MCE441: Intr. Linear Control Systems

Lecture 6: The Transfer Function
Poles and Zeros
Partial Fraction Expansions

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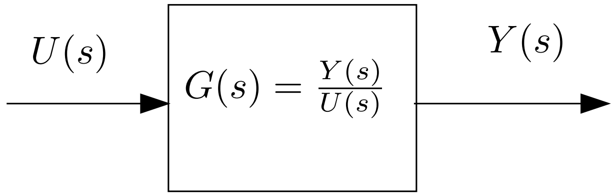
System Transfer Function

- In MCE441, we work with Input/Output systems (black boxes)
- We have learned how to represent the I/O dynamics of electromechanical systems with O.D.E.'s.
- Transfer Functions carry the same information as the I/O diff. eq. in a more convenient format
- Definition:

The transfer function of a system is the ratio of Laplace transforms of output and input, with all initial conditions set to zero.

Transfer Function Definition



- Does not provide information about internal system structure (nor does the I/O ODE)
- E X T E N S I V E L Y used for studying linear system properties and for design
- It equals the Laplace transform of the **impulse response** of a system
- May be experimentally obtained (hammering steel tanks to find cracks)

Example: Galvanometer Model

Let's decouple the galvanometer equations and find the T.F. from applied voltage (V_i) to angular displacement (θ)

Solution

Check that the required transfer function is

$$G(s) = \frac{\Theta(s)}{V_i(s)} = \frac{\alpha}{(Ls^2 + R)(Js^2 + bs + k) + \alpha^2 s}$$

Poles and Zeros

Transfer functions of finite-dimensional (lumped-parameter systems) are always rational functions (ratio of polynomials) of s:

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

- Poles: Roots of the denominator
- Zeros: Roots of the numerator
- **Poles and zeros are the fundamental indicators of dynamic response and stability**
- Factored form:

$$\boxed{G(S) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}}$$

Poles and Zeros in Matlab

- We limit ourselves to *causal* systems: system order$= n \geq m$.
- Matlab example: Find the poles and zeros of the TF:

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{s^4 + 3s^2 + 2s + 1}$$

- ```
  >>num=[1 5 9 7];
  >>den=[1 0 3 2 1];
  >>roots(num) %Matlab comes back with the ans
  >>roots(den) %Matlab comes back with the ans
  ```
- Alternatively, to find the gain as well:
  `>>[z,p,k]=tf2zpk(num,den)`

PFE Decomposition-Real Poles

Suppose $G(s) = \frac{n(s)}{d(s)}$, with $G(s)$ strictly proper ($n > m$):

- To every simple real pole at $s = p$ there corresponds a single term of the form

$$\frac{a}{s-p}$$

  where

$$a = \lim_{s \to p} (s-p)G(p)$$

- To every multiple real pole of multiplicity $k \geq 2$ there correspond k terms:

$$\frac{c_1}{s-p} + \frac{c_2}{(s-p)^2} + \dots + \frac{c_k}{(s-p)^k}$$
  ```

PFE Decomposition-Complex Conjugate Poles

- To every simple pair of complex conjugate poles at $s = \alpha \pm \beta i$ there correspond two terms of the form

$$\frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B}{(s-\alpha)^2+\beta^2}$$

- The quantities a, c_i, A and B are called residues.
- Why by hand? : For complex comjugate case, Matlab gives complex residues.
- We'll skip the case of multiple complex conjugate poles. Rarely found and may be solved using Matlab.

Obtaining the PFE - Recipe

- Find the poles and characterize them as real or complex, single or multiple.
- Determine the structure of the decomposition.
- If single real poles are found, find the residues now.
- Multiply through and equate the coefficients of the numerator to those of the original TF.
- Solve a linear system of equations.
- NOTE: If you found some residues for single real poles, you will have more equations than unknowns. Use the extra equations for verification as an alternative to recombining the PFE.

Examples

Decompose the following TFs into partial fractions:

1. $G(s) = \frac{2}{(s+1)(s+2)}$
2. $G(s) = \frac{2}{(s+1)^3(s+2)}$
3. $G(s) = \frac{(s+1)}{(s^2+2s+4)(s+2)}$

PFE in Matlab

When the TF is proper (but not *strictly* proper), that is, $n \geq m$, we need to perform polynomial division to obtain $G(s) =$constant + strictly proper TF. Matlab finds the residues, poles and multiplicities in one step

```
[r,p,k]=residue(num,den)
```

NOTE: There exist formulas for finding the residues directly. But overall, the method presented involves less computational work.