Example on Root Locus Sketching and Control Design

MCE441 - Spring 05
Dr. Richter

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The following figure represents the system used for controlling the robotic manipulator of a Mars Rover. We wish to:

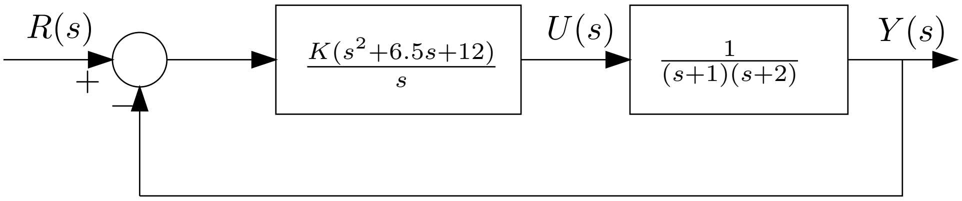

Figure 1: Control System

1. Provide a detailed hand-sketch of the root locus.
2. Find the gain K so that the system responds with an overshoot of 1 percent.
3. Find the gain K so that the system responds with an overshoot of less than 1 percent and a settling time as fast as possible.

Root Locus Sketching
The poles of the open-loop transfer function $G(s)K(s)$ are given by

$$s = 0, \quad s = -1, \quad s = -2$$

and there is a pair of complex zeros at

$$s = -3.25 \pm 1.199i$$

The order of the system (number of poles) is $n = 3$ and the number of zeros is $m = 2$. According to this, $m = 2$ poles go to the zeros as $K \to \infty$ and the remaining pole goes to infinity following an asymptote. The angle of the asymptote is obviously 180°, but in general, we find asymptote angles with the formula

$$\phi_A = \frac{180}{n-m}(2k+1)$$

where $k = 0, 1, ..n - m - 1$. In our case, $n - m - 1 = 0$, so

$$\phi_A = 180^\circ$$

Again, in this case it is unnecessary to find the center of the asymptotes, but in the general case it is obtained by subtracting the sum of zeros from the sum of poles and dividing by $n - m$. In this case

$$\sigma_A = \frac{(-1-2) - (-3.25 + 1.199i - 3.25 - 1.199i)}{1} = 3.5$$

With this information, we still have two possibilities for the shape of the root locus that comply with the basic requirements, as shown in the figure below. In order to determine which one of the possibilities is

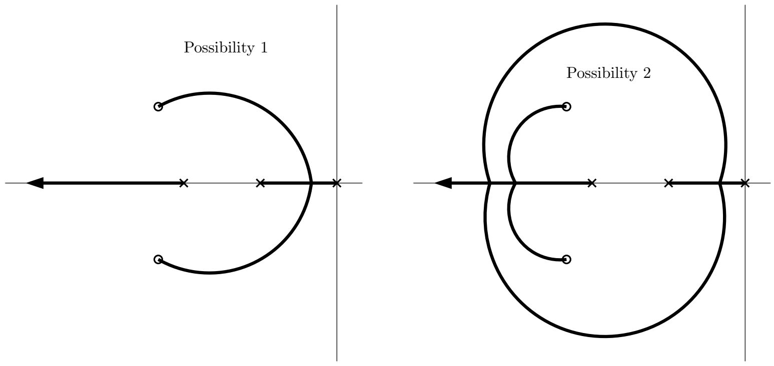

Figure 2: Two possibilities for the root locus

correct, we find the break-in and break-away points. To do this, solve for K from the characteristic equation and call the result $p(s)$:

$$1 + K\frac{s^2 + 6.5s + 12}{s(s+1)(s+2)} \rightarrow K = -\frac{s(s+1)(s+2)}{s^2 + 6.5s + 12} = p(s)$$

Now differentiate $p(s)$ and equate it to zero to find maxima:

$$\frac{dp(s)}{ds} = -\left(\frac{(s^2 + 6.5s + 12)(3s^2 + 6s + 2) - (2s + 6.5)(s^3 + 3s^2 + 2s)}{(s^2 + 6.5s^2 + 12)^2}\right) = 0$$

which reduces to the fourth-order polynomial equation:

$$s^4 + 13s^3 + 53.5s^2 + 72s + 24 = 0$$

with roots

$$s = -5.708, \quad s = -5.1407, \quad s = -1.6580, \quad s = -0.4933$$

Since the point $s = -1.6580$ is not part of the locus, we discard it. The other three points are valid, so we can determine that the sketch on the right of Fig. 2 is the correct one. Although it is not necessary here, we will show how to determine the angles of arrival to the complex zeros. The angle of arrival at a given zero is calculated with

$$\theta_a = 180^\circ + \sum \text{angles from poles} - \sum \text{angles from the other zeros}$$

The following schematic illustrates the calculation: The angle of arrival is calculated as

$$\theta_a = 180 + 159.75 + 151.95 + 136.19 - 90 = 537.89$$

This result must be reduced by an integer number of turns, however. The angle can be expressed as

$$537.89 = 1 \times 360 + 177.89$$

so that 177.89° is the meaningful arrival angle. The root-locus can be obtained in one step by using Matlab:

```
>> num=[1 6.5 12];
>> den=[1 3 2 0];
>> rlocus(num,den)
```

Now we determine the gain K so that the step response has an overshoot of 1 percent. To do this, we notice that the corresponding damping ratio is $\zeta = 0.82$. The angle of the dominant poles must then be $\phi = \cos^{-1}(0.82) = 0.6094$ radians, which is 34.9152°. We draw a line forming that angle with the negative

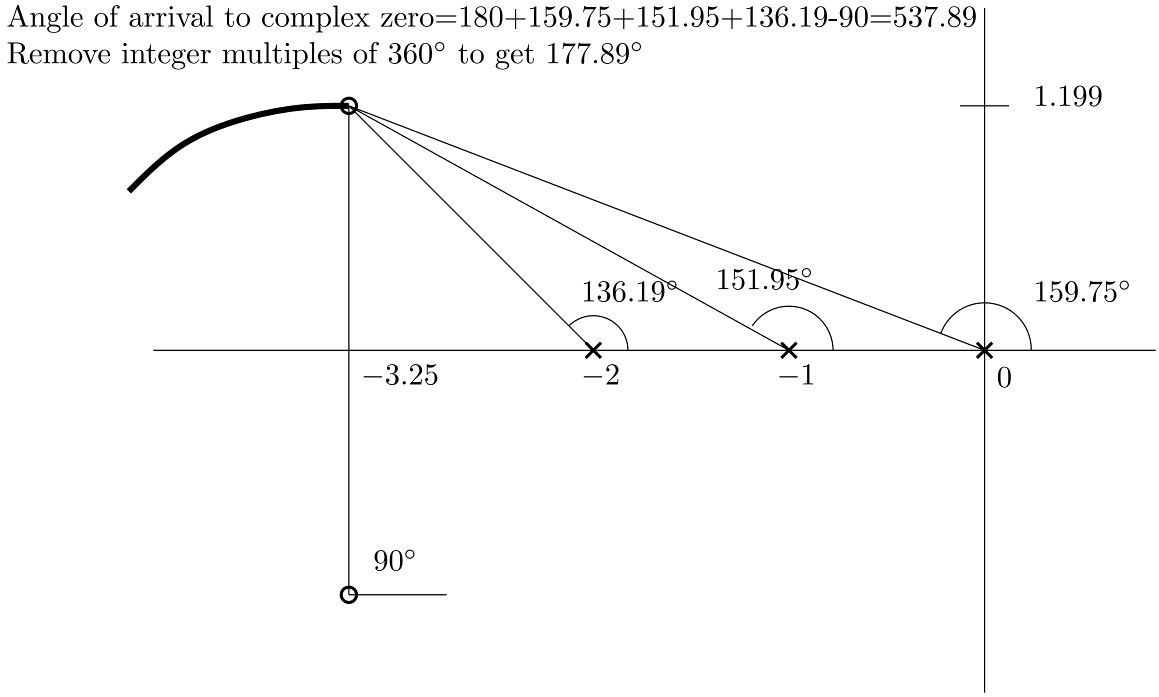

Figure 3: Calculating the angles of arrival and departure

real axis and look for intersections with the root locus. There are two intersections. We can find the gain and poles of the intersections interactively with Matlab. The intersections are

$$s = -4.7 \pm 3.29i, \text{ at } K = 10.1$$

$$s = -0.492 \pm 0.344i, \text{ at } K = 0.0624$$

However, for $K = 10.1$ no pole dominance occurs. Indeed, the other root lies at $s = -3.67$, very close to the complex pair. In addition, the zeros are expected to have an influence, as they have a relatively large time constant. For $K = 0.0624$, the other pole is very close to its open-loop location of -2. This pole and the complex pair of zeros have time constants of 1/2=0.5 and 1/3.25=0.307, while the complex pair of poles have a time constant of 1/0.492=2.0325. The complex poles are 4 times slower than the zeros and 6.5 times slower than the real pole. Dominance is not achieved according to the factor of 8 criterion, but we can expect some approximation to the 1 percent overshoot. Figure 5 shows the responses with approximately 1 percent overshoot and a slow response when $K = 0.0624$, and 15 percent overshoot for $K = 10.1$. Now we wish to select a gain that has a low overshoot but that produces a fast response. By examining the root locus, we see that as the gain increases, the closed-loop complex poles approach the zeros, while the other root stays real and moves to the left. At $K = \infty$, the complex poles match the zeros and a *stable pole-zero cancellation* occurs. When we design control systems, pole-zero cancellations happening on the left-half complex plane are allowed, while those happening on the imaginary axis or on the right-half of the complex plane are not. The reason for this is as follows: If in order to cancel a pair of right-half plane zeros of the plant we introduce a pair of matching right-half plane poles, we have a stable closed-loop system but an unstable controller. Physically, very large electrical signals or mechanical displacements will occur in the elements realizing the control, leading to system failure. In our case the pole-zero cancellation at high gain happens on the left-half plane, so using a high gain is not ruled out. The actual value of the gain to attain the desired overshoot is best found by trial-and-error simulation in Matlab. The value $K = 300$ places the poles at

$$s = -296.46, \quad s = -3.27 \pm 1.20i$$

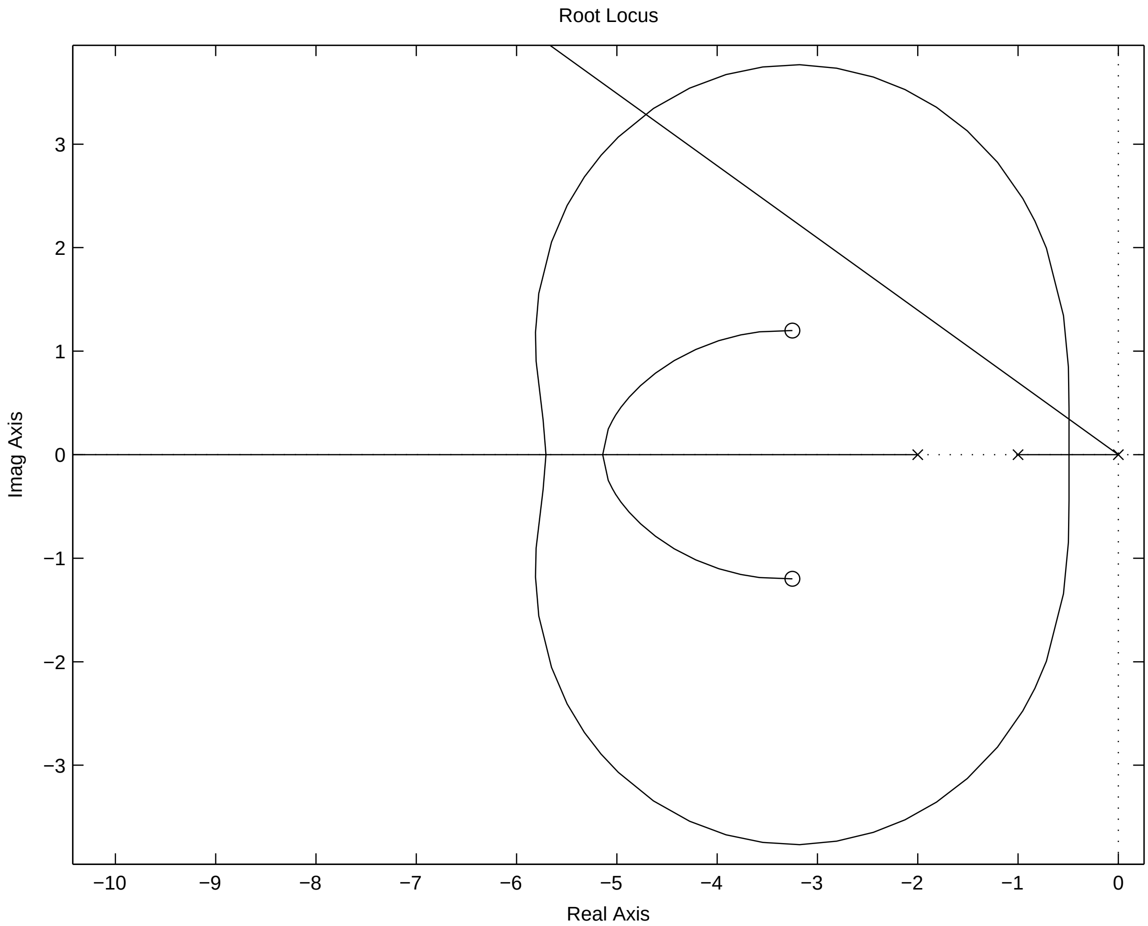

Figure 4: Root Locus and Constant Damping Line

The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = 300\frac{(s-3.25+1.199i)(s-3.25-1.199i)}{(s-3.27+1.199i)(s-3.27-1.199i)(s+296.46)}$$

Note that the complex poles and the zeros almost cancel each other. Figure 6 shows the resulting step response. As a final observation, the saying *there's no free lunch* applies here. Such a high performance is obtained at the cost of very high control effort. A direct simulation can show that output of the controller for $K = 300$ is four orders of magnitude larger than that for $K = 0.0624$. A realistic controller design must take into account limitations in the control magnitude.

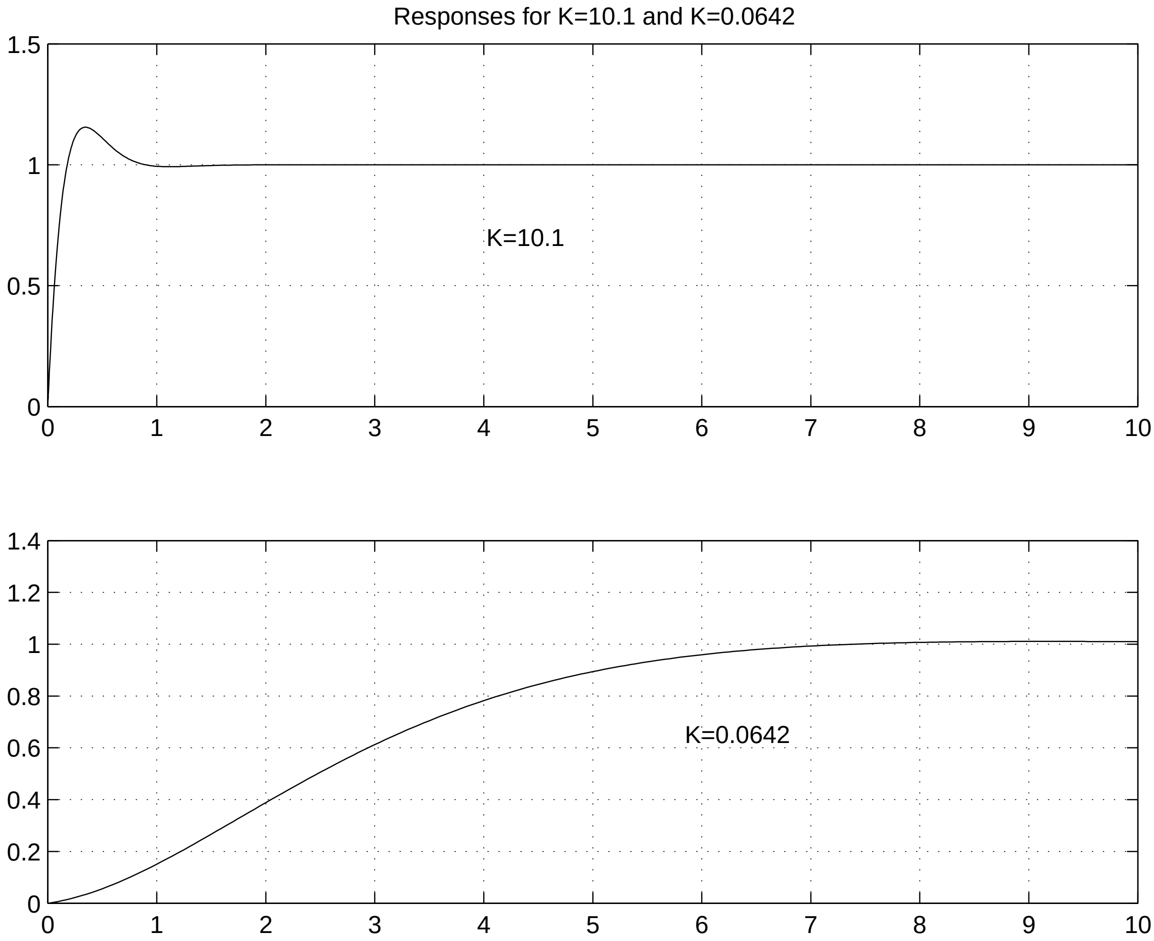

Figure 5: Responses for 1 percent overshoot

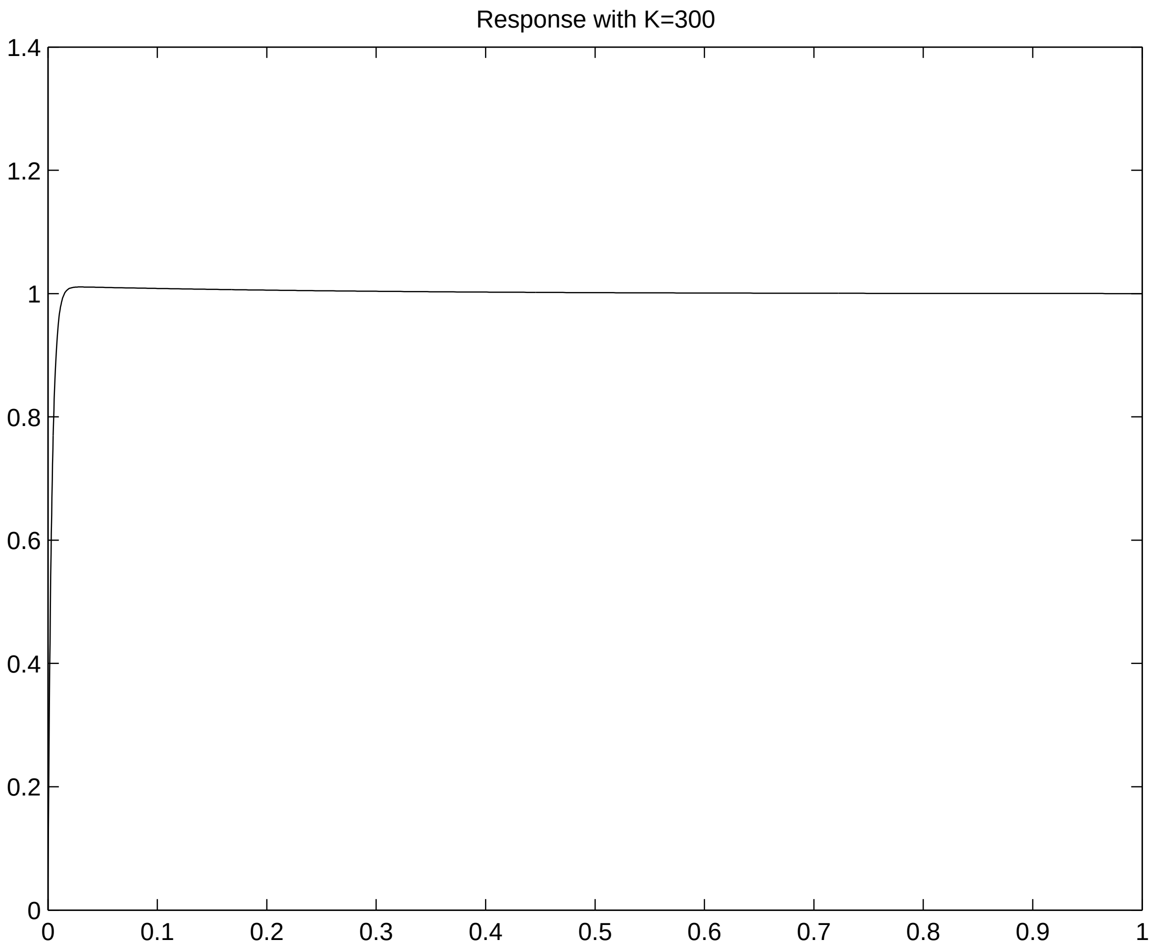

Figure 6: Final Response