MCE/EEC 647/747: Robot Dynamics and Control

Lecture 5: Review of Classical Cascade Compensation

Reading: SHV Chapter 6
Any undergraduate text on classical control.

Mechanical Engineering

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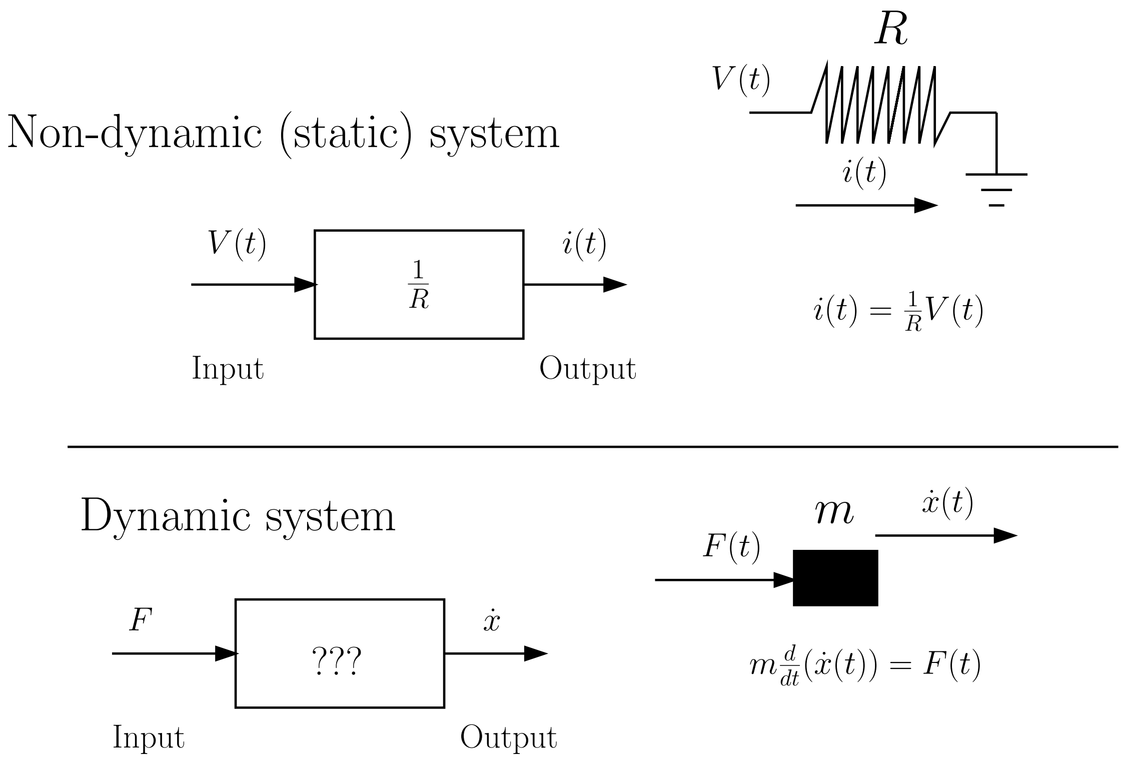
Motivation for the Laplace Transform

- Linear dynamic systems are often represented as I/O differential equations:

$$m\ddot{x} + b\dot{x} + kx = F$$

- An output is usually defined as a function of the system variable and its derivatives. We can define, for instance, $y = x$, $y = \dot{x}$ or other functions.
- We would like to have a more convenient, black-box representation, where the input is simply multiplied by an appropriately defined quantity to give the output.
- The idea is to imitate the linear scaling that occurs in non-dynamic systems.

Systems as Operators



Laplace Transform

- Defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

- s lives in $\mathbb{C}$.
- $f(t)$ needs to be transformable: $\int_0^\infty |f(t)|e^{-\sigma_1} dt < \infty$. Our $f(t)$'s will, no need to check.
- Inverse:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

- We don't need to carry out the integrations. Just use a table of Laplace transform pairs.

Useful Properties

- 1. Linear operator:

$$\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha\mathcal{L}\{f_1\} + \beta\mathcal{L}\{f_2\}$$

- 2. Transform of a derivative:

$$\mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0)..... - f^{(k-1)}(0)$$

- In most cases the initial conditions are zero: $f(0) = f'(0) = f''(0).... = 0$, so

$$\boxed{\mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s)}$$

- This property is the basis for operational calculus. Replaces derivatives by powers of s, reducing differential equations to algebraic equations in the Laplace domain (s variable).

Useful Properties

- 3. Transform of an integral:

$$\boxed{\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}}$$

- The above property is valid for functions such that $f(t) = 0$ for $t < 0$ (OK with us).

Example: If we have $3\ddddot{y} + 2\ddot{y} + y = 3\dot{x}$, taking the transform gives

$$3s^4Y(s) + 2s^2Y(s) + Y(s) = 3sX(s)$$

which allows to solve for Y in the s-domain. If $x(t)$ is given, we find $X(s)$ from a table. Then solve for $Y(s)$ and find the inverse in the table to get $y(t)$, if so desired.

Summary

In summary, the Laplace transform is useful to:

- Decouple O.D.E.'s and eliminate unwanted variables
- Obtain time solutions of O.D.E.'s (rarely needed in control system analysis)
- **Determine the dynamic properties of a system: stability, transient response, sensitivity, etc. without solving the O.D.E.'s**

We should identify s with the differentiation operator

$$\boxed{s \equiv \frac{d}{dt}}$$

and $\frac{1}{s}$ with the integration operator

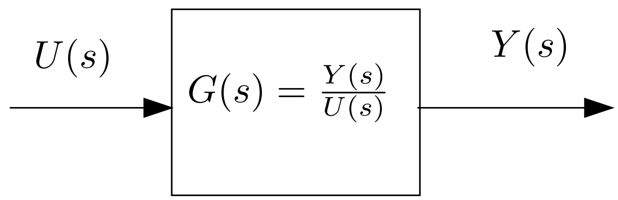
$$\boxed{\frac{1}{s} \equiv \int dt}$$

System Transfer Function

- Transfer functions are black box models. No information is provided regarding internal plant states.
- The concept of transfer function can be extended to multi-input, multi-output systems (MIMO transfer matrix).
- Transfer Functions carry the same information as the I/O diff. eq. in a more convenient format
- Definition:

  The transfer function of a system is the ratio of Laplace transforms of output and input, with all initial conditions set to zero.

Transfer Function Definition



- Does not provide information about internal system structure (nor does the I/O ODE)
- E X T E N S I V E L Y used for studying linear system properties and for design
- It equals the Laplace transform of the **impulse response** of a system
- The impulse respose can be obtained experimentally (hammering steel tanks to find cracks)

Poles and Zeros

Transfer functions of finite-dimensional (lumped-parameter systems) are always rational functions (ratio of polynomials) of s:

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

- Poles: Roots of the denominator
- Zeros: Roots of the numerator
- **Poles and zeros are the fundamental indicators of dynamic response and stability**
- Factored form:

$$\boxed{G(S) = \frac{K(s+z_1)(s+z_2)...(s+z_m)}{(s+p_1)(s+p_2)...(s+p_n)}}$$

Poles and Zeros in Matlab

- We limit ourselves to *causal* systems: system order$= n \geq m$.
- Matlab example: Find the poles and zeros of the TF:

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{s^4 + 3s^2 + 2s + 1}$$

- ```
  >>num=[1 5 9 7];
  >>den=[1 0 3 2 1];
  >>roots(num)
  >>roots(den)
  ```
- Alternatively, to find the gain as well:
  ```
  >>[z,p,k]=tf2zpk(num,den)
  ```


Stability Concepts

- Stability = Bounded Signals
- Many formal approaches to system stability exist. We'll consider one: BIBO (external) Stability.
- Bounded Input - Bounded Output (BIBO) stability: Used for transfer functions. The output must remain bounded whenever the input is bounded.
- No boundedness is required for internal states.
- Mathematically:

A transfer function is BIBO stable if and only if all of its poles have negative real parts
  ```

Open vs. Closed-Loop Stability

In a unity feedback loop with plant $G(s)$ and compensator $K(s)$, the stability of the open-loop system is determined by the poles of $G(s)K(s)$, the *loop transfer function*. In contrast, the closed-loop transfer function (from reference to output) is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

Therefore, closed-loop stability is determined by the poles of the *characteristic equation* $1 + G(s)K(s) = 0$. These poles are called *closed-loop poles*.
The transient response is determined by the closed-loop poles and the zeroes of $G(s)K(s)$.

Example: Run the Simulink system `example1.mdl` and observe the influence of the closed-loop poles and zeroes in the step response.

The Final Value Theorem

- Provides information about the value of $f(t)$ at *steady state* ($t = \infty$)
- The steady-state may not exist: the poles of $sF(s)$ must lie on the open left half of $\mathbb{C}$ (details later)
- If $\lim_{t\to\infty} f(t)$ exists, then

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

- Example: For $F(s) = \frac{1}{s(s+1)}$, we see that the only pole of $sF(s)$ lies on the left half plane. So

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$$

- Check by inversion of $F(s)$

PID Controllers

- Proportional-Integral-Derivative (PID) controllers are widely used in industrial process control.
- The controller has the transfer function

$$G_c(s) = K_p + \frac{K_i}{s} + K_d(s) = \frac{U(s)}{E(s)}$$

- The control law in the time domain is given by

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}$$

- Derivative term approximately implemented:

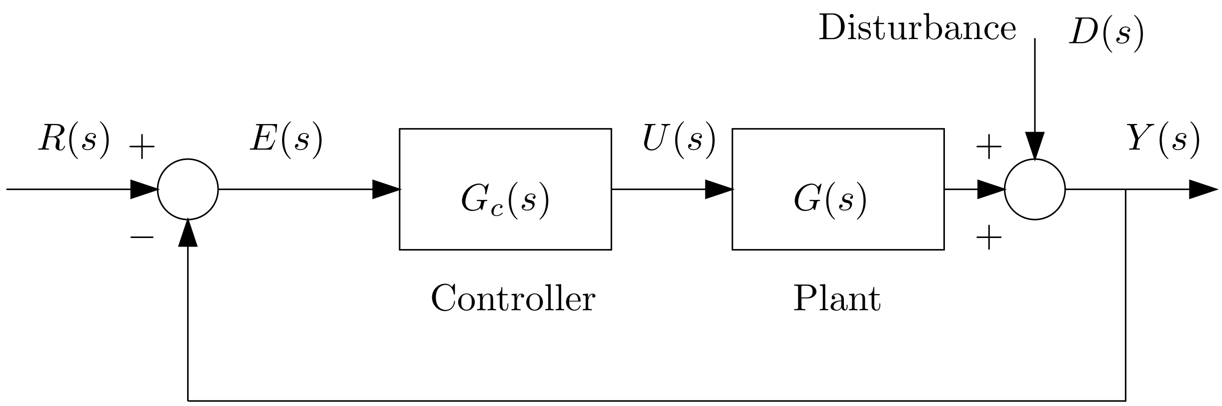
$$G_d(s) = \frac{K_d s}{\tau_d s + 1}, \quad \tau_d \to 0$$

PID Controllers...

- Analog PID boxes can be bought off-the shelf or custom built using R,L, C and op-amps.
- Digital PID can be bought off-the-shelf or programmed according to the application.
- Programmable Logic Controllers (PLC) come with a PID feature.
- PID controllers have limitations.

The Control Objective

The ultimate purpose of using control is to drive the error *nicely* to zero when $r(t)$ and $d(t)$ change in unanticipated ways and when $G(s)$ has uncertain and time-varying parameters and dynamics.



Error-Rejecting Props of P and I actions

Suppose only P-term is used: $u(t) = K_p e(t)$. If a steady-state is reached we have $u_\infty = K_p e_\infty$

e_∞ does not have to be zero

Now suppose an I-term is used: $u(t) = K_i \int_0^t e(\tau) d\tau$. If a steady-state is reached, we have $u_\infty = K_i \int_0^t e_\infty d\tau = K_i e_\infty t$ which must be true for all t.

This is only possible when $e_\infty = 0$.

This powerful property of integral action is one of the key ideas used in controller design.

Example: Stabilization of Double-Integrator

The double integrator plant $G(s) = s^{-2}$ corresponds to a basic mechanical system (mass) driven by force, with position ouput. Suppose a PID controller is to be tuned to achieve zero steady-state error to a step command and closed-loop stability. First, addressing stability, we see that the characteristic equation is given by

$$1 + \frac{K_p + \frac{K_i}{s} + K_d s}{s^2} = 0$$

which reduces to $s^3 + K_d s^2 + K_p s + K_i = 0$. The three roots of this equation must have negative real parts for stability. Although the roots are difficult to write down, the Routh-Hurwitz criterion can be applied to derive the stability condition $K_d K_p > K_i \geq 0$. Note that using $K_i = 0$ (PD control) places a closed-loop pole at zero, however this pole is canceled with a zero at zero.

Stabilization of Double-Integrator...

In terms of steady-state error, we know that

$$Y(s) = T(s)R(s)$$

while $E(s) = R(s) - Y(s)$. The error for a step input $R(s) = 1/s$ can be found as $E(s) = (1 - T(s))/s$. Applying the final value theorem, the following expression gives the steady-state error:

$$e_\infty = 1 - T(0) = 0$$

Therefore, any choice of PID gains such that $K_d K_p > K_i \geq 0$ achieves the desired objective. Gain selection is then done to meet transient response specifications and maintain the control effort within saturation limits.

The use of the I gain is unnecessary in this case, since integral action is already included in the plant dynamics.

Stabilization of Double-Integrator...

For a PD control solution, we can factor the characteristic equation as

$$1 + K_d \frac{s + \frac{K_p}{K_d}}{s^2} = 0$$

and fix the value of the ratio K_p/K_d, which places the zero in the closed-loop system. According to classical theory, a high value of K_p/K_d places the zero in the far left-plane, making its influence negligible. Then we use a root locus using K_d as parameter to choose the dominant closed-loop poles dictating the transient response.

Run `example2.mdl` to observe the effects of the individual gains