MCE/EEC 647/747: Robot Dynamics and Control

Lecture 4.5: Manipulability

Reading: SHV Sect.4.12

Mechanical Engineering
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The Singular Value Decomposition

Let A be the matrix of any linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$. We know that A rotates and changes the length of vectors.
The singular value decomposition (SVD) explains these geometric transformations completely. Fact:

1. Any m-by-n matrix A can be decomposed as

$$A = U \Sigma V^T$$

where U and V are orthogonal and Σ has the following structure:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where: } \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \sigma_p \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq ...\sigma_p > 0$ and $p = \min\{m, n\}$.

Geometric Interpretation

Transformation $u = Av$ takes a vector $v \in \mathbb{R}^n$ and returns a vector $u \in \mathbb{R}^m$. If v is arbitrarily varied under the restriction $||v|| = 1$:

1. The images u describe an ellipsoid.
2. The length of the major axis is σ_1 and σ_p is the length of the minor axis.
3. The right singular vector v_1 results in the maximum amplification ($||u_1||/||v_1|| = \sigma_1$). The image u_1 is the direction of the ellipsoid's major axis.
4. The right singular vector v_n results in the least amplification ($||u_2||/||v_2|| = \sigma_p$). The image u_2 is the direction of the ellipsoid's minor axis.

Matlab: `[U,S,V]=svd(A)`

Example

For an arbitrary 2x2 nonsingular matrix:

1. Use Matlab to vary v with $||v|| = 1$ using the polar coordinate parameterization.
2. Compute and plot the images to visualize the ellipse.
3. Obtain the SVD
4. Calculate Av_1/σ_1 to verify u_1 is obtained. Similarly with Av_2/σ_2.
5. Plot vectors u_1 and u_2 to verify that they coincide with the ellipse's principal axes.

These steps are carried out in `exampleSVD.m`.

From SVD to Manipulability

Regard v as the vector of joint velocities, $v^T = \dot{q}^T = [\dot{q}_1,\ \dot{q}_2.\ ...\dot{q}_n]$. Think of the Jacobian (at a fixed q) as the matrix of a linear transformation: $A = J(q)$ with respect to the world frame basis.

The problem is to determine the set of attainable velocity vectors (linear and angular) under a fixed “joint velocity bugdet”. It’s enough to consider

$$||\dot{q}|| \leq 1$$

since constants other than 1 result in a simple scaling.

This is clearly related to the SVD.

Manipulability Ellipsoid

Let the velocity vector (linear and angular) be ζ (a 6x1 vector). With a full-rank Jacobian, the following is a solution for $\dot{q}$:

$$\dot{q} = J^{+}\zeta$$

Then the set of velocities such that $||\dot{q}|| \leq 1$ is given by

$$\zeta^T (JJ^T)^{-1} \zeta \leq 1$$

This defines an m-dimensional ellipsoid, called *manipulability ellipsoid*

Manipulability Ellipsoid...

If the SVD of $J = U\Sigma V^T$ is used, it is possible to show that the ellipsoid can be described as

$$w^T \Sigma_m^{-2} w \leq 1$$

where $w = U^T \zeta$ (a coordinate transformation to the ellipsoid's principal axes, recall that U is a rotation matrix), and

$$\Sigma_m = \text{ diag } (\sigma_1^{-1}, \sigma_2^{-1}, ...\sigma_m^{-2})$$

This becomes the familiar equation for an ellipsoid:

$$\frac{w_1^2}{\sigma_1^2} + \frac{w_2^2}{\sigma_2^2} + ... + \frac{w_m^2}{\sigma_m^2} \leq 1$$

In other words, the singular values of J are the lengths of the ellipse's axes, and the volume of the ellipsoid (a measure of manipulability) is proportional to $\sigma_1 \sigma_2 ... \sigma_m$.

Manipulability Ellipsoid...

- null($J(q)$) is formed by the right singular vectors corresponding to zero singular values.
- The set of attainable velocity vectors at a given q is col($J(q)$).

Example: Describe the manipulability ellipsoid of the unit length, 2-link planar manipulator at $q_1 = 0$ and $q_2 = \pi/4$ (linear velocity only). Repeat for $q_1 = \pi/4$, $q_2 = 0$.

Manipulability Measures

One measure of a robot's manipulability is given by the ratio of the Jacobian's maximum to the minimum singular value (the *condition number*).

Matlab computes the condition number with `cond`. The larger the condition number, the closest the matrix is to being singular.

Isotropic manipulability is obtained when the condition number is one (ellipsoid is a sphere).

A measure of the volume of the manipulability ellipsoid was introduced by Yoshikawa as

$$\mu(q) = \sqrt{\det(J^T(q)J(q))}$$

When the Jacobian is square, $\mu(q) = \det(J(q))$.

Yoshikawa's Manipulability

$$\mu(q) = \sqrt{\det(J^T(q)J(q))}$$

This measure is convenient since $J^T J$ is always positive semi-definite and provides a measure of the ellipsoid's volume.
Also $\mu(q) = 0$ only when the Jacobian is singular.

However, it is a non-convex measure, which can present difficulties for certain optimization problems. A function $f : X \mapsto \mathbb{R}$ with $X \in \mathbb{R}^n$ is convex in X if $\forall x_1, x_2 \in X$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

$\forall t \in [0, 1]$.

"The image of an interpolation of two points is an interpolation of the images of the points, not an extrapolation."

Some Optimization Problems

- For a 2-link planar manipulator, prove that Yoshikawa's manipulability measure and the condition number are both independent of the first joint coordinate. (Doctoral HW).
- For the planar manipulator, plot the manipulability ellipses at a few values of q_2 for $q_1 = 0$. Determine the best value of q_2 visually.
- Find the best posture (optimal value of q_2) for a given planar manipulator per Yoshikawa's measure by formal maximization. (HW)
- Optimize the geometry of a manipulator to maximize $\mu(q)$ at a given q (potential project)