimension 7, none of the possible simple specializations (that is, setting some of the coefficients to be zero) of resolutions above give a resolution with the desired tail. In fact, they do not even give a projective dimension 7 resolution. So we were lead to ask whether projective dimension 7 and higher binomial resolutions exist. Finding such resolutions turns out to be a daunting task. We searched for examples by checking millions of randomly produced 3-generated binomial ideals using Macaulay 2 [13]. The result was a number of examples having projective dimension 7 and a few having projective dimension 8. We display one of each below. The remainder may be found in [17].

Here is the projective dimension 7 ideal having the smallest number of variables and degrees of generators of the ones we found. It has 7 variables and generators of degree 8.

$$\begin{aligned}
\alpha &= a^3c^3df + ab^2ceg^3 \\
\beta &= ac^2dfg^3 + b^2e^2fg^3 \\
\gamma &= a^3b^2cde + b^3d^2efg
\end{aligned}$$

This ideal has degree 28, regularity 18 and its resolution has the form

$$0 \to R \to R^7 \to R^{22} \to R^{39} \to R^{39} \to R^{18} \to R^3 \to R.$$

Here is the projective dimension 8 ideal found having the smallest number of variables and degrees of generators. It has 10 variables and generators of degree 15.

$$\begin{aligned}
\alpha &= ac^3d^3f^2ghi^4 + bc^2d^3eg^2hi^3j^2 \\
\beta &= b^3c^2efg^2hi^3j^2 + a^2b^4d^2f^2ij^4 \\
\gamma &= a^3c^3def^3g^3h + a^4b^3ef^3ghij
\end{aligned}$$

This ideal has degree 103, regularity 40 and its resolution has the form

$$0 \to R \to R^{10} \to R^{42} \to R^{96} \to R^{130} \to R^{100} \to R^{35} \to R^3 \to R.$$

A projective dimension 9 3-generated binomial ideal has not yet been found. It is unknown whether or not it exists. Further searching could prove fruitful. Another approach, in the manner of Kohn's method in [14], would be to try to find a way to reduce the number of generators while preserving their binomial nature and the projective dimension of the ideal.

6. Further Directions

The method of this paper for finding ideals tail resolution equivalent to the ideal $(x_1, \dots, x_n)$ leads to a number of other questions about tail resolution equivalent ideals. For instance, are there conditions on an ideal that ensure that a 3-generated tail resolution equivalent ideal with monomial or binomial or certain degree generators exists? Is it always possible to find representatives of a tail resolution equivalence class which are generated by binomials? What about ones generated by monomials?

There are also open questions about the particular construction used to generate $\mathrm{G}(n)$. Is it possible to extend this method to all complete intersections or is there something special about the ideal of n variables? Perhaps understanding better the relation between the graphs and the ideals would lead to a more general method. We could also try starting with other graphs. Initial investigations into creating ideals from other graphs, however, were not promising. Also, what if we use this process for constructing 3-generated ideals on some other ideal and end up with a sequence that is not exact? Would the homology of this sequence tell us anything interesting about the ideal or the method?

The author thanks Mike Stillman for all his advice and patience while she was finishing her dissertation. She also thanks André Galligo and everyone in Nice for their kindness during her visit in which she started this project. Finally, thanks to Hal Schenck for encouragement and proofreading.

References

[1] A. Aramova and J. Herzog. Almost regular sequences and Betti numbers. *Amer. J. Math.*, 122:689–719, 2000.

[2] A. Aramova, J. Herzog. and T. Hibi. Ideals with stable Betti numbers. *Adv. Math.*, 162:72–77, 2000.

[3] N. Terai. Alexander duality theorem and Stanley-Reisner rings. *Sūrikaisekikenkyūsho Kōkyūroku*, 1078:174–184, 1999.

[4] W. Bruns. "Jede" endliche freie auflösung ist freie auflösung eines von drei elementen erzeuten ideals. *J. Algebra*, 39:429–439, 1976.

[5] W. Bruns. The Buchsbaum-Eisenbud structure theorems and alternating syzygies. *Comm. Algebra*, 15(5):873–925, 1987.

[6] W. Bruns and J. Herzog. *Cohen-Macaulay Rings.* Number 39 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1993.

[7] D. A. Buchsbaum and D. Eisenbud. What makes a complex exact. *J. Algebra*, 25:259–268, 1973.

[8] D. A. Buchsbaum and D. Eisenbud. Some structure theorems for finite free resolutions. *Adv. in Math.*, 12:84–139, 1974.

[9] T. De Alwis. Pascal type properties of Betti Numbers. *Internat.J. Math. Math. Sci.*, 17:545–552, 1994.

[10] T. De Alwis. Free Minimal Resolutions. *Comm. Algebra*, 21:4575–4585, 1993.

[11] J.A. Eagon and D. G. Northcott. On the Buchsbaum-Eisenbud theory of finite free resolutions. *J. Reine*, 262/263:205–219, 1973.

[12] E. G. Evans and P. Griffith. *Syzygies.* Number 106 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1985.

[13] D. R. Grayson and M. E. Stillman. Macaulay 2. Macaulay 2 homepage, `http://www.math.uiuc.edu/Macaulay2`.

[14] P. Kohn. Ideals generated by 3 elements. *Proc. Amer. Math. Soc.*, 35:55–58, 1972.

[15] V. Reiner and V. Welker. Linear Syzygies of Stanley-Reisner Ideals. *Math. Scand.* 89:117–132, 2001.

[16] C. A. Weibel. *An Introduction to Homological Algebra*. Number 38 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[17] L. Gold. *Some Homological Results in Commutative Algebra*. PhD thesis, Cornell University, 2002.

Gold: Mathematics Department, Texas A&M University, College Station, TX 77843-3368, USA

E-mail address: lgold@math.tamu.edu