Relative Stability Measures

- If it is often useful to quantify the “degree of stability” of a system.
- A closed-loop system may go unstable by adding phase lag or by increasing the gain.
- Two practical measures are introduced:
  - The gain margin is the factor by which the gain has to be multiplied for the system to go unstable.
  - The phase margin is the amount of extra phase lag that makes the system unstable.
- Little phase margin is also an indication of poor transient response (high overshoot).
Margins in Bode and Nyquist

Bode and Nyquist plots of $G(s)K(s)$

Finding Margins

- **Note**

  $$GM = \frac{1}{|G(jw)|} \quad \text{when} \quad \angle G(jw) = -180 \pm 360k, \quad \text{for} \quad k = 0, \pm 1, \pm 2...$$

  There is only one possible $k = k_1$ satisfying the above equation.

  $$PM = \angle G(jw) - (-180 \pm 360k_1) \quad \text{when} \quad |G(jw)| = 1$$

- **That is,** the GM is obtained at phase crossover and the PM is obtained at gain crossover.

- **In Matlab:**
  
  
  $\text{[GM,PM,Wcg,Wcp]}=\text{margin(num,den)}$

- num and den are the coefficients of the open-loop TF. The margins are returned along with the associated crossover frequencies. The GM is NOT returned in dB. The PM is returned in degrees.
Choose $K$ so that the phase margin is $60^\circ$. What is the gain margin?

$$G(s)K(s) = K \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$$

Take $w_n = 1, \quad \zeta = 0.7$

Solution
Steady-State Error: Remember that the ss error depends on the value of $G(s)K(s)$ for $s \to 0$. The larger $G(0)K(0)$, the smaller the sse. In the frequency domain this implies having a high gain at low frequencies. We can make the following statement:

To obtain zero steady-state error for a type $N$ input, the slope of $G(s)K(s)$ when $w \to 0$ must be $-20N \text{ dB/dec}$. If the slope is zero, with a magnitude of $\bar{K}$, then the sse for a step input is $\frac{1}{\bar{K}}$

Overshoot: The closed-loop system becomes more oscillatory as the phase margin decreases (approaching the stability limit). A phase margin between $30^\circ$ and $60^\circ$ results in overshoots between 35 and 10 percent.

Rule-of-Thumb: $\zeta \approx \frac{PM}{100}$

Or use this chart (also, Dorf Fig. 9.21 (11th Ed))
Frequency Domain Specifications

**Overshoot:** The above criteria is based on a pair of complex poles which are dominant. An alternative criterion often used in practical design is the following rule-of-thumb:

The slope of the magnitude at crossover frequency should be close to -20 dB/dec

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Frequency Domain Specifications

**Settling Time:** We can obtain settling time information from the frequency domain by considering the crossover frequency and bandwidth:

- The closed-loop bandwidth is the frequency at which the magnitude has declined 3dB from its low-frequency value. It applies to the closed-loop transfer function. The following applies for $0.3 \leq \zeta \leq 0.8$:

$$w_b = (-1.196\zeta + 1.85)w_n$$

- The crossover frequency $w_c$ is the frequency at which the magnitude crosses the 0dB line. It applies to the loop transfer function $G(s)K(s)$. 


Frequency Domain Specifications

Rules-of-Thumb for Design:

\[ w_c \approx w_b / 1.6 \]

this is valid for \( \zeta \) between 0.3 and 0.8. When a settling time/overshoot specification is given, proceed as follows:

1. Determine the appropriate damping ratio \( \zeta \) and natural frequency \( w_n \) assuming 2nd order dominant poles.
2. Calculate \( w_b \approx (-1.196\zeta + 1.85)w_n \). The target crossover frequency for \( G(s)K(s) \) will be \( w_c = w_b / 1.6 \).

Frequency Domain Specifications

**High-Frequency Roll-Off:** Given in dB/dec, it refers to the slope of \( G(s)K(s) \) for high frequencies. A high roll-off rate is desirable for noise attenuation and robustness against neglected dynamics. The desired roll-off rate is determined from the characteristics and intensity of the noise and from dynamic uncertainty analysis. Physically, by introducing a high roll-off rate, we avoid exciting higher modes of oscillation by limiting the frequency content of the control signal. Additional specifications can be introduced according to expected disturbances and their frequency range.
Example

Specifications:

- Zero steady-state error for step inputs.
- Overshoot less than 35 percent.
- Settling time less than 0.2 sec.
- Noise attenuation: 0.005 for $\omega \geq 300$ rad/sec.

Represent the design requirements in the frequency domain.
Solution
Design in the Frequency Domain

■ Regardless of the method to accomplish it, **loop shaping** is the design methodology.

■ Classical way: lead, lag and lead-lag compensators. These compensators have a fixed structure, motivated by electrical circuits that materialize it.

■ Modern way: $\mathcal{H}_\infty$ synthesis. This is very recent technique which automatically generates a loop shape using optimization. The controls designer is still responsible for selecting appropriate frequency weights that guide the optimization process.

■ Manual loopshaping: Add zeros and poles to bend the loop in the desired way. A good alternative when software is available.

Manual Loopshaping

■ Basic restrictions: 1) Causality: no more zeros than poles; 2) Internal stability: no unstable poles in the controller, even if they get cancelled by a nonminimum-phase zero of the plant.

■ Add zeros to improve the phase at magnitude crossover (correct the PM). However, the magnitude may tend to increase at phase crossover (may spoil the GM). Zeros cannot be added alone due to causality restrictions. A pole must be introduced. When the pole is to the left of the zero in the complex plane, we have a **lead** structure. A zero may be also added to improve the slope at crossover.
Add poles to improve the roll-off rate and to attenuate resonances beyond the desired bandwidth. Poles are also used to maintain causality. When the pole is to the right of the zero, we have a lag structure. The difficulty lies in that each added pole introduces detrimental phase lag.

Use the gain to make final adjustments to the bandwidth and stability margins.

Basic Example

Tune the proportional gain for the plant

\[ G(s) = \frac{1}{s(s+1)(s+2)} \]

using the specifications:

- Overshoot: 35%
- Settling Time: 16 sec.
- Zero steady-state error for step inputs.
Advanced Example

Design a compensator for the previous plant with more stringent specifications:

- Overshoot: 5%
- Settling Time: 1 sec.
- Zero steady-state error for ramp inputs.
The Nichols Chart

The Nichols chart is useful in predicting the magnitude and phase of the closed-loop system from those of the open-loop TF. It is particularly useful for finding the closed-loop bandwidth (which relates to the speed of response) and the maximum magnitude peak (which relates to the damping ratio and overshoot). It also helps in adjusting the open-loop gain so that a target bandwidth and/or overshoot are attained.

Nichols Chart
Example

Determine the closed-loop bandwidth and phase margin for the following open-loop transfer function for $K = 1$. What is the value of $K$ so that the phase margin is now $45^\circ$? What happens to the bandwidth?

$$G(s)K(s) = \frac{K}{s(s^2 + 0.4s + 1)}$$

Solution