MCE441: Intr. Linear Control Systems

Lecture 10: Proportional, Integral and Derivative Actions
Stability Concepts
BIBO Stability and The Routh-Hurwitz Criterion
Dorf, Sections 6.1, 6.2, 7.6

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PID Controllers

- Proportional-Integral-Derivative (PID) controllers are widely used in industrial process control.
- The controller has the transfer function

$$G_c(s) = K_p + \frac{K_i}{s} + K_d(s) = \frac{U(s)}{E(s)}$$

- The control law in the time domain is given by

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}$$

- Derivative term approximately implemented:

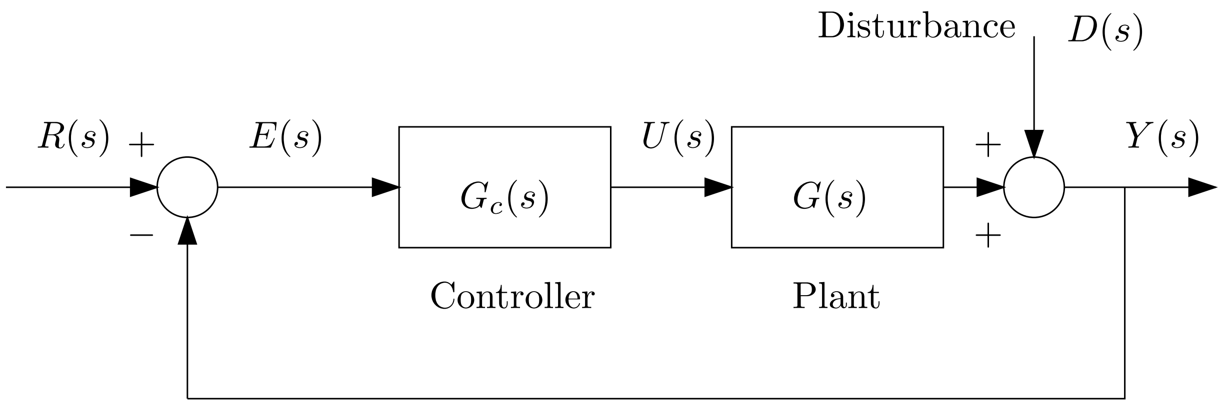
$$G_d(s) = \frac{K_d s}{\tau_d s + 1}, \quad \tau_d \to 0$$

PID Controllers...

- Analog PID boxes can be bought off-the shelf or custom built using R,L, C and op-amps.
- Digital PID can be bought off-the-shelf or programmed according to the application.
- Programmable Logic Controllers (PLC) come with a PID feature.
- PID controllers have limitations.

The Control Objective

The ultimate purpose of using control is to drive the error *nicely* to zero when $r(t)$ and $d(t)$ change in unanticipated ways and when $G(s)$ has uncertain and time-varying parameters and dynamics.



Example: Fly-By-Wire

- $R(t)$: Yoke input: Unanticipated function (the control algorithm cannot store a catalog of possible functions).
- $D(t)$: Turbulence, wind gust (same)
- Structural uncertainty in the plant (unmodeled dynamics): Vibration, friction, voltage drops, temperature effects, aeroelastic effects...
- Time-varying parameters: Weight changes due to fuel consumption, dropped bombs and paratroopers...

Error-Rejecting Props of P and I actions

Suppose only P-term is used: $u(t) = K_p e(t)$. If a steady-state is reached we have $u_\infty = K_p e_\infty$

$$\boxed{e_\infty \text{ does not have to be zero}}$$

Now suppose an I-term is used: $u(t) = K_i \int_0^t e(\tau) d\tau$. If a steady-state is reached, we have $u_\infty = K_i \int_0^t e_\infty d\tau = K_i e_\infty t$ which must be true for all t.

$$\boxed{\text{This is only possible when } e_\infty = 0.}$$

This powerful property of integral action is one of the key ideas used in controller design.

Common perceptions about P , I and D

- The proportional gain is usually associated with better disturbance rejection and faster reponse.
- Always true: **P Action alone does not guarantee a zero steady-state error**.
- The integral gain is usually associated with the ability to drive the error to zero, but at the expense of oscillations.
- Always true: **Unless the pole at $s = 0$ introduced by the I-term is cancelled by the plant, the steady-state error to a step input is zero**.
- The derivative gain is usually associated with dampened oscillations.

Our Approach

- P,I and D are frequently used in combination, and the above perceptions may not necessarily apply.
- We will take a rational approach and use available techniques (root-locus method for perfomance and stability, Routh-Hurwitz criterion for stability) to exactly determine the effects of P, I and D on a case-by-case basis.
- We will present an experimental PID tuning technique: Ziegler-Nichols

Stability Concepts

- Stability = Bounded Signals
- Many formal approaches to system stability exist. We'll consider one: BIBO (external) Stability.
- Bounded Input - Bounded Output (BIBO) stability: Used for transfer functions. The output must remain bounded whenever the input is bounded.
- No boundedness is required for internal states.
- Mathematically:

A transfer function is BIBO stable if and only if all of its poles have negative real parts

The Routh-Hurwitz Criterion

- The polynomial equation leading to the poles is called **characteristic equation**.
- Method Independently developed by A. Hurwitz and E.J. Routh in the late 1800's.
- Allows us to determine the number of roots of the char. eq. with positive real parts.
- It helps at the design stage, when the char. eq. may contain yet to be selected parameters. Then we can use the criterion to determine the ranges of the parameters that will result in a stable system.

Routh-Hurwitz Method

- Remove all roots at $s = 0$, so that $a_n \neq 0$ in the characteristic equation:

$$a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0$$

- Necessary (not sufficient) condition: Coefficients must be all positive or negative. If not satisfied, the system is not stable: Stop unless asked for the number of roots on the rhp.
- If necessary, make the coefficients positive by multiplying the equation by -1.
- Complete the Routh array and look for sign changes in the first column.

The Routh Array

$$\begin{array}{l|c|lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \dots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \dots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

"The number of roots on the r.h.s. equals the number of sign changes in the first column"

Example

Determine the number of roots on the r.h.p. with the Routh-Hurwitz criterion. Check by numerical computation.

$$2s^6 + 3s^5 - s^3 + s^2 + 4s = 0$$

Results for 2nd and 3rd -order Systems

Consider

$$a_0 s^2 + a_1 s + a_2 = 0$$

with $a_i > 0$. The Routh-Hurwitz criterion shows that the condition $a_i > 0$ for $i = 0, 1, 2$ is also *sufficient*.
Now consider

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

with $a_i > 0$. The criterion shows that $a_1 a_2 > a_0 a_3$ guarantees stability.

Shortcuts and Special Cases

Shortcut: Multiplying an entire row by a positive number does not alter the results. Use it to simplify calculations. Two things can happen:

- Special Case A: A first-column term in any row is zero, with either the remaining terms on the row not all zero, or no remaining terms. Then, a pair of purely imaginary roots *may* exist.
- Special Case B: All elements of a given row are zero. Then there are at least two real roots with opposite signs and/or two complex conjugate roots. Call them (real or complex) *symmetric pair of roots*.
- But we're interested in discovering possible roots on the r.h.p...

Special Case A: Carrying On

- Substitute the zero by small positive quantity ε.
- Complete the array in terms of ε.
- If the sign of the element above ε is the same as the one below it, then there **is** a pair of purely imaginary, conjugate roots.
- If the sign of the element above ε is opposite to the one below it, then there is a sign change.

Special Case A: Examples

Consider $s^3 + 2s^2 + s + 2 = 0$ and $s^3 - 3s + 2 = 0$.

Special Case B: Carrying On

- Form an auxiliary polynomial $P(s)$ with the coefficients of the previous row.
- If the symmetric roots are sought, solve $P(s) = 0$.
- The zero row is replaced by the coefficients of $\frac{dP(s)}{ds}$.
- Complete the array and look for sign changes in the first column.

Special Case B: Example

Consider $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$.

Additional Information

- Are purely imaginary roots always detected by the method?
- Answer: Yes, but $P(s)$ may need to be solved.
- For large orders, the above defeats the purpose of the method.
- The method is useful in the determination of safe parameter ranges (stability). See next example.

Example

The characteristic equation of an aircraft roll control system is

$$s^3 + 11.4s^2 + 14s + 14.4k = 0$$

Determine the value of k at which the system becomes unstable.