Asymptotic analysis and domain decomposition for a singularly perturbed reaction–convection–diffusion system with shock–interior layer interactions

S. Shao

Department of Mathematics, Cleveland State University, Cleveland, OH 44115, USA

Received 21 August 2003; accepted 15 November 2005

Abstract

We study an initial-boundary value problem for a singularly perturbed reaction–convection–diffusion system. Asymptotic analysis is used to construct a domain decomposition method for the system to describe the asymptotic nature of the interactions between the boundary layers, interior layers and shock layers. Our results show that the formation of boundary layers and shock layers depends upon initial and boundary data. Impinging shock can thicken interior layers at the point of intersection.

Keywords: Singularly perturbed reaction–convection–diffusion system; Boundary layers; Shock layer; Interior corner layers; Domain decomposition

1. Introduction

Boundary-value problems with critical parameters pose some of the most challenging problems in mathematics and computational science. Those critical parameters, for example, in fluid flow (viscosity), combustion (Lewis number), and superconductivity (Ginzburg–Landau parameter), determine the nature of the solution in some critical way. This is because the solution undergoes dramatic changes over very short distances, which shock and boundary layers, in fluid flow, are visible manifestations of this type of behavior. The fluid dynamical literature is replete with discussions of these important phenomena that are based either on extensive experimental studies or on extensive numerical treatments of the governing differential equations [1,5,9]. However, the remarkably complex character of the shock layers places very strong demands...
Consider an initial-boundary value problem for a singularly perturbed reaction–convection–diffusion system

\begin{align}
&u_t = u_{xx} - v, \quad (x, t) \in (0, 1) \times (0, \infty), \\
&u(x, 0, \epsilon) = \phi_i(x, \epsilon), \quad x \in [0, 1], \\
&u(0, t, \epsilon) = u(1, t, \epsilon) = 0, \quad t \in [0, \infty), \\
&v_t + h(u_x)v_x + g(x, u, u_x)v = \epsilon v_{xx}, \quad (x, t) \in (0, 1) \times (0, \infty), \\
&v(x, 0, \epsilon) = \phi_2(x, \epsilon), \quad x \in [0, 1], \\
&v(0, t, \epsilon) = v_0(t), \quad v(1, t, \epsilon) = v_1(t), \quad t \in [0, \infty),
\end{align}

(1.1)

where $\epsilon$ is a small positive parameter. We assume that function $\phi_i$ is continuous and norm $\|\phi_i\|_\infty$ is bounded in $[0, 1]$ for all $\epsilon > 0$, $i = 1, 2$, where $\|\phi\|_\infty$ is defined as the $L^\infty$-norm of $\phi$. $h(y) = 0$ if and only if $y = 0$, and $g(x, u, u_x) \geq 0$. Finally $h(u_x)$ and $g(x, u, u_x) \in C^{(1)}[0, 1]$ are smooth enough to exclude the coalescence of characteristics of the same family. The equations in (1.1) are related to a physical model of the homogenization process of a passive tracer in a flow with closed mean streamlines. Rhines and Young [15] derived the process of expulsion of the gradient of the passive scalar in which they studied the initial-value problem

\begin{align}
&\theta_t + J(\psi, \theta) = k \Delta^2 \theta, \quad \theta(x, y, 0) = \theta_0(x, y),
\end{align}

(1.2)

where $\psi(x, y, t)$ is the streamfunction of the flow, $\theta$ is the passive scalar, and $k$ is the diffusivity. In a steady plane shear-flow, the tracer equation becomes $\theta_t + U \theta_x = k \Delta^2 \theta$, where $U$ is the velocity of the flow. For certain prescribed initial values of $\theta$, Rhines and Young [15] derived mathematical expressions of the solution of system (1.2) and gave a physical explanation of how rapidly the passive scalar $\theta$ is homogenized by interacting with the velocity field and the diffusivity. However, for small diffusivity $k$, the behavior of the solution $\theta$ to the homogenization process is less well-known for more general initial conditions. The equations in (1.1) are simple analogues of Rhines and Youngs’ model with more general initial condition. Motivated by these applications, Shao [13], Shao and Chin [14] studied system (1.1). An asymptotic guided algorithm was developed in [14] to provide accurate and efficient numerical solutions of system (1.1) with turning points as $\epsilon \rightarrow 0$. The numerical schemes developed in [14] can be used to faithfully capture the essential asymptotic behavior of the solution of system (1.1), except in the case of the presence of shock layers. In this paper we use asymptotic analysis to construct the domain decomposition method in solving (1.1) with shock–interior layer interactions as $\epsilon \rightarrow 0$. Our approach is to show by means of asymptotic analysis (singular perturbation theory) how the boundary layers, interior layers and shock layers are influenced by initial and boundary data of system (1.1), and how the interior layers are influenced by the presence of a shock layer. The goal is to provide a methodology for identifying and characterizing the shock–interior layer interactions for the system as $\epsilon \rightarrow 0$. Our results show that an impinging shock can thicken interior layers at the point of intersection. An illustrative example is given to validate our analysis.
and treatments via asymptotic methods in the design of domain decomposition algorithms. We hope that our results may provide deeper understanding of the related physical problems, such as the homogenization process $\theta$ of (1.2).

The outline of the article is as follows. In Section 2 we review some relevant theoretical results concerning the asymptotic behavior of solutions of system (1.1) with boundary and interior layers. The shock layer formation and the asymptotic structure of the system (1.1) are discussed in Section 3. We conclude the paper in Section 4 with an illustrative example and some final remarks.

2. Boundary layers, interior layers and domain decomposition

In this section, we first review some previous analytical results of system (1.1) [2,6,13,14], then we analyze the asymptotic behavior of the solution $v$ of (1.1) and how to use these results to decompose the domain.

We know that for finite time ($t < T < \infty$), there exists a locally unique solution of (1.1).

It also can be proved that the asymptotic solutions of (1.1) can be expressed in terms of the solutions of the associated system of reduced equations,

\begin{equation}
\begin{align*}
U_t &= U_{xx} - V, \\
V_t + h(U) V_x + g(x, U, U_x)V &= 0,
\end{align*}
\end{equation}

(2.1)

to be supplemented by appropriate boundary and interior layers.

The $U$-equation of (2.1) is an inhomogeneous heat equation driven by $V$. The imposition of initial and boundary conditions for the $U$-equation is obvious: they are

\begin{align*}
U(x, 0) &= \phi_1(x) \quad \text{and} \quad U(0, t) = U(1, t) = 0.
\end{align*}

(2.2)

and the solution of the $U$-equation with the initial boundary conditions (2.2) is well-known,

\begin{align*}
U(x, t) &= \sum_{n=1}^{\infty} 2e^{-(n\pi)^2t} \sin(n\pi x) \\
&\quad \times \left[ \int_0^1 \left\{ \phi_1(x) - \int_0^t V(x, \tau) e^{(n\pi)^2\tau} d\tau \right\} \sin(n\pi x) dx \right].
\end{align*}

(2.3)

While the $V$-equation is a singularly perturbed hyperbolic equation which can exhibit certain layers of rapid change in the solution and its derivatives, the imposition of initial and boundary conditions for the $V$-equation is very subtle. We notice that the $V$-equation (2.1) of the reduced system is a convection–reaction equation whose coefficients depend on $U$ and $U_x$. Thus, the solution $V(x, t)$ of the convection–reaction equation is strongly affected by the form of its convective coefficient, $h(U_x)$. We seek a reduced solution $(U, V)$ of (2.1), which satisfies the original data on the subsets of the parabolic boundary

$B = \{(0, t) : 0 < t < T\} \cup \{(x, 0) : 0 < x < 1\} \cup \{(1, t) : 0 < t < T\},$

where the characteristic curves of the $V$-equation enter the domain $\Omega = (0, 1) \times (0, T)$. This solution of $(U, V)$ of (2.1) is closed to the solution $u(x, t, \epsilon), v(x, t, \epsilon)$ in $\tilde{\Omega}$ except in boundary, interior and shock layer subdomains where second derivatives of $u$ and $v$ become large as $\epsilon \to 0$. Thus, solution $(U, V)$ of (2.1) can be supplemented by appropriate boundary layer solutions, interior layer solutions or shock layer solutions [3,6,13] and [14].
Consider the solution behavior of the $V$-equation of (2.1)

$$V_t + h(U_x)V_x + g(x, U, U_x)V = 0. \tag{2.4}$$

It is a first-order hyperbolic equation and is solvable using the method of characteristics with the characteristic direction given by $[h(U_x), 1]$. We see that $U_x \neq 0$ at both $x = 0$ and $x = 1$ from the $U$-equation of (2.1) and its boundary conditions of (2.2). This suggests that the characteristic curves of the $V$-equation enter $\Omega$ along the boundaries

$$x = 0 \quad \text{if } h(U_x(0, t)) > 0 \quad \text{for some } t > t_0,$$

$$x = 1 \quad \text{if } h(U_x(1, t)) < 0 \quad \text{for some } t > t_0. \tag{2.5}$$

We define

$$\omega(x, t) = tx(x - 1), \quad 0 < x < 1, t > 0,$$

$$\sigma(x, t) = [h(U_x), 1] \cdot \begin{bmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial t} \end{bmatrix} = h(U_x) \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial t},$$

and

$$B = B_{<} \cup B_0 \cup B_{>},$$

where

$$B_{<} = \{(x, t) \in B : \sigma(x, t) < 0\},$$

$$B_0 = \{(x, t) \in B : \sigma(x, t) = 0\},$$

$$B_{>} = \{(x, t) \in B : \sigma(x, t) > 0\}. \tag{2.6}$$

Then we have $\Omega = (0, 1) \times (0, T) = \{(x, t) : \omega < 0\}$ and $[\frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial t}]$ is the exterior normal to the boundary $\{B \setminus \{(0, 0) \cup (1, 0)\}\}$. Therefore, the reduced solution $V$ of (2.4) must satisfy the given initial-boundary data on $B_{<}$ where the characteristics of the $V$-equation (2.4) enter the domain $\Omega$. A boundary layer of width $O(\epsilon)$ occurs on $B_{>}$ and a boundary layer of width $O(\sqrt{\epsilon})$ occurs on $B_0$. In the second case, $B_0$ is itself a characteristic of the $V$-equation (2.4) (see [14]). Since

$$\sigma(x, 0) = [h(U_x(x, 0), 1] \cdot \begin{bmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial t} \end{bmatrix}_{t=0} = [h(U_x(x, 0)), 1] \cdot [0, -1] = -1 < 0, \tag{2.7}$$

this implies the reduced solution $V$ of (2.4) must satisfy the given initial condition. That is, the reduced solution $V$ always satisfies the initial condition

$$V(x, 0) = \phi_2(x). \tag{2.8}$$

For the boundary conditions of the $V$-equation, we have the following results (cf. [14]):

(i) $V(0, t) = v_0(t)$ if $h(U_x(0, t)) > 0$, for all $t > t_0$;

(ii) $V(1, t) = v_1(t)$ if $h(U_x(1, t)) < 0$, for all $t > t_0$;

(iii) If $h(U_x(0, t))$ or $h(U_x(1, t))$ change sign in $[0, T]$, then $V(i, t) = v_i(t)$ provide

$$\hat{h}(U_x(i, t)) > 0 \quad \text{for some } t \in [0, T], i = 0, 1,$$

where

$$\hat{h}(U_x(i, t)) = \begin{cases} h(U_x(0, t)) & \text{if } i = 0, \\ -h(U_x(1, t)) & \text{if } i = 1. \end{cases} \tag{2.9}$$
These results follow from \( \sigma(0, t) < 0 \) if \( h(U(x, 0, t)) > 0 \) and \( \sigma(1, t) < 0 \) if \( h(U(x, 1, t)) > 0 \). In other words, the reduced solution \( V \) cannot satisfy the boundary condition near \( x = 0 \) if \( h(U(x, 0, t)) < 0 \), then there must be a boundary layer near \( x = 0 \); and if \( h(U(x, 1, t)) > 0 \), there is likewise a boundary layer near \( x = 1 \).

Therefore, when \( h(u_x) < 0 \), the boundary layer solution near \( x = 0 \) satisfies the following parabolic equation and boundary conditions:

\[
\begin{align*}
  h(u_x(0, t))v_\zeta + g(\epsilon, 0, u_x(0, t))v &= v_\zeta, \\
  v(0, t) &= v_0(t) - V(0, t), \quad \lim_{\zeta \to \infty} v(\zeta, t) \to 0;
\end{align*}
\]  

(2.10)

with \( \zeta = x/\epsilon \). (2.10) can be solved explicitly, that is the boundary layer near \( x = 0 \) has the form

\[
v(x, t) = (v_0(t) - V(0, t)) \exp \left[ \zeta \left( \frac{1}{2} h_0(t) - \sqrt{h_0(t) + g_0(t) - \frac{1}{4}(h_0(t))^2} \right) \right]
\]

(2.11)

where \( h_0(t) = h(u_x(0, t)) \) and \( g_0(t) = g(t, 0, u_x(0, t)) \). Similarly, when \( h(u_x) > 0 \), the boundary layer solution near \( x = 1 \), with the boundary layer variable, \( \varphi = (1 - x)/\epsilon \) satisfies

\[
\begin{align*}
  -h(u_x(1, t))v_\varphi + g(1 - \varphi \epsilon, 0, u_x(1, t))v &= v_\varphi, \\
  v(0, t) &= v_1(t) - V(1, t), \quad \lim_{\varphi \to \infty} v(\varphi, t) \to 0,
\end{align*}
\]  

(2.12)

and the boundary layer solution at \( x = 1 \) has the form

\[
v(\varphi, t) = (v_1(t) - V(1, t)) \exp \left[ \varphi \left( \frac{1}{2} h_1(t) - \sqrt{h_1(t) + g_1(t) - \frac{1}{4}(h_1(t))^2} \right) \right]
\]

(2.13)

where \( h_1(t) = h(u_x(1, t)) \) and \( g_1(t) = g(t, 0, u_x(1, t)) \).

Because the \( V \)-equation (2.4) prescribes the directional derivative of \( V \) along the characteristic curves, while the initial and the boundary conditions specify the directional derivatives of \( V \) along the coordinate axes, thus, for \( h(u_x) < 0 \), \( \nabla V \) is continuous at the origin if and only if \( v_1 = v(1, t) = \phi_2(1) \) and

\[
0 = \frac{dv_1}{dt} = -h(U_x) \frac{d\phi_2}{dx} \quad \text{which implies } \phi_2(x) = \text{constant};
\]

(2.14)

while for \( h(u_x) > 0 \), \( \nabla V \) is continuous at the origin if and only if

\[
v_0 = \phi_2(0) \quad \text{and } \phi_2(x) = \text{constant},
\]

(2.15)

since, in general, the initial condition of \( \phi_2(x) \) of \( V \) is not equal to the boundary conditions \( v(0, t) = v_0(t) \) at \( x = 0 \) or \( v(0, t) = v_1(t) \) at \( x = 1 \). When \( h(U_x(0, t)) > 0 \), the partial derivatives of the solution \( v \) are large near the origin \((0, 0)\) and remain large in a neighborhood of the characteristic emanating from the origin. The incompatibility \( \phi_2(0) \neq v_0 \) propagates as a discontinuity in the solution \( V \) of (2.4) and (2.8) along this characteristic, then we have an interior corner layer (simply interior layer) of \( O(\epsilon^{1/2} \exp[-(x-F_0(t))/4]) \), where \( \nabla V \) is discontinuous along \( x = F_0(t) \). To recover the effects of the incompatibility, we revise the boundary condition for the reduced problem by setting \( c_0 = v_0(t) - \phi_2(0) \). The solution of the interior layer of \( V \) near the origin \((0, 0)\) satisfies

\[
\begin{align*}
  v_t + h(u_x)v_x + g(x, 0, u_x(0, 0))v &= \epsilon v_{xx}, \\
  v(x, 0, \epsilon) &= 0, \quad v(0, t, \epsilon) = c_0.
\end{align*}
\]

(2.16)
in the subdomain \( IL_0 = \{(x, t) \in \Omega \mid \Gamma_0^{-1}(x) - \mu_1\epsilon^{1/2} < t < \Gamma_0^{-1}(x) + \mu_2\epsilon^{1/2}\} \), where \( \mu_1 \) and \( \mu_2 \) are small positive constants. Therefore, for \( h(u_x) > 0 \), we have

\[
|v(x, t, \epsilon) - V(x, t)| \leq C_0\epsilon + |v_1(t) - V(1, t)| \\
\times \exp \left[ \frac{1 - x}{\epsilon} \left( \frac{1}{2}h_1(t) - \sqrt{h_1(t) + g_1(t) - \frac{1}{4}(h_1(t))^2} \right) \right] \\
+ |v_0 - \phi_2(0)|\epsilon^{1/2} \exp \left[ \frac{(x - \Gamma_0(t))}{\epsilon^{1/2}} \right]
\]  

(2.17)

for \((x, t)\) in \( \Omega \), where \( C_0 \) is a positive constant, and the reduced solution \( V \) has an analytical representation

\[
V(\xi, t) = \begin{cases} 
    v_0(t), & \xi \leq \Gamma_0(t); \\
    \phi_2(\xi) \exp \left\{ \int_0^t -g(\xi, \tau) d\tau \right\}, & \xi \geq \Gamma_0(t),
\end{cases}
\]  

(2.18)

where \((\xi, t)\) are characteristic coordinates such that

\[
\frac{dx}{d\tau} = h(U_x(\xi, t)), \\
x(0) = \xi \quad \text{for} \quad \xi > 0, \\
x(t = \xi) = 0 \quad \text{for} \quad \xi < 0.
\]  

(2.19)

On the other hand, there is an interior layer near \((1, 0)\) if \( h(U_x(1, t)) < 0 \), \( \phi_2(1) \neq v_1 \). Let \( c_1 = v_1(t) - \phi_2(0) \), then the interior layer solution of \( V \) near the corner \((1, 0)\) satisfies

\[
v_t + h(u_x)v_x + g(x, u, u_x) v = \epsilon v_{xx}, \\
v(x, 0, \epsilon) = 0, \quad v(1, t, \epsilon) = c_1.
\]  

(2.20)

in the subdomain \( IL_1 = \{(x, t) \in \Omega \mid \Gamma_1^{-1}(x) - \mu_3\epsilon^{1/2}(x) < t < \Gamma_1^{-1}(x) + \mu_4\epsilon^{1/2}\} \) with \( \mu_3 \) and \( \mu_4 \) small positive constants. Hence, for \( h(u_x) < 0 \), we have

\[
|v(x, t, \epsilon) - V(x, t)| \\
\leq C_1\epsilon + |v_0(t) - V(0, t)| \exp \left[ \frac{x}{\epsilon} \left( \frac{1}{2}h_0(t) - \sqrt{h_0(t) + g_1(t) - \frac{1}{4}(h_0(t))^2} \right) \right] \\
+ |v_1 - \phi_2(1)|\epsilon^{1/2} \exp \left[ \frac{(x - \Gamma_1(t))}{\epsilon^{1/2}} \right]
\]  

(2.21)

for \((x, t)\) in \( \Omega \), where \( \nabla V \) is discontinuous along \( x = \Gamma_1(t) \) near \((1, 0)\). The reduced solution \( V \) in (2.21) has an analytical representation

\[
V(\xi, t) = \begin{cases} 
    v_1(t), & \xi \geq \Gamma_1(t); \\
    \phi_2(\xi) \exp \left\{ \int_0^t -g(\xi, \tau) d\tau \right\}, & \xi \leq \Gamma_1(t),
\end{cases}
\]  

(2.22)

where \((\xi, t)\) are characteristic coordinates defined as in (2.19).

Based on the above analysis we can use it as the theoretical tools to develop a domain decomposition method to obtain the asymptotic-numerical solutions of system (1.1) associated with cases (a) \( h(u_x) < 0 \) and case (b) \( h(u_x) > 0 \), respectively, as \( \epsilon \to 0^+ \) (cf. [14]). The
possible domain decomposition for the asymptotic behavior associated with case (a) is illustrated in Fig. 1, where the domain is subdivided as the subdomain for boundary layer $BL(1)$ near $x = 1$ and the subdomain $IL_0$ for the interior layer along $\Gamma_0$ (Gamma0 in Fig. 1). Similarly, the possible domain decomposition for the asymptotic behavior associated with case (b) is illustrated in Fig. 2, where the domain is subdivided as the subdomain for boundary layer $BL(0)$ near $x = 0$ and the subdomain $IL_1$ for the interior layer along $\Gamma_1$ (Gamma1 in Fig. 2).

3. Shock layer formation

In this section we examine how equations of (1.1) form a shock, and analyze how the interior layers are influenced by the presence of a shock layer. For $x \in [0, 1]$, if $h(U_x)$ of (2.4) changes sign such that (c) $h(U_x(0, t)) > 0$ and $h(U_x(1, t)) < 0$, then the reduced solution $V$ of (2.4) must satisfy both boundary conditions at $x = 0$ and $x = 1$ by (iii) in Section 2. We assume that $h(U_x)$ does not change signs along $x = 0$ and $x = 1$ (there will be boundary layer behaviors if $h(U_x)$ changes signs along $x = 0$ or $x = 1$, see Section 2). Then the two interior layers along
the characteristic curves converge and intersect at \((x_s, t_s) \in \Omega = (0, 1) \times (0, T)\), a shock layer behavior occurs, which is a transition zone of width \(O(\epsilon^{1/2})\) within the interior domain which the solution passes rapidly but smoothly from one inviscid state to another, starts at \((x_s, t_s)\) and sits along a subcharacteristic of the problem (1.1),

\[
\begin{align*}
  u_t &= u_{xx} - v, \\
  u(x, 0, \epsilon) &= \phi_1(x, \epsilon), \\
  u(0, t, \epsilon) = u(1, t, \epsilon) &= 0, \\
  v_t + h(u_x)v_x + g(x, u, u_x)v &= \epsilon v_{xx}, \\
  v(x, 0) &= \phi_2(x), \\
  v(0, t) = v_0(t), & \quad v(1, t) = v_1(t).
\end{align*}
\]

In order to understand the formation of the shock layer and the solution behavior of \(v\) of (1.1) in the case with a shock, we define, for the values \((x, t, v)\) in the domain \(\hat{\Omega}\),

\[
\hat{\Omega} = \bar{\Omega} \times \{v : |v(x, t, \epsilon) - V(x, t)| \leq \rho(t)\},
\]

where \(\Omega = (0, 1) \times (0, T)\), \(\rho(t)\) is defined by

\[
\rho(t) = \delta + \max\{|v_0(t) - V(0, t)|, |v_1(t) - V(1, t)|\},
\]

where \(\delta\) is a small positive number, and the shock layer solution \(V\) is the leading term in an asymptotic expansion of the solution \(v\) of (1.1), or say, a solution of the reduced equation of (1.1)

\[
\begin{align*}
  V_t + h(U_x)V_x + g(x, U, U_x)V &= 0, \\
  V(x, 0) &= \phi_2(x), \\
  V(0, t) = v_0(t), & \quad V(1, t) = v_1(t).
\end{align*}
\]  

To simply the discussion, we focus on the interaction between a shock layer and interior layers of system (1.1) \(\epsilon \to 0\). The analysis of the boundary–interior layer interactions for case (c) are similar to the analysis given in Section 2. A possible asymptotic behavior associated with case (c) is illustrated in Fig. 3, where domain \(\Omega\) is subdivided into the different
subdomains with \( \Omega = \Omega_L \cup \Omega_R \cup \Omega_{LS} \cup \Omega_{RS} \cup S \cup \Omega_{cL} \cup \Omega_{cR} \cup \Omega_{LI} \cup \Omega_{RI} \cup \gamma_L \cup \gamma_R \) (gammaS, gammaL, gammaR and (xS, tS) in Fig. 3 are defined as in \( \gamma_S, \gamma_L, \gamma_R \) and \((x_s, t_s)\), respectively). The definitions of all the subdomains are given in conditions (C2)–(C3) below. To estimate the size of the shock layer and interior layers, we have the following conditions:

(C1) We assume that \( g(t, u, u_x) \geq 0 \) for \((x, t, u)\) in the domain \( \Omega = \bar{\Omega} \times R \) and \([v_t - h(u_x)v_x + g(x, u, u_x)v]_t \geq m > 0 \) for some positive constant \( m \).

(C2) Assume that the system (1.1) has a smooth solution \((\hat{u}, \hat{v})\) with a shock layer of \( O(\varepsilon^{1/2}) \) along a simple curve \( S \) defined implicitly by \( \gamma_s(x, t) = 0 \); \( S \) interacts with both interior layers along characteristic curves \( \gamma_L \) and \( \gamma_R \), respectively. \( \gamma = \gamma_s \cup \gamma_L \cup \gamma_R \) is a piecewise smooth curve. It is clear that the shock layer starts at the intersection point \((x_s, t_s)\) of \( \gamma_L \) and \( \gamma_R \), which is also the turning point where \( h(U_x(x_s, t_s)) = 0 \). It implies \( U_x(x_s, t_s) = 0 \) by the assumption that \( h(y) = 0 \) if and only if \( y = 0 \). If, in addition, \( g(x, u, u_x) = 0 \), we have \( V_t(x_s, t_s) = 0 \) from the V-equation (2.4).

(C3) We assume that there exist smooth functions \( \gamma_s(x, t), \gamma_L(x, t) \) and \( \gamma_R(x, t) \) such that

\[
\begin{align*}
\Omega_{LS} &= \{(x, t) : \gamma_s(x, t) < 0, t \geq t_s\} \cap \Omega, \\
\Omega_{RS} &= \{(x, t) : \gamma_s(x, t) > 0, t \geq t_s\} \cap \Omega, \\
S &= \{(x, t) : \gamma_s(x, t) = 0, t \geq t_s\} \cap \Omega, \\
\Omega_L &= \{(x, t) : \gamma_L(x, t) < 0, t < t_s\} \cap \Omega, \\
\Omega_{cL} &= \{(x, t) : \gamma_L(x, t) > 0, t < t_s\} \cap \Omega, \\
\Omega_{LI} &= \{(x, t) : \gamma_L(x, t) = 0, t < t_s\} \cap \Omega, \\
\Omega_R &= \{(x, t) : \gamma_R(x, t) < 0, t < t_s\} \cap \Omega, \\
\Omega_{cR} &= \{(x, t) : \gamma_R(x, t) > 0, t < t_s\} \cap \Omega, \\
\Omega_{RI} &= \{(x, t) : \gamma_R(x, t) = 0, t < t_s\} \cap \Omega,
\end{align*}
\]

with \( \Omega = \Omega_L \cup \Omega_R \cup \Omega_{LS} \cup \Omega_{RS} \cup S \cup \Omega_{cL} \cup \Omega_{cR} \cup \Omega_{LI} \cup \Omega_{RI} \cup \gamma_L \cup \gamma_R \) (cf. Fig. 3) and that

\[
\begin{align*}
h(U_x) \frac{\partial \gamma_s}{\partial x} + \frac{\partial \gamma_s}{\partial t} &\geq 0 \quad \text{in } \Omega_{LS}^\delta, \\
h(U_x) \frac{\partial \gamma_L}{\partial x} + \frac{\partial \gamma_L}{\partial t} &\geq 0 \quad \text{in } \Omega_L^\delta, \\
h(U_x) \frac{\partial \gamma_R}{\partial x} + \frac{\partial \gamma_R}{\partial t} &\geq 0 \quad \text{in } \Omega_{cR}^\delta,
\end{align*}
\]

and

\[
\begin{align*}
h(U_x) \frac{\partial \gamma_s}{\partial x} + \frac{\partial \gamma_s}{\partial t} &\leq 0 \quad \text{in } \Omega_{RS}^\delta, \\
h(U_x) \frac{\partial \gamma_L}{\partial x} + \frac{\partial \gamma_L}{\partial t} &\leq 0 \quad \text{in } \Omega_{cL}^\delta, \\
h(U_x) \frac{\partial \gamma_R}{\partial x} + \frac{\partial \gamma_R}{\partial t} &\leq 0 \quad \text{in } \Omega_R^\delta,
\end{align*}
\]

where \( \Omega_k^\delta = \bar{\Omega}_k \cap \{(x, t) : \text{dist}(x, t, \gamma_j) < \delta\}, k = L, R, SL, SR, cL, cR; j = L, R, s. \)
Then $v$-equation in (1.1) has a shock layer solution

$$
\hat{v}(x,t,\epsilon) = \begin{cases} 
V_L, & \text{in } \Omega_{LS}, \\
V_R, & \text{in } \Omega_{RS}, \\
V_c, & \text{in } \Omega_{cL} \cup \Omega_{cR},
\end{cases}
$$

(3.4)

where $\Omega_{LS}$, $\Omega_{RS}$, $\Omega_{cL}$ and $\Omega_{cR}$ are defined in (C3), and $IL(\gamma_L)$, $IL(\gamma_R)$ and $SL(\gamma_s)$ interior layers and boundary layers, respectively, and are illustrated in Fig. 3. $V_c$ is the solution of the initial value problem (2.4) and (2.19) defined on $\Omega_{cL} \cup \Omega_{cR}$ and has the form

$$
V_c(\xi,t) = \phi_2(\xi) \exp \left\{ \int_0^t -g(\xi,\tau)d\tau \right\},
$$

(3.5)

where $(\xi,t)$ is the characteristic coordinate defined by (2.19). Thus, solution $\hat{v}(x,t,\epsilon)$ must satisfy

$$
\tilde{v}(x,t,\epsilon) = \lim_{\epsilon \to 0} \hat{v}(x,t,\epsilon) = \begin{cases} 
V_L, & \text{in } \Omega_{LS}, \\
V_R, & \text{in } \Omega_{RS}, \\
V_c, & \text{in } \Omega_{cL} \cup \Omega_{cR},
\end{cases}
$$

(3.6)

Therefore, the shock layer $V$-solution of (3.1) satisfies the reduced $v$-equation by virtue of the boundary conditions of $v$ in (1.1) in domain $\Omega_{LS} \cup S \cup \Omega_{RS}$. The reduced solution $V$ of (3.1) is supplemented by a shock layer and two interior layers. The shock layer connects the two branches $V_L$ and $V_R$, where $V_L(0,t) = v_0(t)$ and $V_R(1,t) = v_1(t)$.

We note that the entropy condition states that

$$
h((U_x)^{(1)}) > 0 > h((U_x)^{(2)}),
$$

(3.7)

where $(U_x)^{(1)}$ and $(U_x)^{(2)}$ are the corresponding derivative of the shock layer $U$-solution of (2.1) with respect to $x$, respectively (cf. [8]). That is,

$$
\hat{U}_x(x,t) = U_x^{(i)}, \quad i = 1, 2,
$$

(3.8)

defined and smooth on $\partial_l \cup S$ with $\Omega_1 = \Omega_{LS}$ and $\Omega_2 = \Omega_{RS}$. Condition (3.7) matches our previous condition in (iii) in Section 2. Hence, we have

$$
[h((U_x)^{(1)}(x,t)),1] \cdot \left[ \frac{\partial \gamma_x}{\partial x}, \frac{\partial \gamma_s}{\partial t} \right] > 0 \quad \text{for } (x,t) \in \Omega_{LS} \cup \gamma_s,
$$

(3.9)

$$
[h((U_x)^{(2)}(x,t)),1] \cdot \left[ \frac{\partial \gamma_x}{\partial x}, \frac{\partial \gamma_s}{\partial t} \right] < 0 \quad \text{for } (x,t) \in \Omega_{RS} \cup \gamma_s.
$$

Condition (3.9) tells us that the $V_L$-subcharacteristics of the reduced Eq. (2.4) leaves $\Omega_L$ nontangentially if $[h((U_x)^{(1)}(x,t)),1] \cdot \left[ \frac{\partial \gamma_x}{\partial x}, \frac{\partial \gamma_s}{\partial t} \right] > 0$; while if $[h((U_x)^{(2)}(x,t)),1] \cdot \left[ \frac{\partial \gamma_x}{\partial x}, \frac{\partial \gamma_s}{\partial t} \right] < 0$ then the $V_R$-subcharacteristics of (2.4) leave $\Omega_R$ nontangentially. To study the interaction of the shock layer and the interior layers in a neighborhood of the intersection point $(x_s,t_s)$, we let

$$
\tilde{v}(x,t,\epsilon) = \begin{cases} 
V_L, & \text{in } \Omega_{LS}, \\
V_R, & \text{in } \Omega_{RS}, \\
V_c, & \text{in } \Omega_{cL} \cup \Omega_{cR},
\end{cases}
$$

(3.10)

$$
+ s(x,t,\epsilon) + w_L(x,t,\epsilon) + w_R(x,t,\epsilon),
$$
where $V_L$, $V_R$ and $V_c(x, t)$ are defined as before in (3.6) and (3.5), $s(x, t, \epsilon)$ is the shock layer of $v$ of order $O(|\gamma_s(x, t)|(m_1/\epsilon)^{1/2})$ for some positive constant $m_1$, and the functions $w_L$ and $w_R$ are the left and right interior layer solutions, respectively. That is, $w_L$ is the solution of the initial-boundary value problem
\begin{align*}
    w_t + h(u_x)w_x + g(x, u, u_x)w = \epsilon w_{xx}, \quad t < t_s, x < x_s \\
    w(x, 0, \epsilon) = 0, \quad w(0, t, \epsilon) = v_0(t) - \phi_2(0),
\end{align*}
(3.11)
in the subdomain $\Omega_{LI} = \{(x, t) \in \Omega_L \cup \Omega_{cL} \mid \gamma_L^{-1}(x) - \mu_1\epsilon^{1/2} < t < \gamma_L^{-1}(x) + \mu_2\epsilon^{1/2}\}$, where $\mu_1$ and $\mu_2$ are small positive constants, while $w_R$ is the solution of equations
\begin{align*}
    w_t + h(u_x)w_x + g(x, u, u_x)w = \epsilon w_{xx}, \quad t > t_s, x > x_s \\
    w(x, 0, \epsilon) = 0, \quad w(1, t, \epsilon) = v_1(t) - \phi_2(1),
\end{align*}
(3.12)
in the subdomain $\Omega_{RI} = \{(x, t) \in \Omega_R \cup \Omega_{cR} \mid \gamma_R^{-1}(x) - \mu_3\epsilon^{1/2} < t < \gamma_R^{-1}(x) + \mu_4\epsilon^{1/2}\}$, where $\mu_3$ and $\mu_4$ are small positive constants, respectively, which match (2.16) and (2.20) in the neighborhood of $(x_s, t_s)$. To estimate the size of the shock layer and interior layers, we choose $m_1$ such that $0 < m_1 < m^{1/2}$, where $m$ is defined as in (C1). Therefore, in the neighborhood of $\gamma_s$, $\gamma_L$ and $\gamma_R$, the solution $v$ of (1.1) approximated by $\tilde{v}$ defined in (3.6) satisfies the estimate
\begin{align*}
    v(x, t, \epsilon) = \tilde{v}(x, t, \epsilon) + O(|V_L - V_c| \exp[-m_1\gamma_L^2(x, t)/\epsilon^{1/2}]) \\
    + O(|V_R - V_c| \exp[-m_1\gamma_R^2(x, t)/\epsilon^{1/2}]) \\
    + O(|V_L - V_R| \exp[-m_1\gamma_s^2(x, t)/\epsilon^{1/2}]) \\
    \times O((1/2)(\epsilon/m_1)^{1/2} \left| \frac{\partial V_L}{\partial n} - \frac{\partial V_c}{\partial n} \right| \exp[-|\gamma_L(x, t)|(m_1/\epsilon)^{1/2}]) \\
    + O((1/2)(\epsilon/m_1)^{1/2} \left| \frac{\partial V_R}{\partial n} - \frac{\partial V_c}{\partial n} \right| \exp[-|\gamma_R(x, t)|(m_1/\epsilon)^{1/2}]) \\
    + O((1/2)(\epsilon/m_1)^{1/2} \left| \frac{\partial V_L}{\partial n} - \frac{\partial V_R}{\partial n} \right| \exp[-|\gamma_s(x, t)|(m_1/\epsilon)^{1/2}]) \\
    + O(\epsilon^{1/2}).
\end{align*}
(3.13)
where $\frac{\partial}{\partial n} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial t}) \cdot \hat{n}$. We notice that in the neighborhood of the point $(x_s, t_s)$ where the shock layer intersects with the interior layers, the interior layers are thicker (of order $(m_1^{-1}\epsilon)^{1/2}$) as the result of this intersection comparing with the interior layers (2.16) and (2.20) that without the shock layer. We now state the following theorem.

**Theorem.** Assume that the reduced problem (2.1), (2.2) and (2.8) has a solution $V(x, t)$ and $U(x, t)$, and if conditions (C1)–(C3) hold, then there exists an $\epsilon_0 > 0$ such that the system (1.1) has a solution $(u(x, t, \epsilon), v(x, t, \epsilon))$ with
\begin{align*}
    u(x, t, \epsilon) = \sum_{n=1}^{\infty} 2e^{-((\pi n)^2 t)} \sin(n\pi x) \\
    \times \left[ \int_0^1 \left\{ \phi_1(x) - \int_0^t v(x, \tau, \epsilon)e^{(\pi n)^2 \tau} d\tau \right\} \sin(n\pi x) dx \right]
\end{align*}
(3.14)
and \( v(x, t, \epsilon) \) satisfies the estimate
\[
|v(x, t, \epsilon) - \bar{v}(x, t, \epsilon)| = O(|VL - V_c| \exp[-m_1y_L^2(x, t)/\epsilon^{1/2}]) + O(|VR - V_c| \exp[-m_1y_R^2(x, t)/\epsilon^{1/2}]) + O(|VL - VR| \exp[-m_1y_s^2(x, t)/\epsilon^{1/2}]) + O((1/2)\epsilon/m_1)^{1/2} \frac{\partial V_L}{\partial n} - \frac{\partial V_c}{\partial n} \exp[-|y_L(x, t)|(m_1/\epsilon)^{1/2}]) + O((1/2)\epsilon/m_1)^{1/2} \frac{\partial V_R}{\partial n} - \frac{\partial V_c}{\partial n} \exp[-|y_R(x, t)|(m_1/\epsilon)^{1/2}]) + O((1/2)\epsilon/m_1)^{1/2} \frac{\partial V_L}{\partial n} - \frac{\partial V_R}{\partial n} \exp[-|y_s(x, t)|(m_1/\epsilon)^{1/2}]) + O(\epsilon^{1/2}).
\]
whenever \( 0 < \epsilon \leq \epsilon_0 \), in the given domain, where \( \bar{v} \) is defined in (3.6).

**Remarks.** A double shock or a multiple shock may occur in case (c). Its analysis is similar to the above. The analysis can be used to develop a domain decomposition method for obtaining the asymptotic–numerical solutions of system (1.1) with shock–interior layer interactions as \( \epsilon \to 0 \). An illustrative example will be provided in the next section.

**4. An illustrative example and final remarks**

In this section, we will show that the analysis in the previous sections can be used to design a numerical method that faithfully captures the local asymptotic behavior of solutions of system (1.1) with shock–interior layer interactions. Before proceeding to provide an illustrative example to validate our analysis and treatments via asymptotic method in the design of domain decomposition algorithms for the solution of system (1.1) as \( \epsilon \to 0 \), we introduce an automated shock detection and the domain decomposition algorithm.

**Shock detection.** To identify a shock numerically, we need to know the behavior of \( h(U_x(x, t)) \). By the analysis in the previous sections, an automated shock detection can be built into the algorithm to monitor the shock layer in subdomain \( \Omega_{LS} \cup \Omega_{RS} \), where \( \Omega_{LS} \) and \( \Omega_{RS} \) are defined as in Section 3. The magnitude of the Jacobian
\[
J = \frac{\partial x}{\partial \xi}
\]
(4.1)
can be used as a shock detection. For region \( \Omega_{L}^\delta \), in a neighborhood of a shock the corresponding characteristics are getting asymptotically closed together [3,4]. That is, the characteristics to the left are travelling faster than the characteristics to the right. This effect is reflected as a diminution of the size of the Jacobian and it can also be observed by taking the partial derivative of \( J \) with respect to \( \xi \) to obtain
\[
\frac{\partial J}{\partial \tau} = J \frac{\partial h(U_x)}{\partial x}, \quad \text{or} \quad \frac{\partial J}{\partial t} = J \frac{\partial h(U_x)}{\partial x}.
\]
(4.2)
The initial condition for (4.2) is \( J = 1 \) on \( x = 0 \) by (2.1). \( J \) is decreasing as a function of \( t \) in the regions where \( \frac{\partial h(U_x)}{\partial x} < 0 \) and in the regions of converging characteristics. Therefore, inequality
\[
\frac{\partial h(U_x)}{\partial x} < 0
\]
(4.3)
gives rise to converging characteristics in this region $\Omega_L^{\delta}$, while for $\frac{\partial h(U)}{\partial x} > 0$ we have $J \geq 1$ (see [3]). Based on these ideas, a tolerance TOL can be specified. In the region where $J < TOL$ there are significant effects of the shock. The region inside the curve, $J = TOL$, will be the shock layer subdomain. This region is known with $h(U) = 0$ as the characteristic curves converge.

We monitor the magnitude of the Jacobian $J$ as the characteristics are computed through (4.2). This quantity is tracked and recorded at time level $t$. Thus, we obtain the values $x_l(t)$ and $x_r(t)$ which are the $x$ coordinates of the leftmost and rightmost characteristics respectively at which the magnitude of $J < TOL$. The collection of these coordinates comprises the boundary of the shock formation region. Then the shock-adaptive Godunov scheme [11,16] is introduced to cater for the shock propagation inside the shock region, we solve Eq. (2.4) with both boundary conditions of $V$ obtained from the shock formation region.

**Domain decomposition algorithm.** In the spirit of [14], we use asymptotic analysis to suggest a domain decomposition method to solve the system (1.1) with shock–interior interactions, which analysis and computation complement and enhance each other’s performance. The asymptotic-domain decomposition algorithm developed in [14] is used to capture the essential asymptotic behaviors of the solution of (1.1), except in the case of the presence of the shock layers. When the shock layer occurs, an automated shock detection built into the asymptotic-domain decomposition algorithm can be used to locate the shock layer region, the adaptive Godunov scheme [16] is then introduced to cater the shock propagation inside the shock region and the shock–interior layer interactions.

We follow the practice of the domain decomposition community by introducing overlaps between neighboring domains and apply Schwarz’s alternating method to transfer information between them. That is, we introduce an iterative scheme with two sub-stages: an inner and outer iteration. The outer iteration is concerned with communicating information between contiguous subdomains, while the inner iteration is concerned with finding with each subdomain an accurate and efficient local solution to the governing system of equations (cf. [14]).

To be specific, in the subdomains not containing boundary, interior and shock layers, the following iterative method is suggested:

$$
\frac{\partial u^k}{\partial t} - \frac{\partial^2 u^k}{\partial x^2} = -v^{k-1},
$$

$$\frac{\partial v^k}{\partial t} + h \left( \frac{\partial u^k}{\partial x} \right) \frac{\partial v^k}{\partial x} + g \left( x, u^k, \frac{\partial u^k}{\partial x} \right) = \epsilon \frac{\partial^2 v^{k-1}}{\partial x^2},
$$

The computational steps of solution algorithm for (4.4) and (4.5) are as follows. Let $U, V$ be the solutions at $(x_m, t_m)$ of (4.4), then $U^1 = \phi_1(x_m), V^1 = \phi_2(x_m), U^n = U(x_m, t_n)$, and $V^n = V(x_m, t_n)$, for $n > 1$. Then we perform the following computation to advance the solution from $t_{n-1}$ to $t_n$.

(1) For small $t_n - t_{n-1}$, we have

$$
(U_{\phi})_x (x_m, t_n, \epsilon) = \frac{1}{8\pi(t_n - t_{n-1})^3} \int_0^1 \exp \left( -\frac{(x_m - \bar{x})^2}{4(t_n - t_{n-1})} \right) (x_m - \bar{x}) U^{n-1} d\bar{x}
$$

$$+ O \left( \frac{\exp[-\delta^2/(t_n - t_{n-1})]}{t_n - t_{n-1}} \right),
$$

(4.6)
where $\delta$ is some constant independent of $t_n - t_{n-1}$; while for large $t_n - t_{n-1}$, we use

$$
(U_{\phi})_x(x_m, t_n, \epsilon) = 2 \sum_{k=1}^{p} (kp) \exp[-(kp)^2(t_n - t_{n-1})] 
\times \cos(k\pi x_m) \int_0^1 \sin(k\pi \bar{x}) U^{n-1} d\bar{x},
$$

(4.7)

where $M$ is the number of equally spaced points in $[0, 1]$ and $p = \sqrt{M}$.

(2) The inhomogeneous diffusion equation with homogeneous initial and boundary conditions is solved by ray-mod representation of the solution (see [14]) next:

$$
(U^k_{\phi})_x(x_m, t_n, \epsilon) = \frac{1}{\sqrt{\pi}} \left\{ \sqrt{t_n - t_{n-1}} - \int_0^{\sqrt{t_n - t_{n-1}}} \exp\left(-\frac{1}{2z^2}\right) dz \right\} 
\times V^{k-1} \left\{ 1 - \frac{t_n}{3} (V^{k-1}_x - V^{k-1}_{xx}) \right\} + O((t_n - t_{n-1})^{5/2}).
$$

(4.8)

This computation is required for each inner iteration. Then

$$
U^n_x = U_x(x_m, t_n, \epsilon) = (U_{\phi})_x(x_m, t_n, \epsilon) + (U^k_{\phi})_x(x_m, t_n, \epsilon).
$$

(4.9)

(3) The first order inhomogeneous hyperbolic equation is solved in

$$
\frac{\partial x}{\partial t} = h(U_x(\xi, t)).
$$

(4.10)

Now we specify a tolerance TOL and compute the shock detection $J$ through (4.2). We monitor the magnitude of $J$. In the case without a shock layer ($J > TOL$), we obtain

$$
V^n = \phi_2(\xi_m) \exp\left\{ - \int_{t_{n-1}}^{t_n} g(\xi_m, \tau) d\tau \right\} + \int_{t_{n-1}}^{t_n} \Gamma(\xi_m, \tau') \exp\left\{ - \int_{\tau'}^{t_n} g(\xi_m, \tau) d\tau \right\} d\tau',
$$

(4.11)

from (4.10), where $\Gamma(\xi, t)$ is the inhomogeneous term.

In the case where shock layers develop in the interior of the domain, where $J \leq TOL$, we track quantity $J$ and record at time level $t$. The region inside the curve $J = TOL$ will be the shock layer subdomain. The values $\xi_l(t)$ and $x_r(t)$ are the $x$ coordinates of the leftmost and rightmost characteristics, respectively, at which $J < TOL$. The collection of these coordinates comprises the boundary of the shock formation region. Then the adaptive Godunov scheme is introduced to cater the shock propagation inside the shock region. The numerical scheme of the shock-adaptive Godunov method used here is in the same spirit as [16]. That is, we form a semi-global central scheme with accuracy proportional to the number of cells to the nearest discontinuity.

(4) The convergence of the inner iteration scheme is checked. From the above inner iteration scheme, we obtain the solutions $U^n$ and $V^n$ in subdomain $\Omega_{cL} \cup \Omega_{cR}$. The detailed information can be found in [14].

One virtue of domain decomposition methods is that they enable in each subdomain the employment of methods of solution in accordance with its solution behavior. The sign of $h(U_x(\xi, t))$ and the location of zeros in (4.10) determine the location of the boundary and interior layers. In the case of the presence of a shock layer where $J \leq TOL$, the adaptive Godunov scheme is introduced to cater for the shock propagation inside the shock region. The width of
interior layers near the shock–interior layer intersection is of $O(m_1^{-1} \epsilon^{1/2})$ (cf. (3.13)), where $0 < m_1 < m^{1/2}$ with $m$ is defined in (C1). The widths of the the overlap regions of the subdomains are $\alpha \epsilon$ and $\beta \epsilon^{1/2}$ for the boundary layers and interior layers, where $\alpha$ and $\beta$ are parameters to be determined by computer experiments. Numerical matching is performed. The solutions in subdomains $\Omega_{cL} \cup \Omega_{cR}, \Omega_L, \Omega_R, \Omega_{LS}$ and $\Omega_{RS}$ provide the boundary data needed for solving the interior layer equations (3.12) and (3.11), respectively, and provide the boundary data for the adaptive Godunov schemes to capture the shock layer. Therefore, in different subdomains $\Omega_L, \Omega_R, \Omega_{LS}, \Omega_{RS}, S, \Omega_{cL}, \Omega_{cR}, \Omega_{L1}, \Omega_{R1}, \gamma_L$ and $\gamma_R$, different approximating equations and different numerical methods can be used to produce an accurate and efficient numerical method.

**Example.** Let $h(u_x) = u_x$, $g(x, u, u_x) = 0$, $\phi_1(x, \epsilon) = x(1 - x)$, $\phi_2(x, \epsilon) = \sin x$, $v_0 = -0.6$ and $v_1 = 1.0$, then we have

\[
\begin{align*}
  u_t &= u_{xx} - v, \quad (x, t) \in (0, 1) \times (0, \infty), \\
  u(x, 0, \epsilon) &= x(1 - x), \quad x \in [0, 1], \\
  u(0, t, \epsilon) &= u(1, t, \epsilon) = 0, \quad t \in [0, \infty), \\
  v_t + u_x v_x &= \epsilon v_{xx}, \quad (x, t) \in (0, 1) \times (0, \infty), \\
  v(x, 0, \epsilon) &= \sin x, \quad x \in [0, 1], \\
  v(0, t, \epsilon) &= -0.6, \quad v(1, t, \epsilon) = 1.0, \quad t \in [0, \infty).
\end{align*}
\]

Using the asymptotic numerical schemes described in the above, let $\epsilon = 10^{-4}$, we obtain the numerical solutions $v(x, t, \epsilon)$ and $u(x, t, \epsilon)$ of (4.12). The results are plotted in Figs. 4 and 5, respectively. We notice that the solution $v(x, t, \epsilon)$ exhibits rich asymptotic behaviors with the boundary layers, interior layers and the shock–interior layer interactions, which match our analysis. In addition, the solution $v(x, t, \epsilon)$ also exhibits more interesting features that it has a double shock layer with two shock–interior layer interactions for this particular example. The solution $u(x, t, \epsilon)$ exhibits the corresponding asymptotic behaviors and a much shaper interior corner layer near $(0, 0)$ when $t$ is comparatively small ($t < 6$). In other numerical experiments, we found that the shock profiles of both of the solutions $v(x, t, \epsilon)$ and $u(x, t, \epsilon)$ seem to move very slowly as $\epsilon$ decreases. It is difficult to see that the graphs are really different.

**Final remarks.** We study an initial-boundary value problem for a singularly perturbed reaction–convection–diffusion system (1.1) with a shock layer and interior layer interaction.
We show that asymptotic analysis can be used to construct a domain decomposition method to solve system (1.1) with the asymptotic nature of shock–interior layer interactions. Our results show that the formation of boundary layers and shock layers depends upon initial and boundary data. An impinging shock can thicken interior layers at the point of intersection. The numerical example also shows that some complicated situations may occur during the shock propagation. The suggested asymptotic-numerical method is able to cater for the asymptotic behavior of the solution of the system (1.1) with shock–interior layer interactions.

Acknowledgements

This research was supported by the Research and Creative Activities Grant No. 02100-01822 and by Cleveland State University’s College of Arts and Sciences’ Faculty Scholarly Travel Award.

References


