MCE503: Modeling and Simulation of Mechatronic Systems

Lecture 7: Manual Equation Generation

Reading: KMR Chapter 5

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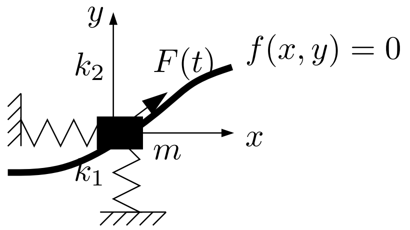
Preliminaries

- Several representations exist that carry the same information about system dynamics.
- A general representation is a system of differential-algebraic equations (DAE)
- Example: Mass under geometric constraint:



$$\begin{aligned} F\cos\theta - k_1 x - N\sin\theta &= m\ddot{x} \\ F\sin\theta - k_2 y + N\cos\theta &= m\ddot{y} \\ f(x,y) &= 0 \\ \tan\theta = \frac{dy}{dx} & \end{aligned}$$

Four equations (2 differential, 2 algebraic); 4 unknowns (x,y, θ, N).

Review of state-space equations

- Often, one can solve for the unknowns in the algebraic equations. Substitution leads to a system of ODE's.
- Often, algebraic constraints are absent. Only ODE's appear, but may involve derivatives of any order.
- A state space representation consists only of first order ODE's:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, ...x_n, u_1, u_2, ...u_m) \\
\dot{x}_2 &= f_2(x_1, x_2, ...x_n, u_1, u_2, ...u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, ...x_n, u_1, u_2, ...u_m)
\end{aligned}$$

State space equations (SSE)

- Each state derivative is written as a function of inputs (input derivatives allowed) and states (no derivatives allowed)
- Methods exist to obtain state-space equations from other representations (transfer functions, high-order ODEs)
- Here we obtain the SSE *directly from bond graph*. Choice of states is fixed: p and q corresponding to I and C.
- p and q are called *energy variables*. e and f are *co-energy variables.*
- A clever setup: p and q have integration built into their definitions. Finding expressions for $\dot{q}$ and $\dot{q}$ should be an entirely algebraic procedure, given that C and I relate $e = \dot{p}$ and $f = \dot{q}$ statically to q and p, respectively. This is, really, the central idea for bond graphs!!!

On the dimension of the state space

- Suppose $x = x_1, x_2, ...x_n$ is a candidate set of states and let the state equations be compactly written as $\dot{x} = f(x, u)$, where $f(x, u)$ is a vector field mapping $\mathbb{R}^{n+m}$ into $\mathbb{R}^n$.
- Suppose that r functions exist s.t. $h_i(x_1, x_2, ...x_n, u_1, u_2, ...u_m) = 0$, $i = 1, 2..r$.
- This implies that r states are dependent on the remaining ones. Therefore the state space has dimension less than n. (only $n - r$ states are necessary).
- Fact: *each C or I for which integration causality cannot be assigned introduces algebraic (static) dependence between its energy variable and the energy variables of other C's or I's (states) or the inputs. The dimension of the state space equals the number of C's and I's minus the number of elements requiring derivative causality.*
- Proving that this will always be the case in general requires material on vector field algebra (extending concept of dependence to the nonlinear case).

Example: Mechanical system

Case I: “All integral and fully determined” causality

- We mean: It’s possible to assign integral causality to all C’s and I’s, and causality propagates to each bond.
- Assign power directions and number the bonds.
- Assign proper causalities to the sources.
- Assign integration causality to each C and I (assumed possible). Propagate causality following allowable settings for GY, TF, O and 1. We’re assuming that all other bonds receive causality assignment by propagation.
- Identify the set of states: p’s and q’s from all C’s and I’s. Identify the set of inputs (from sources).
- Write the constitutive equations for each element according to the causal assignment (output on the left-hand side).
- Write expressions for state derivatives. Substitute to keep just states and inputs. Help the process by following causality.

Several examples

The electric vehicle example illustrates the process, including nonlinear constitutive relations.

Case II: “All integral but not fully determined” causality

- In this case it’s possible to assign integral causality to all C’s and I’s, but causality does not propagate to each bond. This indicates an *algebraic loop* is present.
- An algebraic loop is an algebraic equation relating an effort or flow variable with itself, other efforts and flows and inputs, for example:

$$e_i = \phi_l(e_i, e_j, f_j, u_j)$$

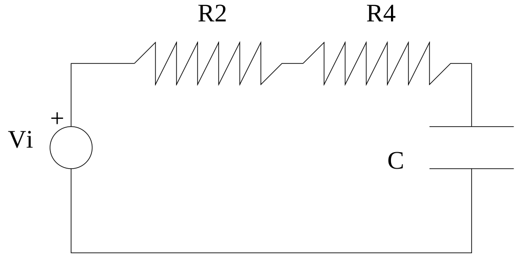
- Algebraic loops, when found, are inherent to the physical model. They will appear regardless of the analytical approach (bond graph or conventional).
- The bond graph approach reveals exactly how many loops will be present before actually writing a single equation.
- The number of algebraic loops equals the number of elements for which arbitrary causal assignments were required.

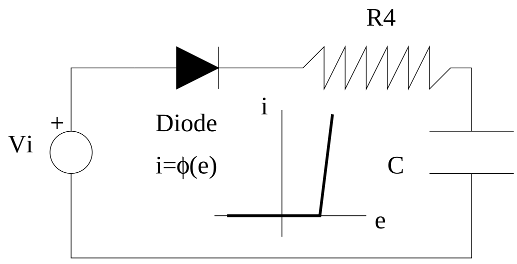
Case II: Revealing the algebraic loop

- As in Case I, states are given by all q’s and p’s.
- The process is the same as before, except for the e’s and f’s associated with the elements that received arbitrary causal assignments.
- *An algebraic equation must be written for each one of the variables which are the output of such elements*.
- Sometimes the algebraic equations have no closed-form solution. In these cases, the mathematical description of the system remains as a system of DAEs (differential-algebraic equations) and numerical methods have to be used for solution.

Several examples

Please follow the example of KMR, Fig. 5.10 (4th ed.). Also, consider the following simple examples:





Case III: "Not all integral but fully determined" causality

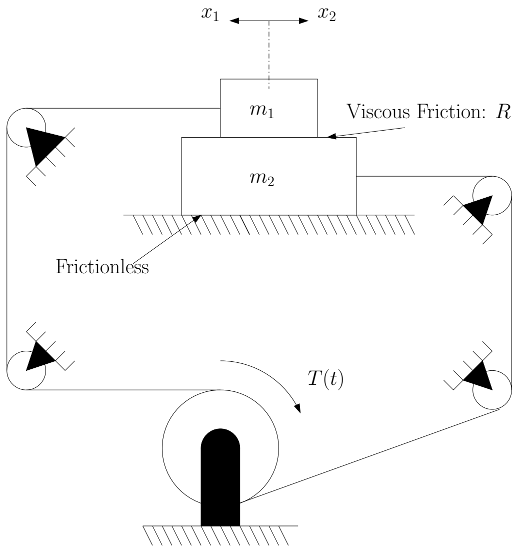
- In this case it's not possible to assign integral causality to all C's and I's, but causality propagates to each bond.
- The presence of derivative causalities indicates that some state variables are algebraically linked to others. Therefore, they must not be included as states.
- **Basic principle:** The number of states equals the number of C's and I's having integral causality.
- The process is the same as before, but the state variables for the elements with derivative causalities must be expressed in terms of the other state. This is a necessary step in the formulation of the state equations.

Case III: “Not all integral but fully determined” causality

- At some point, the derivative of a q or p will appear in the equation derivation, corresponding to an element with derivative causality. Since such elements do not contribute a state equation, the required derivative must be found separately.
- Start with the constitutive law for the element having derivative causality, but write it “backwards” (input on the left-hand side).
- Then follow causality in the standard way until the required algebraic relationship is found. Then take the derivative and return to the main derivation process.
- Derivative causality can usually be avoided by adding small inertias and large stiffnesses.

Several examples

Carefully follow Sect. 5.4.1 in KMR (4th ed.). In addition consider the following example (Fall 05 exam)



Case IV: “Not all integral and not fully determined” causalit

- This case is a combination of the previous two and it is the most difficult situation that can be encountered, especially in the nonlinear case.
- The bond graph method, however, guides the formulation of equations in a way that is most convenient for computer solution.
- As a highly-recommended exercise, make up the simplest bondgraph having one derivative causality and one algebraic loop.
  Assume linear constitutive equations and derive the state equations.

Output Variables

- Usually, we are not interesting in looking at (plotting) all of the state variables.
- The selected group of variables to be plotted, analyzed or used as feedback in control systems is called the output.
- The choice of outputs is up to the user. Equations must be written for these outputs.
- The procedure to find the outputs is not unlike that of finding the state equations.
- In the linear case, a complete state-space description has the form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$