

## Lecture 10: Euler-Lagrange Modeling of Robotic Manipulators

Reading: SHV Ch.7

Mechanical Engineering

Hanz Richter, PhD

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### Formulating the Kinetic Energy

The kinetic energy of a rigid body undergoing general motion (translation + rotation) can be expressed as the sum of two components:

**Energy due to velocity of the center of mass:** This component is simply

$$\frac{1}{2}mv^T v$$

where  $m$  is the mass of the body and  $v$  is the velocity of the center of mass relative to the world frame.

**Energy due to rotation about the center of mass:** This component can be expressed as

$$\frac{1}{2}w^T \mathcal{I} w$$

where  $w$  is the angular velocity referred to the world frame and  $\mathcal{I}$  is a geometry, mass and configuration-dependent term called the *inertia tensor*.

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# Kinetic Energy...

**To compute  $v$ , we rigidly attach a coordinate frame to the center of mass. Then we find the Jacobian. This also allows us to find  $w$ .** If  $R$  is the transformation frame between the body-attached frame and the world frame, and  $I$  is the inertia tensor expressed in the body-attached frame then

$$\mathcal{I} = RIR^T$$

**Note that  $I$  depends only on the geometry and mass distribution of the body (independent of  $q$ ). Tables exist for various shapes.**

The inertia tensor in the body frame is defined as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

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# Kinetic Energy...

The components of the inertia tensor are

$$\begin{aligned} I_{xx} &= \int \int \int (y^2 + z^2) \rho(x, y, z) dx dy dz \\ I_{yy} &= \int \int \int (x^2 + z^2) \rho(x, y, z) dx dy dz \\ I_{zz} &= \int \int \int (x^2 + y^2) \rho(x, y, z) dx dy dz \\ I_{xy} = I_{yx} &= - \int \int \int xy \rho(x, y, z) dx dy dz \\ I_{xz} = I_{zx} &= - \int \int \int xz \rho(x, y, z) dx dy dz \\ I_{yz} = I_{zy} &= - \int \int \int yz \rho(x, y, z) dx dy dz \end{aligned}$$

where  $\rho(x, y, z)$  is the density distribution (mass per unit volume).

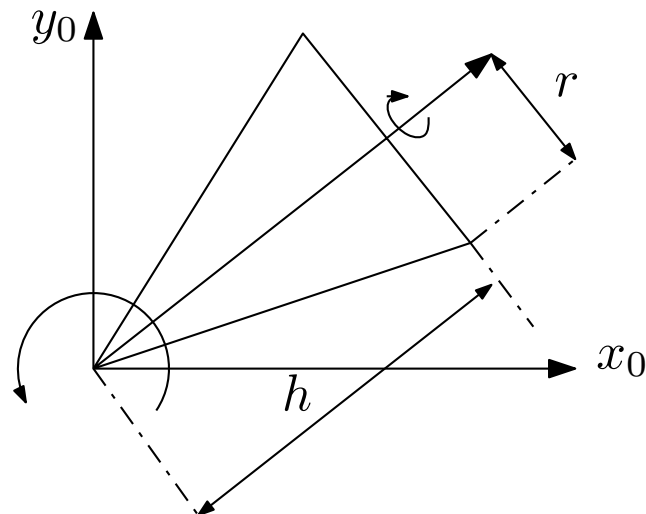
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# Kinetic Energy...

The terms  $I_{xx}$ ,  $I_{yy}$  and  $I_{zz}$  are called *principal moments of inertia*. The other terms are called *products of inertia*. When the body has uniform density ( $\rho$  constant) and symmetry about one of the axes, the product of inertia involving that axis is zero.

Example:

Find the kinetic energy of a cone with uniform density which rotates as shown in the figure. Use the methods developed in this course.



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## Kinetic Energy of a Manipulator

Remember that the velocity of the center of mass and the angular velocity of link  $i$  can be obtained by using

$$v_i = J_{v_i}(q)\dot{q} \text{ and } w_i = J_{w_i}(q)\dot{q}$$

**The upper Jacobian  $J_{v_i}$  must be computed by replacing  $o_n$  in Eq. (4.57) by  $r_{ci}$ , the position vector of the center of mass of link  $i$ .**

As derived in SHV, the kinetic energy is found as

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

where  $D(q)$  is the *inertia matrix*:

$$D(q) = \left[ \sum_{i=1}^n m_i J_{v_i}(q)^T J_{v_i}(q) + J_{w_i}(q)^T R_i(q) I_i R_i(q)^T J_{w_i}(q) \right]$$

**The inertia matrix  $D(q)$  is symmetric and positive-definite for all  $q$**

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# Potential Energy of a Manipulator

The potential energy of the  $i^{th}$  link is obtained as

$$P_i = m_i g^T r_{ci}$$

where  $g$  is the vector specifying the acceleration of gravity in the world frame,  $m_i$  is the mass of the link and  $r_{ci}$  is the position vector of the center of mass.

The total potential energy is just the sum over all links:

$$P = \sum_{i=1}^n P_i$$

The above applies to rigid links. When links are elastic, the elastic potential energy needs to be included.

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## Equations of Motion

As derived in SHV, application of the Euler-Lagrange equations to each link results in a system of coupled differential equations. In matrix form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

Here  $D(q)$  is the inertia matrix introduced earlier,  $g(q)$  is the gradient of the potential energy ( $g_k = \frac{\partial P}{\partial q_k}$ ) and  $\tau$  is the vector of joint inputs (forces or torques). The elements of matrix  $C(q)$  are defined by

$$c_{kj} = \sum_{i=1}^n c_{ijk} \dot{q}_i$$

where  $c_{ijk}$  are the *Christoffel Symbols* defined as

$$c_{ijk} = \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_j} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\}$$

The quantities  $d_{ij}$  are the entries of the inertia matrix  $D(q)$ . Note that for a fixed

$k, c_{ijk} = c_{jik}$  (no need to compute all of the  $c_{ijk}$ .)

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# Example

Carefully follow the derivation of the model for the planar elbow manipulator (p.259). In HW5 you will obtain the model for the three-link case.

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## Model Properties: Skew Symmetry and Passivity

The matrix  $N(q, \dot{q}) = \dot{D}(q) - 2C(q, \dot{q})$  is skew-symmetric. This property will be useful when deriving certain control laws.

The *passivity property* states that the mechanical work spent when using joint forces/torques  $\tau$  to produce joint velocities  $\dot{q}$  is lower-bounded by (cannot be less than) certain number  $-\beta$ :

$$\int_0^T \dot{q}^T(\zeta) \tau(\zeta) d\zeta \geq -\beta$$

with  $\beta \geq 0$  and an arbitrary time  $T > 0$ . Physically, a passive system only dissipates energy. In fact, in a manipulator  $\dot{q}^T \tau = \dot{H}$  is power, where  $H$  is the energy. Integrating we see that

$$\int_0^T \dot{q}^T(\zeta) \tau(\zeta) d\zeta = H(T) - H(0) \geq -H(0)$$

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## Model Properties: Bounds on Inertia Matrix

For any frozen configuration  $q$ , the inertia matrix  $D(q)$  has  $n$  positive eigenvalues  $0 < \lambda_1(q) \leq \lambda_2(q) \leq \dots \leq \lambda_n(q)$ . It is possible to show (we do it now) that

$$\lambda_1(q)I \leq D(q) \leq \lambda_n(q)I$$

Show this by diagonalizing  $D(q)$  and using the fact that similarity transformations preserve eigenvalues (and therefore sign-definiteness).

When the manipulator contains only revolute joints,  $D(q)$  contains only sines and cosines, which are bounded by  $\pm 1$ . Then  $D(q)$  can be bounded by explicit constants:  $\lambda_m I \leq D(q) \leq \lambda_M I$

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## Model Properties: Linearity in the Parameters

The dynamic equations of an  $n$ -link manipulator can be written as

$$Y(q, \dot{q}, \ddot{q})\Theta = \tau$$

where  $Y(q, \dot{q}, \ddot{q})$  is an  $n$ -by- $l$  matrix function called the *regressor* and  $\Theta$  is an  $l$ -vector of parameters. The minimum number  $l$  of parameters is difficult to establish, as it is to find  $Y$ .

For many common configurations, the parameterization can be looked up in books/research literature. This property is crucial for advanced schemes like adaptive control and passivity-based control.

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