Consider the following nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= x_3 - x_1x_2 + u \\
\dot{x}_3 &= x_1x_3 - u
\end{align*}
\]

1. Find the single equilibrium point for \( u_0 = 1 \).

2. Linearize the system about the equilibrium point. Find matrices \( A \) and \( B \) of the linearized system

\[
\Delta x = A\Delta x + B\Delta u
\]

3. Determine the stability of the linearized system about \( x_0 \). Also determine controllability.

4. Choose a feedback gain \( K \) so that the closed-loop poles of the linearized system under the control \( \Delta u = -K\Delta x \) are \([-1 - 2 - 3]\).

5. Build a Simulink diagram that simultaneously applies the control \( u = u_0 + \Delta u = u_0 - K(x - x_0) \) to the nonlinear system and the control \( \Delta u = -K\Delta x \) to the linearized system. Simulate using consistent initial conditions (use \((\Delta x)_0\) for the linearized system and \(x_0 + (\Delta x)_0\) for the nonlinear system). Show that the nonlinear system can be unstable when operated away from \( x_0 \).

6. Find the state equations for the nonlinear system under the control \( u = u_0 - K(x - x_0) \). Verify that \( x_0 \) is still an equilibrium point.

7. Find \( P \) for a quadratic Lyapunov function \( V = \frac{1}{2}(x - x_0)^T P(x - x_0) \) with \( P \) determined from the equation

\[
(A - BK)^T P + P(A - BK) + Q = 0
\]

(start with \( Q = I \)).

8. Write a program that scans a neighborhood of \( x_0 \) using a spherical sweep to determine the sign of \( \dot{V} = \dot{x}^T P(x - x_0) + (x - x_0)^T P\dot{x} \). Run the program to determine a radius of attraction for \( x_0 \) as determined by \( V \). Test for \( \dot{V} > 0 \) at each point.

9. Pick a point inside your estimated region of attraction. Use it as initial condition in your Simulink model to verify that stability holds. Comment on the results.

10. Pick a point just outside your estimated radius of attraction and use it as an initial condition for simulation. Does the point belong to the true region of attraction?

Note: It is highly suggested to use the Symbolic Toolbox for various tasks:

- Finding the state derivatives of the nonlinear system under the action of \( u = u_0 + \Delta u = u_0 - K(x - x_0) \)
- Finding the expression for $\dot{V}$ by chain-rule differentiation of $V$ and substitution of the above state derivatives.

- You may simply copy-paste the long expression for $\dot{V}$ into the spherical sweep program (use \texttt{vpa(Vdot,8)} and copy-paste the result). Also, the state derivatives may be copied-pasted into a Matlab function for Simulink.

- Note that you must use the non-conjugate transpose $x.'$ when transposing symbolic matrices and vectors.

- Bonus points: Can you vary $Q$ to obtain a better radius of attraction? -Show a numerical example.

**How to turn in the homework:**

Any hand calculations must be explained. Please scan any handwritten sheets. Email the instructor, attaching the hand calculations, m-file(s) and Simulink files (.mdl). Show all steps (commands and outputs formatted using \texttt{vpa}) when using the Symbolic Toolbox Please save the Simulink files as version 7.6 or earlier (Matlab version 2010b or earlier). You are encouraged to consult the instructor on issues related to Matlab/Simulink.