Modeling of Continuous Beams: Finite-Mode Approach

Reading: KMR Chapter 10

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The Dirac Delta Function

- The Dirac Delta function $\delta(x)$ is a useful mathematical construct. It will allow us to locate point forces and moments in a continuous beam.
- The delta function cannot be evaluated. It can be intuitively understood as a pulse at 0 having infinite height but zero width, so that its integral between $[-\infty, \infty]$ is unity.
- The *sifting property*

$$\int_{-\infty}^{\infty} \delta(x - x^*) f(x) dx = f(x^*)$$

is perhaps the most useful.

- The derivative of δ satisfies:

$$\int_{-\infty}^{\infty} \delta'(x - x^*) f(x) dx = -f'(x^*)$$

Beam Equations with Concentrated Forces and Moments

- The wave equation (used in longitudinal beam vibration) with n_f concentrated forces at locations x_i is

$$\rho \frac{\partial^2 \xi}{\partial t^2} - E\frac{\partial^2 \xi}{\partial x^2} = \sum_{i=1}^{n_f} F_i(t)\delta(x - x_i)/A$$

- The Euler-Bernoulli equation (used in traverse beam vibration) with n_f concentrated forces at locations x_i and n_m concentrated moments at locations x_j is

$$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = \sum_{i=1}^{n_f} F_i(t)\delta(x - x_i) - \sum_{j=1}^{n_m} M_j(t)\delta'(x - x_j)$$

Longitudinal Beam Vibrations

- The relevant equation is

$$\rho \frac{\partial^2 \xi}{\partial t^2} - E\frac{\partial^2 \xi}{\partial x^2} = \sum_{i=1}^{n_f} F_i(t)\delta(x - x_i)/A$$

- First, the unforced problem is considered (right-hand side equal to zero). The separation of variables assumes that the displacement is a product of two functions, one a function of x only and the other a function of t only:

$$\xi(x,t) = Y(x)f(t)$$

- Substituting into the PDE and re-arranging gives:

$$\frac{1}{f}\frac{d^2 f}{dt^2} = \frac{E}{\rho}\frac{1}{Y}\frac{d^2 Y}{dx^2}$$

- Since the l.h.s is a function of t only and the $r.h.s$ is a function of x only, both sides must be constant for the equation to hold.

Longitudinal Beam Vibrations...

- Let $\frac{1}{f}\frac{d^2 f}{dt^2} = -w^2$. Then the following system of ODE's results:

$$\begin{aligned} \frac{d^2 f}{dt^2} + w^2 f &= 0 \\ \frac{d^2 Y}{dx^2} + \frac{\rho}{E} w^2 Y &= 0 \end{aligned}$$

- The spatial equation has a solution:

$$Y(x) = A\cos(kx) + B\sin(kx)$$

  where $k^2 = \frac{\rho}{E} w^2$.

- The values of A and B are determined from the boundary conditions (clamped-free, clamped-pinned, etc).

Clamped-Free Beam

- As explained in KMR, it is convenient to take the force as acting at an infinitesimal distance from the end of the beam, so that the stress at the boundary is zero:

$$\sigma(L,t) = E\frac{\partial \xi(L,t)}{\partial x} = 0$$

- This means $\frac{\xi(L,t)}{\partial x} = \frac{dY}{dx}(L)f(t) = 0$ for all t, implying that $\frac{dY}{dx}(L) = 0$.
- Also, the clamped end has zero displacement, so $\xi(0,t) = Y(0)f(t) = 0$ for all t, implying that $Y(0) = 0$.
- These two conditions on Y give $A = 0$ and

$$Bk\cos(kL) = 0$$

- The above is called *frequency equation*. The solutions are

$$k_n L = (2n-1)\frac{\pi}{2}, \text{ for } n = 1, 2, 3... \text{ that is:}$$

$$w_n = \sqrt{\frac{E}{\rho}}\frac{2n-1}{L}\frac{\pi}{2}$$

Mode Shapes and Orthogonality

- We obtain a series of *mode shape functions*

$$Y_n(x) = B_n \sin\left((2n-1)\frac{\pi}{2}\frac{x}{L}\right)$$

where B_n can be taken as one.

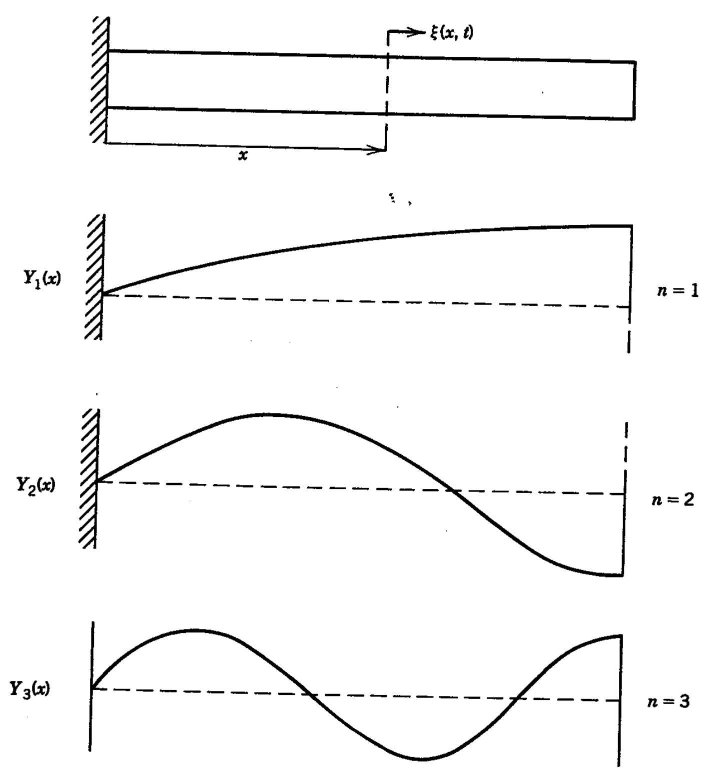
- **The mode shapes are scaled by the time-varying factor $f(t)$ to produce the vibration.**
- The mode shape functions are *orthogonal*. That is, if we define an inner product in an appropriate function space as

$$< F, G > \triangleq \int_0^L F(x)G(x)dx$$

then we have

$$< Y_n, Y_m >= \begin{cases} 0 & \text{when } n \neq m \\ \frac{L}{2} & \text{when } n = m \end{cases}$$

First 3 Mode Shapes



Clamped-Free Beam with End Force

- It can be shown that both forced and unforced responses can be expressed as a weighted sum of mode shapes and time functions. So we take

$$\xi(x,t) = \sum_{n=1}^{\infty} Y_n(x)\eta_n(t)$$

- Substitute this into the forced version of the wave equation to obtain

$$\sum_n \rho A Y_n \ddot{\eta}_n - \sum_n AE\frac{d^2Y_n}{dx^2}\eta_n = F(t)\delta(x-L)$$

- Multiply the equation by $Y_m(x)$ and integrate to obtain

$$\sum_n \left(\int_0^L \rho A Y_n Y_m dx\right)\ddot{\eta}_n + \sum_n \left(\int_0^L \rho A Y_n Y_m dx\right) w^2\eta_n = \int_0^L F(t)\delta(x-L)Y_m dx$$

- Orthogonality and the sifting property of δ result in

$$m_m\ddot{\eta}_m + K_m\eta_m = F(t)Y_m(L)$$

where $m_m = \rho AL/2$ is the *modal mass* and $K_m = m_m w_m^2$ is the *modal stifness*. Note $K_m \neq k_n$.

Clamped-Free Beam with End Force...

- The forced response is then given by

$$\xi(x,t) = \sum_{m=1}^{\infty} Y_m(x)\eta_m(t)$$

where only a finite number of modes is retained.

- A bond graph representation of the equations is very informative. Define the modal momentum as $p_m = m_m\dot{\eta}_m$ and the modal displacement as $q_m = \eta_m$. Then the 2nd-order ODE for η_m can be re-written in state-space form as

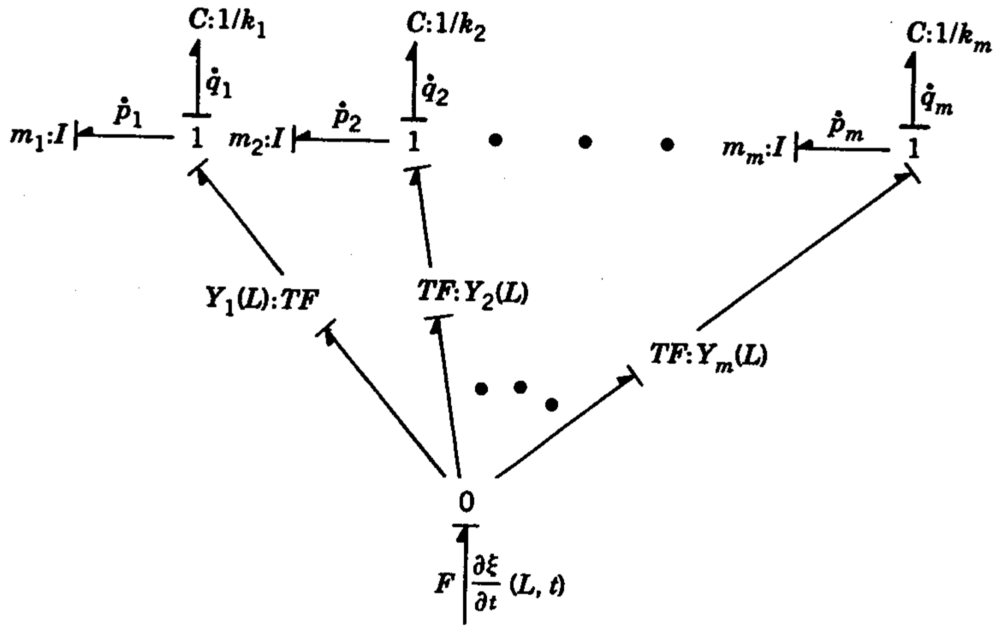
$$\begin{aligned}\dot{p}_m &= -K_m q_m + F(t)Y_m(L)\\ \dot{q}_m &= \frac{1}{m_m}p_m\end{aligned}$$

- Note that the flow at the location of the force is given by

$$\frac{\partial\xi(L,t)}{\partial t} = Y_1(L)\frac{p_1}{m_1} + Y_2(L)\frac{p_2}{m_2} + ... + Y_m(L)\frac{p_m}{m_m}$$

- This suggests the bond graph structure shown next.

Bond Graph for Longitudinal Beam Vibrations

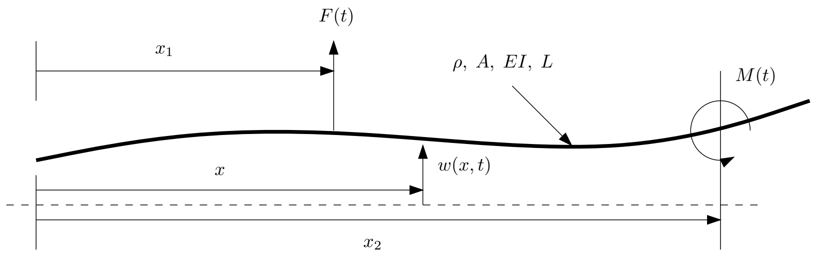


Observations

- *What if the force is located elsewhere or if there are more forces?* - Just include sources connecting each force to all 1-junctions using a transformer. The modulus should be $\frac{1}{Y_m(x_i)}$ where m is the mode number and x_i is the location of the force.
- *How to find the flow at a location where there is no force?* -Proceed as above with a fictitious (zero) force.
- *What if the boundary conditions change?* -The core bond graph does not change. We just need to use the appropriate mode shapes and associated modal masses, stiffnesses and frequencies.
- Tables of pre-calculated mode shapes and frequencies are widely available. We have to be careful with the normalization used in each case. Here we used $B_n = 1$, but sometimes the $Y_m(x)$ are normalized by the area under the curve between 0 and L.
- The above feature makes the use of BG particularly attractive for interacting systems. See KMR Figs. 10.11, 10.12, 10.13 (4th ed.).

Transverse Beam Vibrations

- The approach is to consider force-free boundaries to obtain the model. Later, the actual boundaries are accounted for by using the correct mode shapes (read KMR).
- We consider a beam with one concentrated force and one concentrated moment (slightly different than KMR).



Transverse Beam Vibrations...

- The relevant equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = F_1(t)\delta(x - x_1) - M(t)\delta'(x - x_2)$$

- We continue with the separation of variables assumption that the displacement is a product of two functions, one a function of x only and the other a function of t only:

$$w(x,t) = Y(x)f(t)$$

- Substituting into the PDE and re-arranging gives:

$$\frac{EI}{\rho A}\frac{1}{Y}\frac{d^4 Y}{dx^4} = -\frac{1}{f}\frac{d^2 f}{dt^2}$$

- Since the l.h.s is a function of x only and the $r.h.s$ is a function of t only, both sides must be constant for the equation to hold.

Transverse Beam Vibrations...

- Let $\frac{1}{f}\frac{d^2 f}{dt^2} = -w^2$. Then the following system of ODE's results:

$$\begin{aligned} \frac{d^2 f}{dt^2} + w^2 f &= 0 \\ \frac{d^4 Y}{dx^4} - \frac{\rho A}{EI} w^2 Y &= 0 \end{aligned}$$

- The spatial equation has a solution:

$$Y(x) = A\cosh(kx) + B\sinh(kx) + C\cos(kx) + D\sin(kx)$$

  where $k^4 = \frac{\rho A}{EI} w^2$.

- The values of the constants are determined, up to a scaling factor, from the boundary conditions (clamped-free, clamped-pinned, etc).

Mode Shape Normalization Issue

- Unfortunately, KMR does not devote any discussion to the scaling issue. The boundary conditions for the free-free case are: no moment and no shear at $x = 0$ and $x = L$, which translate into:

$$\begin{aligned} \frac{\partial^2 w(0,t)}{\partial x^2} &= \frac{d^2 Y(0)}{dx^2} = 0 \\ \frac{\partial^2 w(L,t)}{\partial x^2} &= \frac{d^2 Y(L)}{dx^2} = 0 \\ \frac{\partial^3 w(0,t)}{\partial x^3} &= \frac{d^3 Y(0)}{dx^3} = 0 \\ \frac{\partial^3 w(L,t)}{\partial x^3} &= \frac{d^3 Y(L)}{dx^3} = 0 \end{aligned}$$

- The above can determine only three of the four constants!. The remaining one is chosen to give $Y(x)$ a scale. The choice varies from book to book. Work the equations to see how KMR chooses the scaling.

Mode Shapes and Frequency Equation

- The frequency equation

$$\cosh(k_n L)\cos(k_n L) = 1$$

holds for free-free beams, regardless of the scaling method. This equation is solved numerically for $k_n L$. The first nonzero values are $k_1 L = 0.1776338$ and $k_2 L = 4.730040744862704$.

- When L is known, we can find k_n and the corresponding natural frequencies using

$$w_n = \frac{EI}{\rho A}\frac{(k_n L)^4}{L^4}$$

- Now, using KMR's scaling, the mode shape functions become

$$Y_n(x) = (\cos KnL - \cosh k_n L)(\sin k_n x + \sinh k_n x) - (\sin k_n L - \sinh k_n L)(\cos k_n x + \cos k$$

- In the free-free mode case, $k_n = 0$ is a solution to the frequency equation, corresponding to the *rigid-body modes* (translation and rotation)

$$Y_{00} = 1$$

$$Y_0 = x - \frac{L}{2}$$

Mode Shapes and Orthogonality

- As in the longitudinal case, the mode shape functions are *orthogonal*. That is,

$$\int_0^L Y_n(x)Y_m(x)dx = 0 \text{ when } n \neq m$$

for all modes, including the rigid ones.

- The value of the integral when $n = m$ depends on the choice of scale. Find what the value is for the $Y(x)$ used in KMR.
- The second derivative of the mode shape also satisfies orthogonality. This is useful for the applied moments.
- The normalization of mode shapes affects the compatibility of the finite-mode approach with conventional static beam formulas. The book "Mechanics of Vibration" by Bishop contains tables of pre-calculated mode shapes using a compatible normalization. An excerpt has been posted on the course website.

Free-Free Beam with Moment and Force

- The forced response can be expressed as a weighted sum of mode shapes and time functions. So we take

$$w(x,t) = \sum_{n=0}^{\infty} Y_n(x)\eta_n(t)$$

- Substitute this into the forced version of the wave equation. Use the orthogonality of the modes and the sifting property of δ and its derivative.
- Orthogonality and the sifting property of δ result in

$$\left(\int_0^L \rho A Y_n^2 dx\right)\ddot{\eta}_n + \left(\int_0^L \rho A Y_n^2 dx\right) w^2 \eta_n = F(t)Y_n(x_1) + M(t)Y_n'(x_2) \qquad (1)$$

- The rigid-body modes are treated separately. Take the translation mode: $w = 0,\ Y(x_1) = 1,\ Y'(x_2) = 0$ in the above to obtain

$$m\ddot{\eta}_{00} = F_1(t)$$

where m is the total mass of the beam. Now take the rotation mode: $w = 0,\ Y(x_1) = x_1 - \frac{L}{2},\ Y'(x_2) = 1$ to obtain

$$J_g\ddot{\eta}_0 = F_1(t)(x_1 - \frac{L}{2}) + M(t)$$

Free-Free Beam with Moment and Force...

- Returning to Eq. 1, define the modal mass and stiffness as

$$m_n \triangleq \int_0^L \rho A Y_n^2(x) dx$$

$$K_n \triangleq m_n w_n^2$$

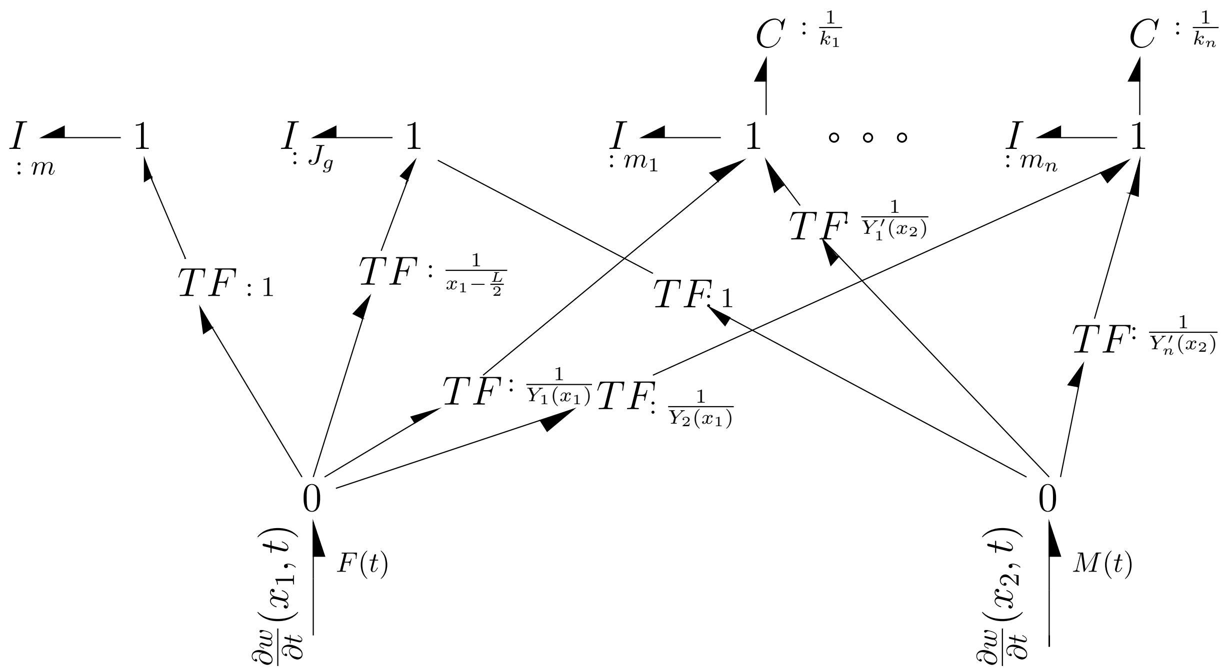
Note $K_n \neq k_n$

- A bond graph representation of the equations is obtained as in the longitudinal case, with the addition of the rigid-body modes. Define the modal momentum as $p_m = m_m\dot{\eta}_m$ and the modal displacement as $q_m = \eta_m$.
- The flow at the location of the external forces or moments (can be fictitious to allow output at other locations) is again given by

$$\frac{\partial w(x_i,t)}{\partial t} = Y_1(x_i)\frac{p_1}{m_1} + Y_2(x_i)\frac{p_2}{m_2} + ... + Y_m(x_i)\frac{p_m}{m_m}$$

- Note that Eq. 1 indicates that the transformer modulus for moments is the derivative of the mode shape evaluated at the location of the moment. This suggests the bond graph structure shown next.

Bond Graph for Free-Free Beam with Force and Moment



Observations

- Multiple forces and moments are handled by adding sources and transformers. Each source connects to each 1-junction. An exception is the rigid translation mode, which only receives input from force (no moment) sources.
- Each set of boundary conditions has its own frequency equation. You can solve such equations with Matlab's `fsolve` command.
- External sources can be replaced by bonds to other subsystems directly. The core bond graph for the beam remains. The same is true when boundary conditions change. We only need to change the values of the transformer modulus.
- See KMR Figs. 10.15, 10.16 (4th ed.).
- The non-dimensional evaluation of the modal masses is incorrect in KMR. It should be

$$\frac{m_n}{m} = \int_0^1 Y_n^2(Lz)dz$$

  under the change of variables $x = Lz$. Check for yourself!
- The tables by Bishop normalize the modes so that all modal masses are equal to ρAL.

Numerical Example

A cantilever beam with dimensions 181.32 by 15.52 by 0.68 mm has a 0.5 N-mm moment applied at 52.29 mm from the fixed end. The moment has an axis parallel to the width of the beam. The beam is made of steel $(\rho = 7860$ kg/m$^3, \ E = 162.727$ GPa). A deflection sensor is installed at 179 mm from the fixed end.

- Obtain the bond graph for two modes (there are no rigid body modes here!)
- Calculate the first two natural frequencies and determine the modal masses and stiffnesses. Use Bishop's data.
- Obtain the state equations from the bond graph in symbolic form.
- Find the symbolic transfer function from applied moment to displacement at sensor location.
- Use the above transfer function and the Final Value Theorem to predict the static tip deflection when the moment is a step input applied at the tip. Compare the result with standard table data.
- Use the numerical values to plot the frequency response (Bode plot) using the actual sensor and moment locations.