MCE503: Modeling and Simulation of Mechatronic System

Lecture 8: Fields and Junction Structures - Part I

Reading: KMR Chapter 7

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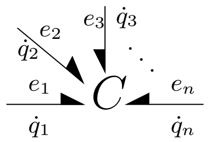
Fields

- A field is a multiport generalization of the basic one-ports and two ports studied before.
- *Fields allow many power ports in a given element.*
- C, I, multiport transformers and junction structures will be power-conserving.
- Unlike the 1-port case, the constitutive equations of multiport C and I will be restricted, so that energy conservation holds.
- Example of a C-field: a massless elastic beam with applied moment and force (efforts). The corresponding flows could be the deflection and rotation at the tip.
- Example of an I-field: A rigid spacecraft with thrusters for x, y and z (gravitational effects not considered).

C-Fields

- The symbol for a C-field is



- The energy stored in the C-field is given by

$$dE = \sum_{i=0}^{n} e_i f_i dt$$

- This can be integrated and expressed as

$$E(q) = \int_{q_0}^{q} e^T(q) dq$$

where $e(q)$ and q are column vectors containing the individual efforts and displacements.

- $e = e(q)$ represents the constitutive equation of the C-field in *stiffness* form. The form $q = q(e)$ is called the *compliance* form.

Energy Conservation in C-Fields

- Remember the fundamental theorem of Calculus to verify that

$$\frac{\partial E}{\partial q_i} = e_i(q)$$

- If the energy function $E(q)$ is assumed to be smooth enough to have second derivatives, we have:

$$\frac{\partial^2 E}{\partial q_j \partial q_i} = \frac{\partial^2 E}{\partial q_i \partial q_j}$$

which implies

$$\frac{\partial e_i(q)}{\partial q_j} = \frac{\partial e_j(q)}{\partial q_i}$$

which places a restriction on the form of $e(q)$. These constraints are called the *Maxwell reciprocal relations*.

- In the linear case, the constitutive law in stiffness form is $e = Kq$, where K is an n-by-n stiffness matrix. The above restriction implies that K must be symmetric.

- Since the inverse of a symmetric matrix is symmetric, the compliance matrix will also be symmetric.

Explicit vs. Implicit Fields

- The following applies to any field, not just C, which will be used to introduce the concept.
- In an *explicit* field, the constitutive law is specified in mathematical form. Information about the physical nature and internal structure of the field is not given. For example:

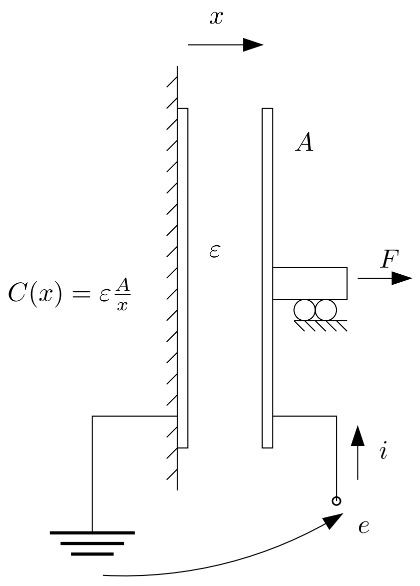
$$\begin{aligned} q_1 &= 3e_1 - e_2 \\ q_2 &= -e_1 + e_2 \end{aligned}$$

represents a legitimate C-field, but we don't know where it came from.

- An *implicit* field is the opposite: details about the system are given and the ports are identified. If a legitimate field, it must yield a constitutive equation satisfying the Maxwell reciprocal relations.

Example of an Implicit C-Field

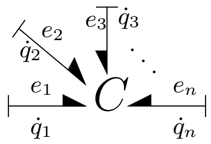
Obtain a C-field model of a movable-plate capacitor transducer.



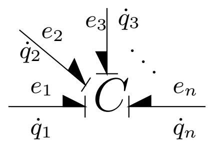
See also the examples in KMR.

Causality in C-Fields

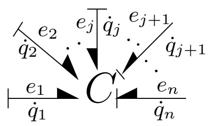
- Causality can be fully integral, fully derivative, or a mix.
- The symbol for integral causality is



while for derivative causality it is



and for mixed causality:



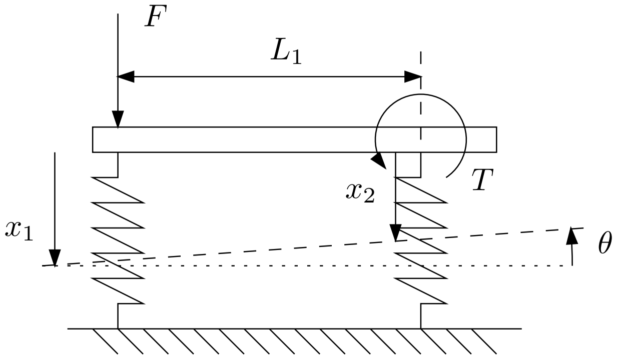
Causality in Explicit C-Fields

- In explicit fields, one allowable causality is implied by the form in which the equations are given. For instance, $e = e(q)$ indicates that fully integral causality is possible. *Even if we knew the exact form of the functions* e_i, *it is difficult to determine if other causal forms are possible* (a question of invertibility of a series of functions).
- Explicit fields with mixed causality (following the symbol) are given as

$$
\begin{aligned}
e_1 &= e_1(q_1, q_2, ...q_j, e_{j+1}, e_{j+2}, ...e_n) \\
e_2 &= e_2(q_1, q_2, ...q_j, e_{j+1}, e_{j+2}, ...e_n) \\
\vdots & \quad \vdots \\
q_{j+1} &= q_{j+1}(q_1, q_2, ...q_j, e_{j+1}, e_{j+2}, ...e_n) \\
q_{j+2} &= q_{j+2}(q_1, q_2, ...q_j, e_{j+1}, e_{j+2}, ...e_n) \\
\vdots & \quad \vdots \\
q_n &= q_n(q_1, q_2, ...q_j, e_{j+1}, e_{j+2}, ...e_n)
\end{aligned}
$$

Causality in Implicit C-Fields

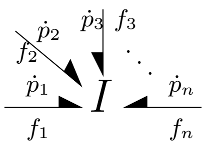
- The allowable causalities of implicit fields (in general) are obtained following the conventional rules for all bond graphs.
- Example: Determine if the following system behaves as a C-field from the indicated ports and determine all possible causalities. Work the small rotations case first. Then consider the nonlinear case. Are the possible causalities maintained?



massless bar
small rotations
cubic springs $e = kq^3$

I-Fields

- The symbol for an I-field is



- The energy stored in the I-field is given by

$$dE = \sum_{i=0}^{n} e_i f_i dt$$

- This can be integrated and expressed as

$$E(p) = \int_{p_0}^{p} f^T(p) dp$$

where $f(p)$ and p are column vectors containing the individual flows and and momenta.

- $f = f(p)$ represents the constitutive equation of the I-field. The opposite form is $p = p(f)$.

Energy Conservation in I-Fields

- Remember the fundamental theorem of Calculus to verify that

$$\frac{\partial E}{\partial p_i} = f_i(p)$$

- If the energy function $E(p)$ is assumed to be smooth enough to have second derivatives, we have:

$$\frac{\partial^2 E}{\partial p_j \partial p_i} = \frac{\partial^2 E}{\partial p_i \partial p_j}$$

  which implies

$$\frac{\partial f_i(q)}{\partial p_j} = \frac{\partial f_j(q)}{\partial p_i}$$

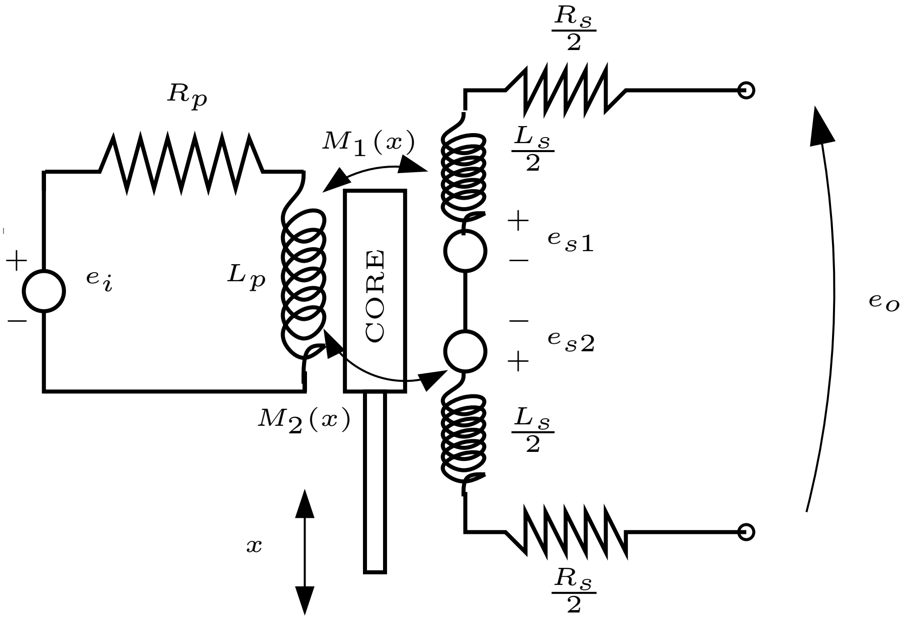
  which places a restriction on the form of $f(p)$. These constraints are called the *Maxwell reciprocal relations* for the I-field.

- In the linear case, the constitutive law can be expressed as $p = If$, where I is an n-by-n *mass* or *inertia* matrix. The above restriction implies that I must be symmetric.

- Since the inverse of a symmetric matrix is symmetric, the I^{-1} matrix will also be symmetric. This matrix defines the opposite constitutive relation $f = I^{-1}p$.

Example of an Implicit I-Field

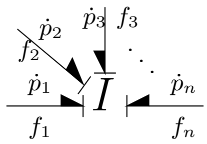
Obtain an I-field model for the LVDT (linear variable differential transformer)



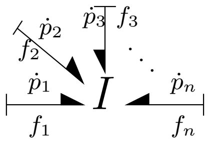
See also the examples in KMR.

Causality in I-Fields

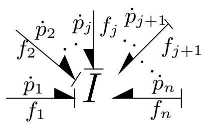
- Causality can be fully integral, fully derivative, or a mix.
- The symbol for integral causality is



while for derivative causality it is



and for mixed causality:



Causality in Explicit I-Fields

- In explicit fields, one allowable causality is implied by the form in which the equations are given. For instance, $f = f(p)$ indicates that fully integral causality is possible.
- Explicit I-fields with mixed causality (following the symbol) are given as

$$\begin{aligned}
f_1 &= f_1(p_1, p_2, ...p_j, f_{j+1}, f_{j+2}, ...f_n) \\
f_2 &= f_2(p_1, p_2, ...p_j, f_{j+1}, f_{j+2}, ...f_n) \\
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p_{j+1} &= p_{j+1}(p_1, p_2, ...p_j, f_{j+1}, f_{j+2}, ...f_n) \\
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p_n &= p_n(p_1, p_2, ...p_j, f_{j+1}, f_{j+2}, ...f_n)
\end{aligned}$$