Motivation

- The transient response of a control system is dictated by the location of the closed-loop poles and zeros.
- A compensator (controller) is used to alter the pole-zero structure according to needs.
- Two levels of design:
  1. Find the appropriate control gains given a controller structure
  2. Choose an appropriate controller structure (add zeros and poles)
- The method introduced by W.R. Evans is a systematic approach.
Characteristic Equation and Root Locus

The characteristic equation of a feedback system is given by

\[ 1 + G(s)K(s) = 0 \]

where \( K(s) \) may be placed in the feedback path or in the forward path (MCE441 notes and textbook). We will concentrate on the single-gain root locus. Write the char.eq. as

\[ 1 + K \frac{(s + z_1)(s + z_2)...(s + z_m)}{(s + p_1)(s + p_2)...(s + p_n)} = 0 \]

The root locus is a graph showing the location of the poles and zeros of the closed-loop TF as the gain \( K \) is continuously varied from 0 to \( \infty \).

Obtaining a Root Locus

- Find the char. eq. and manipulate the expression to isolate the tunable gain:

\[ 1 + K \frac{(s + z_1)(s + z_2)...(s + z_m)}{(s + p_1)(s + p_2)...(s + p_n)} = 0 \]

- Choose between hand-sketching or Matlab.
- Hand-sketching: This skill provides great insight and simple sketches can be quickly obtained.
- Matlab: For complex systems and to fine-tune the gain. Also good for automated design.
Root Locus Sketching Rules - Level I

- Plot the open-loop poles and zeros. **The zeros stay put:** Changing the gain will not change the location of the zeros.
- The root locus is always symmetrical with respect to the real axis.
- The portions of locus in the real axis lie to the left of an odd number of poles and zeros.
- \( m \) branches start at the open-loop pole locations \( (K = 0) \) and end at the \( m \) zeros \( (K = \infty) \). The remaining \( (n - m) \) poles escape to infinity along asymptotes.
- Crossings of the imaginary axis may be found using the Routh-Hurwitz criterion.

Example 1

Sketch the root locus for the following control and plant transfer functions:

\[
G(s) = \frac{s + 3}{s - 1}, \quad K(s) = K \frac{s + 1}{s + 2}
\]
Example 2

Obtain a Level-I root locus for the following control and plant transfer functions:

\[ G(s) = \frac{s + 3}{s - 1}, \quad K(s) = K \frac{1}{s + 2} \]

Example 3

Obtain a Level-I root locus for the following char.eq.:

\[ s^4 + 4s^3 + 8s^2 + (8 + k)s + 4k = 0 \]
Root Locus Sketching - Level II

- Find the $n - m$ asymptote angles in degrees with
  \[
  \phi_A = \frac{180(2k + 1)}{n - m} \quad k = 0, 1, 2...n - m - 1
  \]

- Find the center of asymptotes on the real axis with
  \[
  \sigma_A = \frac{\sum_{j=1}^{n} (-p_j) - \sum_{i=1}^{m} (-z_i)}{n - m}
  \]

- Find the break-in/breakaway points:
  1. Solve for $K$ from the char. eq: $K = p(s)$.
  2. Solve:
     \[
     \frac{dp(s)}{ds} = 0
     \]
Root Locus Sketching - Level II

- Find the angles of departure from complex poles /arrival at complex zeros:

\[ \text{angle}(G(s)K(s)) = 180 \pm k360 \quad \text{at} \quad s = p_j \text{ or } z = z_i \]

- Find all crossings with the imaginary axis

- To evaluate the gain at a given location \( s_1 \), substitute the value of \( s_1 \) into the char. eq:

\[
1 + K|G(s_1)K(s_1)| = 0
\]

Example

Sketch a Level II Root Locus for \( G(s)K(s) = \frac{K}{s(s^2 + 6s + 25)} \)
Example

Sketch a Level II Root Locus for $G(s)K(s) = \frac{K(s+0.4)}{s^2(s+3.6)}$

Solution
Solution

**Root Locus using Matlab**

- **Basic command:** `rlocus(num, den)`
- `num` and `den` are the numerator and denominator of $G(s)K(s)$ excluding the tunable gain $K$.
- Current version has interactive “point and get” capabilities for obtaining the roots and gains at specific locations
- The `rlocfind` command can be used to find matching gains and poles:
  
  ```matlab
  [k, poles] = rlocfind(sys, p)
  ```
  
  The closest match is returned when the locus does not go through the requested pole location(s).