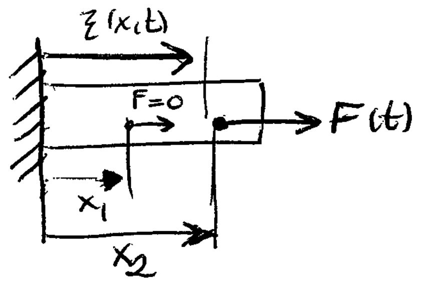
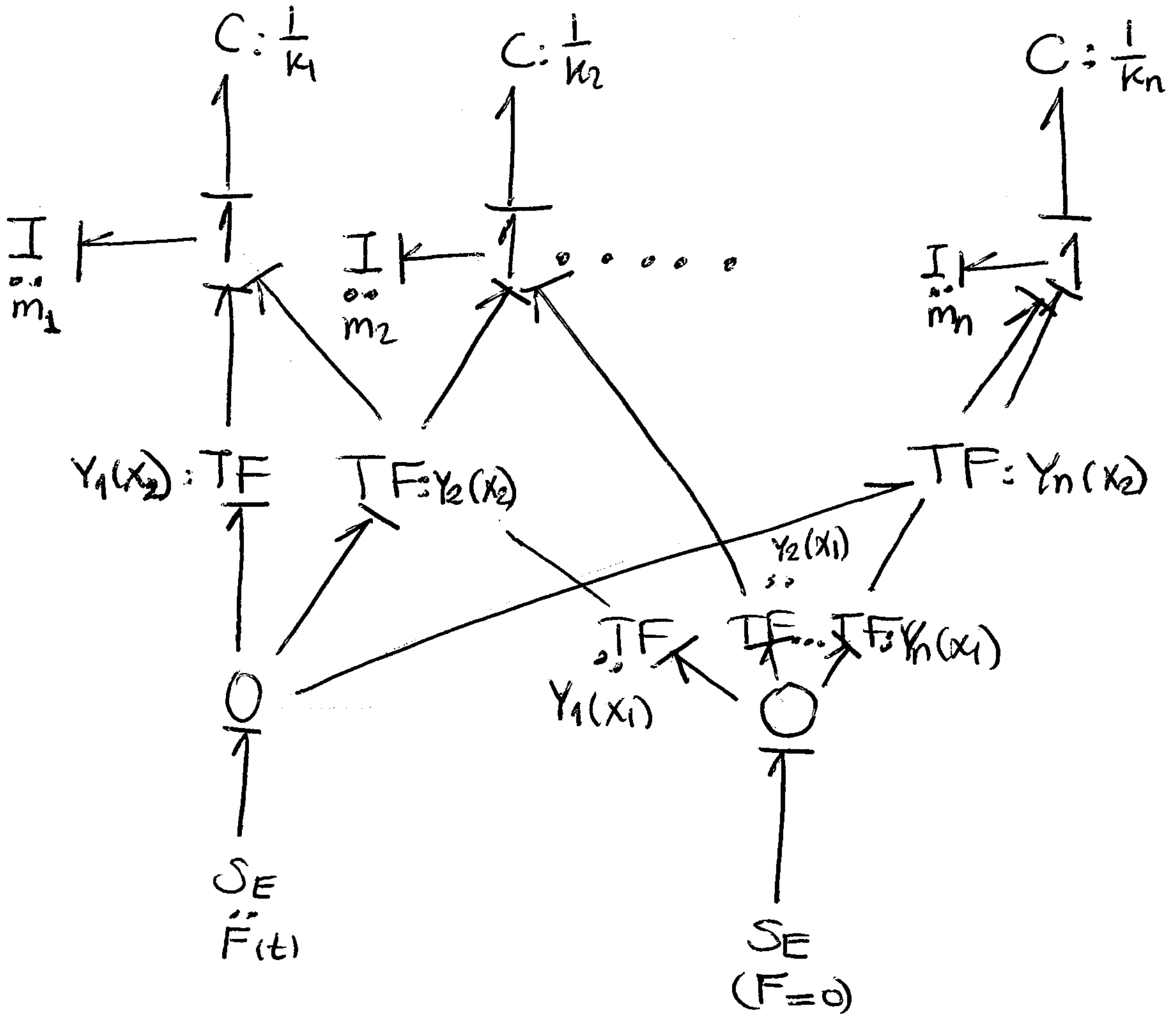
More on finite-mode modeling of distributed beams.

- Obtaining output at any point; longitudinal case
  Just include a force input equal to zero at the desired location:



– Force acting at $x = x_2$
– Flow output desired at $x = x_1$



The bond graph quickly gets messy with increased number of modes or outputs.

See also KMR, Fig. 10.13

Instead, we can simply write

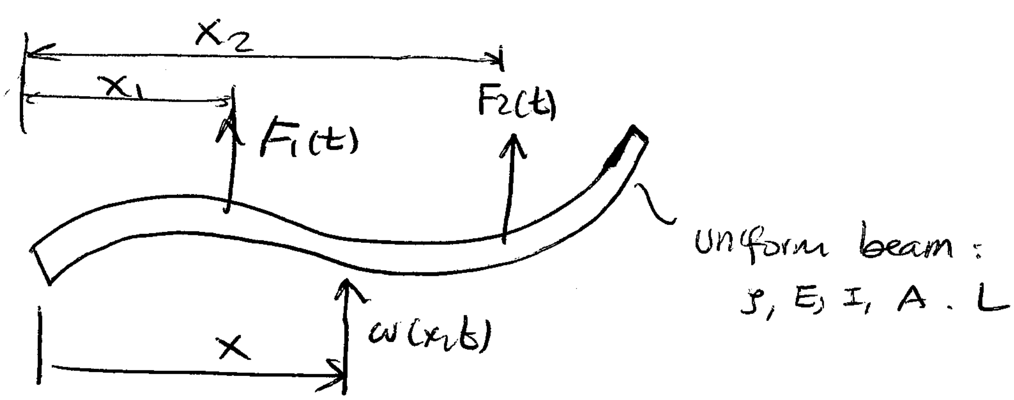
$$\frac{\partial \xi}{\partial t}(x_1, t) = Y_1(x_1)\frac{p_1}{m_1} + Y_2(x_1)\frac{p_2}{m_2} + \cdots Y_n(x_1)\frac{p_n}{m_n}.$$

Bernoulli-Euler Beam by Finite Modes.

-Summary-

We'll skip the derivation, since it's similar to the one for transversal vibrations. For details, refer to KMR, sect. 10.2.

Take the following configuration:



(force-free boundaries)

The equation of motion is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = F_1\,\delta(x-x_1) + F_2\,\delta(x-x_2)$$

The mode shape functions are given by

$$Y_n(x) = (\cos k_n L - \cosh k_n L)(\sin k_n x + \sinh k_n x) - (\sin k_n L - \sinh k_n L)(\cos k_n x + \cosh k_n x)$$

The values of k_n are obtained from the frequency equation $\cosh k_n L \cos k_n L = 1$.

The frequency is related to k_n as:

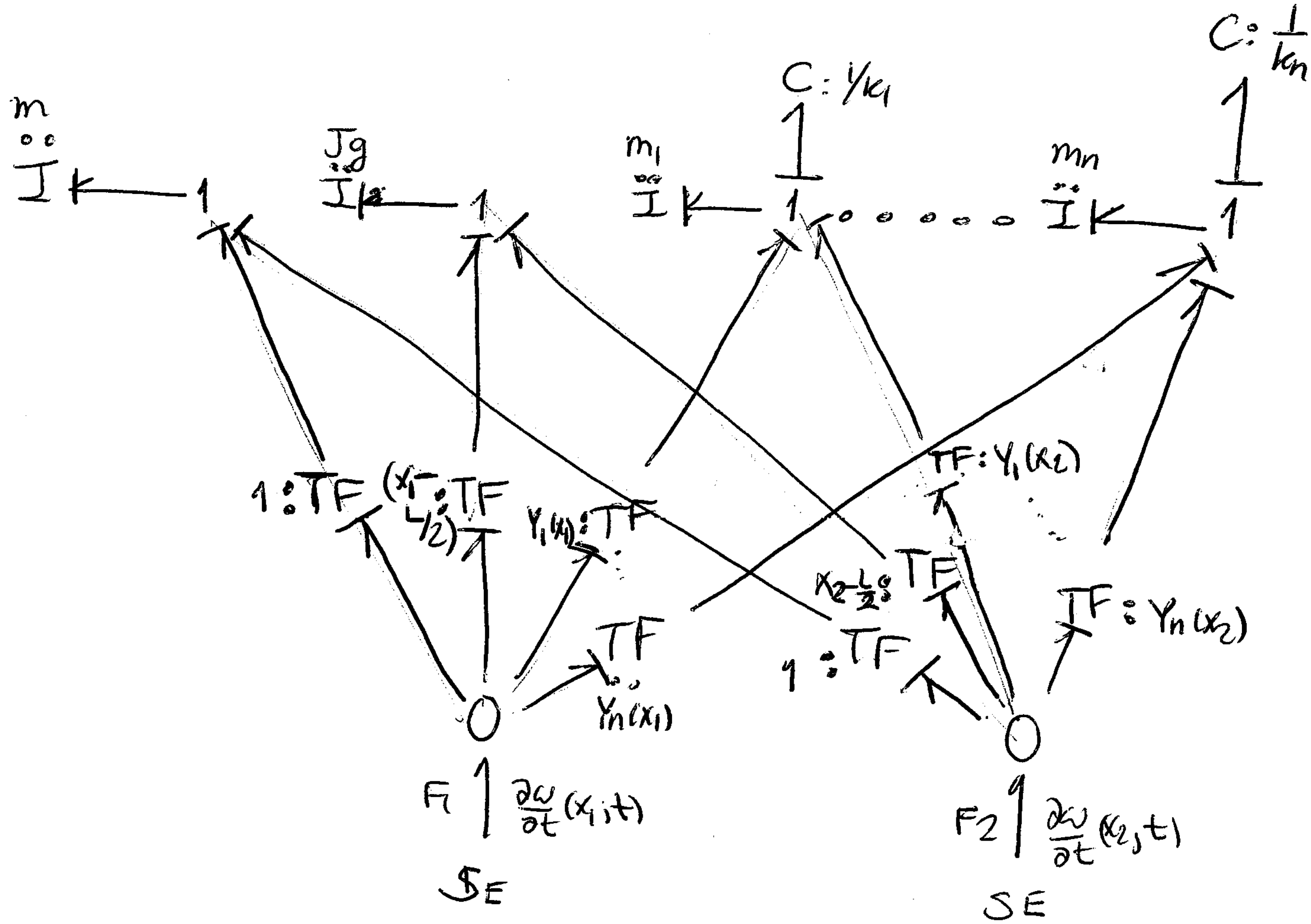
$$\omega_n^2 = \frac{EI}{\rho A}\frac{(k_n L)^4}{L^4} = \frac{EI k_n^4}{\rho A}$$

In the case of force-free boundaries, $k_n = 0 = \omega_n$ is a valid solution for the frequency equation.

This gives the rigid-body mode shapes

$$Y_{00} = 1 \quad \text{(pure translation)}$$
$$Y_0 = x - \frac{L}{2} \quad \text{(pure rotation about C.M.)}$$

Assuming m to be the total beam mass and J_g the beam's moment of inertia about the center of mass, we have the bondgraph:



The modal mass is $m_n = \int_0^L \rho A Y_n^2 dx$, $n = 1, 2..$

and the modal stiffness : $k_n = m_n \omega_n^2$

Important notes :

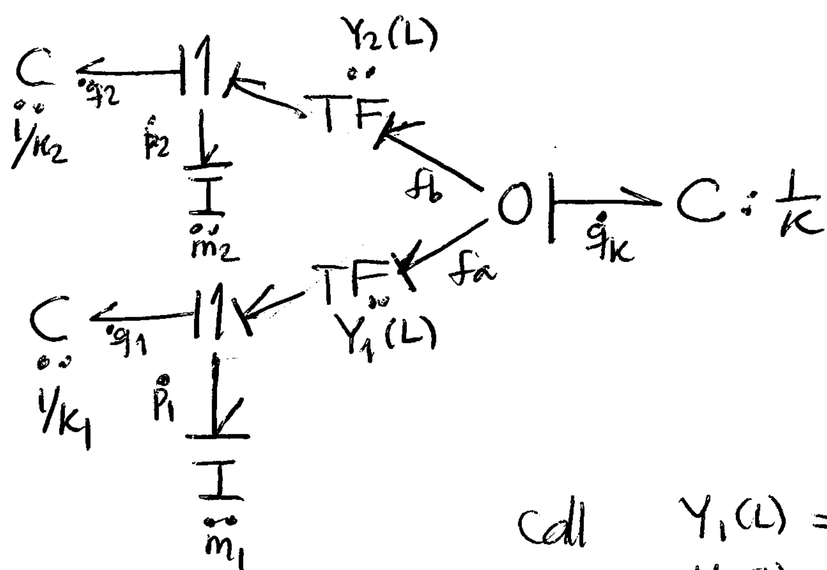
1) The above bond graph represents all structures exhibiting normal-mode behavior. Particular cases are obtained from calculating the correct mode shapes, modal masses and modal stiffnesses. These can be obtained analytically, from tables, or experimentally.

2) If a moment is applied instead of a force, the corresponding transformer modulus is the mode shape slope (instead of value) evaluated at the location of the moment.

3) It is quite simple to construct models where the beams interact with other subsystems. (see KMR, Fig 10-16).

Problem 10-10

a)



call $Y_1(L) = Y_1$
$Y_2(L) = Y_2$

Equation derivation:

$$x = \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_k \end{bmatrix} \qquad \text{co-}x: \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \\ e_k \end{bmatrix} \qquad \text{no inputs.}$$

$$f_1 = \frac{p_1}{m_1} \; ; \; f_2 = \frac{p_2}{m_2} \; ; \; e_k = q_k K.$$

$$\dot{p}_1 = Y_1 K q_k - q_1 K_1$$

$$\dot{p}_2 = Y_2 k q_k - q_2 k_2$$

$$\dot{q}_1 = \frac{p_1}{m_1}$$

$$\dot{q}_2 = \frac{p_2}{m_2}$$

$$\dot{q}_k = -f_b - f_a = -Y_2 \frac{p_2}{m_2} - Y_1 \frac{p_1}{m_1}$$

In state-space form:

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -k_1 & 0 & Y_1 k \\ 0 & 0 & 0 & -k_2 & Y_2 k \\ \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 & 0 & 0 \\ -\frac{Y_1}{m_1} & -\frac{Y_2}{m_2} & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_k \end{bmatrix}$$

The solutions are $\lambda = 0$

and $$\lambda = \omega_n^2 = \frac{\dfrac{k_1 y_2^2}{m_1 m_2} + \dfrac{k_2 y_1^3}{m_1 m_2}}{\dfrac{y_1^2}{m_1} + \dfrac{y_2^2}{m_2}}$$

The mode shapes are $Y_n(x) = \sin\left((2n-1)\frac{\pi}{2}\frac{x}{L}\right)$

so $m_1 = \int_0^L \rho A Y_1^2 dx = m_2 = \frac{\rho A L}{2}$

$$k_1 = m_1 \omega_1^2 = m_1 \frac{E}{\rho}(2\times1-1)^2 \frac{\pi^2}{4L^2} = \frac{\rho A L}{2\rho}\frac{E\pi^2}{4L^2} = \frac{AE\pi^2}{8L}$$

$$k_2 = m_2 \omega_2^2 = m_2 \frac{E}{\rho}(2\times2-1)^2 \frac{\pi^2}{4L^2} = \frac{\rho A L}{2} \times \frac{E\pi^2 \times 9}{\rho \times 4L^2} = \frac{9AE\pi^2}{8L}$$

$y_1(L) = \sin\left(\frac{\pi}{2}\right) = 1$

$y_2(L) = \sin\left(\frac{3\pi}{2}\right) = -1$

so $$\omega_n^2 = \frac{\dfrac{(AE\pi^2 + 9AE\pi^2)4}{8L \times \rho^2 A^2 L^2}}{\dfrac{2}{\rho A L} + \dfrac{2}{\rho A L}} = \left(\frac{5}{4}\right)\left(\frac{E}{\rho}\right)\left(\frac{\pi}{L}\right)^2$$

while the actual frequency is $\omega_{true}^2 = \left(\frac{E}{\rho}\right)\left(\frac{\pi}{L}\right)^2$

The error in frequency is about 12%. with only two modes.

The characteristic equation is $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & 0 & k_1 & 0 & -kY_1 \\ 0 & \lambda & 0 & k_2 & -kY_2 \\ -\frac{1}{m_1} & 0 & \lambda & 0 & 0 \\ 0 & -\frac{1}{m_2} & 0 & \lambda & 0 \\ \frac{y_1}{m_1} & \frac{y_2}{m_2} & 0 & 0 & \lambda \end{vmatrix} = 0$$

by hand or using Maple or Mathematica:

$$\lambda^5 + \lambda^3\left(\frac{ky_1^2}{m_1} + \frac{ky_2^2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_2}\right) +$$

$$\left(\frac{kk_2y_1^2}{m_1m_2} + \frac{kk_1y_2^2}{m_1m_2} + \frac{k_1k_2}{m_1m_2}\right)\lambda = 0$$

Dividing by k:

$$\frac{\lambda^5}{k} + \left(\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2}\right)\lambda^3 + \left(\frac{k_1}{m_1k} + \frac{k_2}{m_2k}\right)\lambda^3$$

$$+ \left(\frac{k_2y_1^2}{m_1m_2} + \frac{k_1y_2^2}{m_1m_2}\right)\lambda + \frac{k_1k_2}{km_1m_2}\lambda = 0$$

Letting $k \to \infty$ we keep

$$\lambda\left[\left(\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2}\right)\lambda^2 + \frac{k_2y_1^2}{m_1m_2} + \frac{k_1y_2^2}{m_1m_2}\right] = 0$$