MCE441: Intr. Linear Control Systems

Lecture 5: The Laplace Transform

Dorf, Sect. 2-4.

Cleveland State University
Mechanical Engineering
Hanz Richter, PhD

Instructor

Motivation

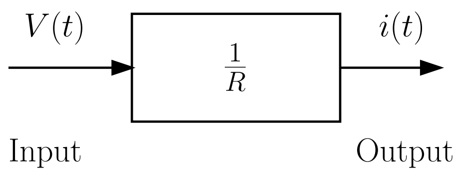
- Linear dynamic systems are often represented as I/O differential equations:

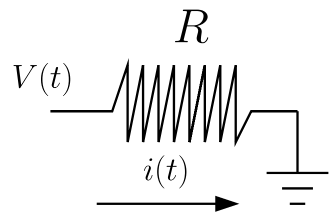
$$m\ddot{x} + b\dot{x} + kx = F$$

- An output is usually defined as a function of the system variable and its derivatives. We can define, for instance, $y = x$, $y = \dot{x}$ or other functions.
- We would like to have a more convenient, black-box representation, where the input is simply multiplied by an appropriately defined quantity to give the output.
- The idea is to imitate the linear scaling that occurs in non-dynamic systems.

Systems as Operators

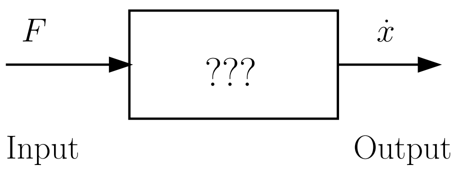
Non-dynamic (static) system

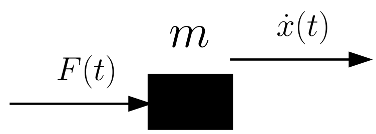




$i(t) = \frac{1}{R}V(t)$

Dynamic system





$m\frac{d}{dt}(\dot{x}(t)) = F(t)$

Laplace Transform

- Defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

- s lives in $\mathbb{C}$.
- $f(t)$ needs to be transformable: $\int_0^\infty |f(t)|e^{-\sigma_1}dt < \infty$. Our $f(t)$'s will, no need to check.
- Inverse:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

- We don't need to carry out the integrations. Just use a table of Laplace transform pairs.

Useful Properties

- 1. Linear operator:

$$\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha\mathcal{L}\{f_1\} + \beta\mathcal{L}\{f_2\}$$

- 2. Transform of a derivative:

$$\mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - s^{k-1}f(0) - s^{k-2}f'(0)\ldots.. - f^{(k-1)}(0)$$

- In most cases the initial conditions are zero: $f(0) = f'(0) = f''(0)\ldots. = 0$, so

$$\boxed{\mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s)}$$

- This property is the basis for operational calculus. Replaces derivatives by powers of s, reducing differential equations to algebraic equations in the Laplace domain (s variable).

Useful Properties

- 3. Transform of an integral:

$$\boxed{\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}}$$

- The above property is valid for functions such that $f(t) = 0$ for $t < 0$ (OK with us).

Example: If we have $3\ddddot{y} + 2\ddot{y} + y = 3\dot{x}$, taking the transform gives

$$3s^4Y(s) + 2s^2Y(s) + Y(s) = 3sX(s)$$

which allows to solve for Y in the s-domain. If $x(t)$ is given, we find $X(s)$ from a table. Then solve for $Y(s)$ and find the inverse in the table to get $y(t)$, if so desired.

Summary

In summary, the Laplace transform is useful to:

- Decouple O.D.E.'s and eliminate unwanted variables
- Obtain time solutions of O.D.E.'s (rarely needed in control system analysis)
- **Determine the dynamic properties of a system: stability, transient response, sensitivity, etc. without solving the O.D.E.'s**

We should identify s with the differentiation operator

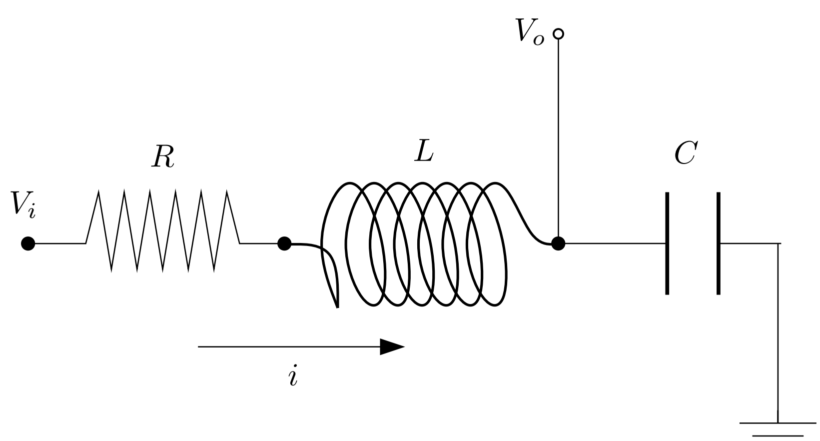
$$\boxed{s \equiv \frac{d}{dt}}$$

and $\frac{1}{s}$ with the integration operator

$$\boxed{\frac{1}{s} \equiv \int dt}$$

Example

Let's use the Laplace transform method to find the I/O diff. eq., from V_i to V_o (eliminate i):



Example

Using the tables

Use the Laplace transform to solve the O.D.E.

$$\ddot{y} + 5\dot{y} + 6y = 2$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = -2$.

Using the tables

Solution: $y(t) = \frac{1}{3}(1 + 2e^{-3t})$. In general, the inversion of $Y(s)$ requires a partial fraction expansion (PFE): Next lecture.