

Population Distributions in Biogeography-Based Optimization Algorithms with Elitism

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Abstract—Biogeography-based optimization (BBO) is an evolutionary algorithm that is based on the science of biogeography. Biogeography is the study of the geographical distribution of organisms. In BBO, problem solutions are represented as islands, and the sharing of features between solutions is represented as migration between islands. This paper develops a Markov analysis of BBO, including the option of elitism. Our analysis gives the probability of BBO convergence to each possible population distribution for a given problem. We compare our BBO Markov analysis with a similar genetic algorithm (GA) Markov analysis. Analytical comparisons on three simple problems show that with high mutation rates the performance of GAs and BBO is similar, but with low mutation rates BBO outperforms GAs. Our analysis also shows that elitism is not necessary for all problems, but for some problems it can significantly improve performance.

Index Terms—biogeography-based optimization, evolutionary algorithms, probability, combinatorics, Markov analysis

I. INTRODUCTION

Mathematical models of biogeography describe the migration, speciation, and extinction of species. The science of biogeography began with empirical descriptions by 19th century naturalists such as Alfred Wallace [1] and Charles Darwin [2]. Eugene Munroe was the first to introduce mathematical models of biogeography in 1948 [3], [4], and Robert MacArthur and Edward Wilson were the first to extensively develop and publicize them in the 1960s [5], [6].

Islands that are well suited as habitats for biological species are said to have a high island suitability index (ISI). Features that correlate with ISI include rainfall, topographic diversity, area, temperature, etc. The variables that characterize these features are called suitability index variables (SIVs). SIVs are the independent variables of the island, and ISI is the dependent variable.

Islands with a high ISI tend to have a large number of species, and those with a low ISI have a small number of species. Islands with a high ISI have many species that emigrate to nearby islands because of the accumulation of random effects on its large populations. Emigration occurs as animals ride flotsam, fly, or swim to neighboring islands.

Biogeography-based optimization (BBO) was first presented in [7] and is an example of how a natural process can be modeled to solve optimization problems. This is similar to

what has occurred in the past few decades with genetic algorithms, artificial immune systems, simulated annealing, particle swarm optimization, and other areas of computer intelligence. Suppose that we have some problem, and that we also have several candidate solutions. A good solution is analogous to an island with a high ISI, and a poor solution is like an island with a low ISI. High ISI solutions are more likely to share their features with other solutions, and low ISI solutions are more likely to accept shared features from other solutions. This approach to problem solving is called biogeography-based optimization. As with every other evolutionary algorithm (EA), each solution also typically has some probability of mutation, although mutation is not an essential feature of BBO.

The goals of this paper are three-fold. Our first goal is to present an overview of BBO, which we do in Section II. Our second goal is to use Markov analysis to obtain the limiting distribution of BBO populations, which we do in Section III. Our third goal is to confirm the theory with simulation, compare BBO Markov theory with GA Markov theory, and analyze the effect of elitism on BBO performance, which we do in Section IV. We provide some concluding remarks and directions for future work in Section V.

II. BIOGEOGRAPHY-BASED OPTIMIZATION

A. Biogeography

In this subsection we first give a brief overview of biogeography. Figure 1 illustrates a model of species abundance in a single island. The emigration rate μ out of the island, and the immigration rate λ into the island, are functions of the number of species S on the island. The maximum possible immigration rate I occurs when there are zero species on the island. As the number of species increases, the island becomes more crowded, fewer species are able to successfully survive immigration, and the immigration rate decreases. The largest possible number of species that the island can support is S_{max} , at which point the immigration rate is zero.

If there are no species on the island then the emigration rate is zero. As the number of species increases, the island becomes more crowded, representative individuals of species are more likely to leave the island, and the emigration rate increases. The maximum emigration rate E occurs when the island contains the largest number of species that it can support.

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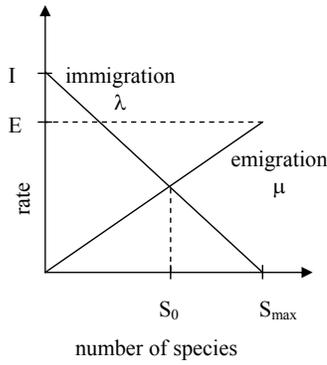


Fig. 1. Linear migration relationships for an island, adapted from [6]. The equilibrium number of species is S_0 , at which point the immigration and emigration rates are equal.

We have shown the immigration and emigration curves in Figure 1 as straight lines, but in general they might be nonlinear. Nevertheless this simple model gives a general description of the process of immigration and emigration.

B. Biogeography-Based Optimization Algorithms

Suppose that we have a problem and a population of candidate solutions that are represented as vectors. Further suppose that we have some way of assessing the goodness of the solutions. Good solutions are analogous to islands with a high ISI, and poor solutions are analogous to islands with a low ISI. Note that ISI is the same as “fitness” in other population-based optimization algorithms.

In biogeography, species migrate between islands. However, in BBO we instead migrate solution features (SIVs) between islands. We base the migration probabilities on a curve similar to that shown in Figure 1, but for the sake of simplicity we assume that all solutions (islands) have identical migration curves with $E = I$. Figure 2 illustrates the migration curves along with two solutions. ISI_1 represents a poor solution and ISI_2 represents a good solution. The immigration rate for ISI_1 will therefore be higher than the immigration rate for ISI_2 , and the emigration rate for ISI_1 will be lower than the emigration rate for ISI_2 .

We use the migration rates of each solution to probabilistically share features between solutions. This can be implemented in several different ways, but in this paper we use the original BBO formulation [7], which is called partial immigration-based BBO in [8]. In this approach, for each feature in each solution, we use the immigration curve to probabilistically decide whether or not to immigrate. If immigration is selected for a given solution feature, then the emigrating island is selected probabilistically (e.g., using μ -based roulette wheel selection). Figure 3 is a conceptual description of one generation of this approach, where we use the notation $y_k(s)$ to denote the s th feature of the k th population member. Migration and mutation of the entire population take place before any of the solutions are replaced in the population, which requires the use of the temporary population vector w in Figure 3. A

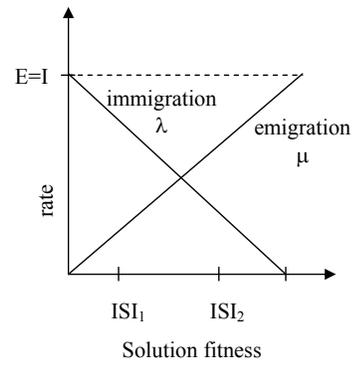


Fig. 2. Illustration of two candidate solutions to some problem using symmetric immigration and emigration curves. ISI_1 is a relatively poor solution and ISI_2 is a relatively good solution. ISI_1 has a high immigration and a low emigration rate, and ISI_2 has a low immigration and a high emigration rate.

qualitative comparison between BBO and other EAs for some common benchmark functions is given in [7].

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Let  $y$  denote the population of solutions
 $w \leftarrow y$  (note that  $w$  is a temporary population vector)
For each island  $w_k$ 
  For each SIV  $s$ 
    Use  $\lambda_k$  to probabilistically decide whether to
    immigrate to  $w_k(s)$ .
    If immigrating then
      Use  $\mu$  to probabilistically select the
      emigrating island  $y_j$ .
       $w_k(s) \leftarrow y_j(s)$ 
    end if
    Probabilistically decide whether to mutate  $w_k(s)$ 
  next SIV
next island
 $y \leftarrow w$ 

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Fig. 3. One generation of the partial immigration-based BBO algorithm. y is the entire population of solutions, y_k is the k th solution, and $y_k(s)$ is the s th feature of y_k .

C. Elitism

As with other population-based algorithms, we often incorporate elitism in order to retain the best solutions in the population from one generation to the next. We use z to denote the number of top individuals in the population that have a zero probability of immigration. If elitism is not used, then $z = 0$ and the immigration probability λ is a linear function of fitness and is positive for all fitness values. This is shown in Figure 4(a). If elitism is used, then $z > 0$ and the immigration probability λ is zero for the top z individuals in the search space. This is shown in Figure 4(b).

III. MARKOV ANALYSIS

In this section we derive the probabilities for a BBO transition from some population distribution at one generation to another distribution at the next generation. First we set up the notation.

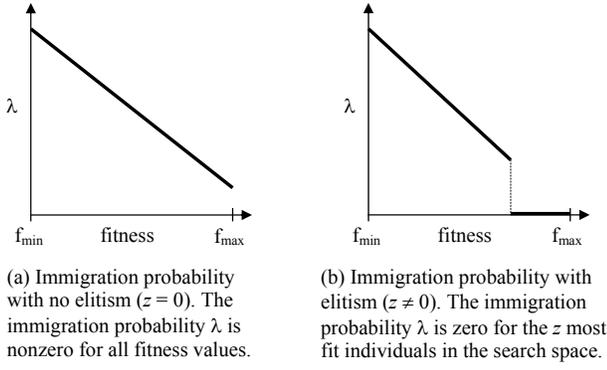


Fig. 4. Immigration probability λ as a function of fitness.

Suppose that we have a problem with a binary search space. The search space consists of n possible bit strings x_i , each x_i containing q bits. The cardinality of the search space is $n = 2^q$. We use N to denote the population size, and we use v to denote the population vector. That is, v_i is the number of x_i individuals in the population. We see that

$$\sum_{i=1}^n v_i = N \quad (1)$$

The entire population consists of N bit strings. We use y_k to denote the k th individual in the population, and we order them in the same order as x_i . That is,

$$\text{Population} = \{y_1, \dots, y_N\} = \underbrace{\{x_1, x_1, \dots, x_1\}}_{v_1 \text{ copies}} \underbrace{\{x_2, x_2, \dots, x_2\}}_{v_2 \text{ copies}} \dots \underbrace{\{x_n, x_n, \dots, x_n\}}_{v_n \text{ copies}} \quad (2)$$

This can be written more compactly as

$$\begin{aligned} y_k &= x_m \text{ for } k = 1, \dots, N \\ m &= \min r \text{ such that } \sum_{i=1}^r v_i \geq k \end{aligned} \quad (3)$$

If we need to denote the generation number, we use an additional subscript. For example, $y_k(s)_t$ is the value of the s th bit of the k th individual at generation number t .

We use λ_i to denote the immigration probability of x_i , and μ_i to denote the emigration probability of x_i . We use the notation $x_i(s)$ to denote the s th bit of solution x_i . We use the notation $\mathcal{J}_i(s)$ to denote the set of population indices j such that the s th bit of x_j is equal to the s th bit of x_i . That is,

$$\mathcal{J}_i(s) = \{j : x_j(s) = x_i(s)\} \quad (4)$$

A. Migration

For each SIV (bit), y_k has v_m chances of being selected for immigration, each chance with probability λ_m . If the s th SIV of y_k is not selected for immigration during generation t , then

$$y_k(s)_{t+1} = x_m(s) \text{ (no immigration)} \quad (5)$$

That is, $y_k(s)$ does not change from generation t to generation $t+1$. However, if the s th SIV of y_k is selected for immigration during generation t , then the probability that $y_k(s)_{t+1}$ is equal to $x_i(s)$ is proportional to the combined emigration rates of all individuals whose s th feature is equal to $x_i(s)$. This probability can be written as

$$\Pr(y_k(s)_{t+1} = x_i(s)) = \frac{\sum_{j \in \mathcal{J}_i(s)} v_j \mu_j}{\sum_j v_j \mu_j} \text{ (immigration)} \quad (6)$$

We can combine (5) and (6), along with the fact that the probability of immigration to $y_k(s)$ is equal to λ_m , to obtain

$$\Pr(y_k(s)_{t+1} = x_i(s)) = (1 - \lambda_m) \mathbf{1}_0(x_m(s) - x_i(s)) + \lambda_m \frac{\sum_{j \in \mathcal{J}_i(s)} v_j \mu_j}{\sum_j v_j \mu_j} \quad (7)$$

where $\mathbf{1}_0(\cdot)$ is the indicator function on the set $\{0\}$. Since there are q SIVs in each island, the probability that immigration results in $y_{k,t+1}$ being equal to x_i , given that the population is described by the vector v , is denoted as $P_{ki}(v)$ and can be written as

$$\begin{aligned} P_{ki}(v) &= \Pr(y_{k,t+1} = x_i) \\ &= \prod_{s=1}^q \left[(1 - \lambda_m) \mathbf{1}_0(x_m(s) - x_i(s)) + \lambda_m \frac{\sum_{j \in \mathcal{J}_i(s)} v_j \mu_j}{\sum_j v_j \mu_j} \right] \end{aligned} \quad (8)$$

$P_{ki}(v)$ can be computed for each $k \in [1, N]$ (i.e., each solution in the population) and each $i \in [1, n]$ (i.e., each element of the search space) in order to form the $N \times n$ matrix $P(v)$. The k th row of $P(v)$ corresponds to the k th iteration of the outer loop in Figure 3. The i th column of $P(v)$ corresponds to the probability of obtaining island x_i during an outer loop iteration.

The BBO algorithm entails N trials (i.e., N iterations of the outer loop in Figure 3), where the probability of the i th outcome on the k th trial is given as $P_{ki}(v)$. We use u_i to denote the total number of times that outcome i occurs after all N trials have been completed, and $u = [u_1 \dots u_n]^T$. Then the probability that we start with a population vector v and obtain a population vector u at the next generation is given by the generalized multinomial theorem [9] as follows.

$$\begin{aligned} \Pr(u|v) &= \sum_Y \prod_{k=1}^N \prod_{i=1}^n [P_{ki}(v)]^{J_{ki}} \\ Y &= \left\{ J \in \mathbf{R}^{N \times n} : J_{ki} \in \{0, 1\}, \right. \\ &\quad \left. \sum_{i=1}^n J_{ki} = 1 \text{ for all } k, \sum_{k=1}^N J_{ki} = u_i \text{ for all } i \right\} \end{aligned} \quad (9)$$

Some assumptions were made in the above analysis that we now explicitly note. First, all of the new islands are created before any islands are replaced in the population. This is clear from the use of the temporary population vector w in

Figure 3. Second, an island can emigrate an SIV to itself. This is analogous to crossover in GAs when an individual is allowed to cross with itself. Third, the migration rates λ and μ are independent of the population distribution. That is, absolute fitness values are used to obtain λ and μ , as opposed to rank-based fitness.

B. Mutation

The previous section considered only migration. In this section we add the possibility of mutation. We use U to denote the $n \times n$ mutation matrix, where U_{ij} is the probability that x_j mutates to x_i . The probability that the k th immigration trial followed by mutation results in x_i is denoted as $P_{ki}^{(2)}(v)$. This can be written as

$$\begin{aligned} P_{ki}^{(2)}(v) &= \sum_{j=1}^n U_{ij} P_{kj}(v) \\ P^{(2)}(v) &= P(v)U^T \end{aligned} \quad (10)$$

where the elements of $P(v)$ are given in (8). We can write the probability of transitioning from population vector v to population vector u after one generation of both migration and mutation as

$$\Pr^{(2)}(u|v) = \sum_Y \prod_{k=1}^N \prod_{i=1}^n [P_{ki}^{(2)}(v)]^{J_{ki}} \quad (11)$$

where Y is given in (9). We can use standard Markov tools [15] with this matrix to find the limiting distribution of the BBO population.

The Markov transition matrix Q is obtained by computing (11) for each possible v vector and each possible u vector. Q is therefore a $T \times T$ matrix, where T is the total number of possible population distributions. That is, T is the number of possible $n \times 1$ integer vectors v whose elements sum to N and each of whose elements $v_i \in [0, N]$. It is shown in [10] that

$$T = \binom{n+N-1}{N} = (n+N-1)\text{-choose-}N \quad (12)$$

Other mathematically equivalent expressions for T are given in [11].

IV. RESULTS

The theory of the preceding section was confirmed with simulations in [12]. In this section we first compare the BBO population distribution with GAs, and then analyze the effect of elitism on BBO performance.

A. Analytical Comparison with Genetic Algorithms

Consider a genetic algorithm with fitness-proportional (roulette wheel) selection, followed by mutation, followed by single point crossover. We use v_i to represent the i th element of the population vector v , f_i is the fitness of x_i , $G_j^s(v)$ is the probability of obtaining individual x_j by selection alone, U_{ij} is the probability of obtaining x_i from x_j by mutation, $G_i^{sm}(v)$ is the probability of obtaining individual x_i by selection and mutation combined, $r(i, j, k)$ is the probability that x_i and x_j

cross to form x_k , and $G_k^{smc}(v)$ is the probability of obtaining individual x_k by selection, mutation, and crossover combined. These quantities are obtained in [10], [13], [14], [15] as

$$\begin{aligned} G_j^s(v) &= \frac{v_j f_j}{\sum_j v_j f_j} \\ G_i^{sm}(v) &= \sum_j U_{ij} G_j^s(v) \\ G_k^{smc}(v) &= \sum_j r(i, j, k) G_i^{sm}(v) G_j^{sm}(v) \end{aligned} \quad (13)$$

Equation (13) can be used with the multinomial theorem [16] to obtain the probability that population vector v transitions to u after one generation.

$$\Pr_G(u|v) = N! \prod_i \frac{[G_i^{smc}(v)]^{u_i}}{u_i!} \quad (14)$$

In this section we use Equation (11) to obtain the limiting population distributions of BBO, and (14) to obtain the limiting population distributions of a GA. Due to the exponential increase of matrix sizes with problem size, investigation was limited to three-bit problems ($n = 8$) with a population size of four ($N = 4$), results in 330 possible population vectors as shown by (12). The three fitness functions we investigated are

$$\begin{aligned} F_1 &= (2 \ 4 \ 6 \ 8 \ 6 \ 4 \ 2 \ 1) \\ F_2 &= (1 \ 2 \ 3 \ 2 \ 1 \ 2 \ 3 \ 2) \\ F_3 &= (3 \ 2 \ 1 \ 4 \ 1 \ 2 \ 3 \ 3) \end{aligned} \quad (15)$$

where the fitness values are in binary order of the population members. That is, for problem F_1 , individual 000 has a fitness of 2, individual 001 has a fitness of 4, ..., and individual 111 has a fitness of 1. For the BBO, we use $\mu_i = F_{1i}/10$ for F_1 , $\mu_i = 3F_{2i}/10$ for F_2 , and $\mu_i = 2F_{3i}/10$ for F_3 . We use $\lambda_i = 1 - \mu_i$ and no elitism for all of the problems.

Tables I-III show comparisons between theoretical GA and theoretical BBO results. The crossover probability used in the GA was 0.9 throughout in this paper. The tables show the probability of obtaining a uniform optimal population, and the probability of obtaining a population which does not have any optimal individuals.

TABLE I
OPTIMIZATION RESULTS FOR F_1 . THE BETTER-PERFORMING RESULT IS SHOWN IN **red bold font** IN EACH ROW.

Mutation Rate	Population Vector	Probability	
		GA	BBO
0.1	Uniform Optimal	0.0564	0.0509
	No Optima	0.3990	0.3285
0.01	Uniform Optimal	0.5429	0.6649
	No Optima	0.2873	0.0943
0.001	Uniform Optimal	0.6989	0.9057
	No Optima	0.2815	0.0652

Several things are notable about the results of Table I-III. First, as the mutation rate decreases, the probability of a uniform optimal population increases, and the probability of no optima decreases. This is true for both the GA and BBO.

TABLE II
OPTIMIZATION RESULTS FOR F_2 . THE BETTER-PERFORMING RESULT IS SHOWN IN **red bold font** IN EACH ROW.

Mutation Rate	Population Vector	Probability	
		GA	BBO
0.1	Uniform Optimal	0.13274	0.1156
	No Optima	0.2427	0.1483
0.01	Uniform Optimal	0.69384	0.8004
	No Optima	0.1663	0.0137
0.001	Uniform Optimal	0.83198	0.9743
	No Optima	0.1526	0.0053

TABLE III
OPTIMIZATION RESULTS FOR F_3 . THE BETTER-PERFORMING RESULT IS SHOWN IN **red bold font** IN EACH ROW.

Mutation Rate	Population Vector	Probability	
		GA	BBO
0.1	Uniform Optimal	0.0385	0.0377
	No Optima	0.4558	0.3752
0.01	Uniform Optimal	0.4667	0.6206
	No Optima	0.3772	0.1363
0.001	Uniform Optimal	0.6180	0.8771
	No Optima	0.3640	0.0938

Second, the GA outperforms BBO only when the mutation rate is high (10% per bit), and even then the probability of a uniform optimal population is only slightly higher in the GA than in BBO. Third, in every other performance comparison in the tables, BBO far outperforms the GA.

B. Elitism Analysis

Next we explore the effect of elitism on BBO performance. As explained in Section II-C, elitism is implemented by setting the immigration probability of the z most fit individuals in the search space to zero. Note that this differs from the implementation of elitism in practical problems. In practical problems, we do not know the most fit individuals in the search space, we only know the most fit individuals in the current population. However, the analysis in this section can be approximated in a practical problem by setting the immigration probability to zero if the fitness exceeds some problem-dependent threshold.

Another difference between our analysis and most practical implementations is that we use elitism only to prevent immigration, but not to prevent mutation. Our analysis could be extended with future work to prevent mutation, which would require a modification of the mutation matrix in Section III-B.

Figure 5 shows the theoretical probabilities that a BBO population converges to a uniform optimal population for problem F_1 , for various mutation rates, and for various values of z (number of elite individuals in the search space). Figures 6 and 7 show corresponding results for problem F_2 and problem F_3 . We see that the addition of elitism significantly improves convergence results for F_1 and F_3 , but only slightly for F_2 . But if we use too many elites in the population, convergence becomes worse.

Figure 8 shows the theoretical probabilities that BBO converges to a population that does not have any optima for

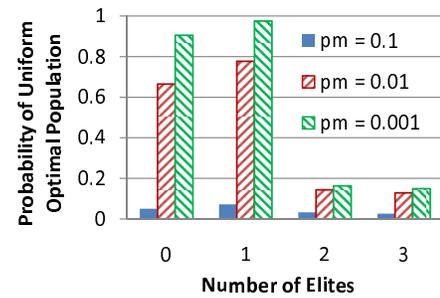


Fig. 5. Probability of BBO achieving a uniform optimal population for problem F_1 , where p_m is the mutation probability per bit. An elitism parameter of 1 significantly improves convergence to the optimum, but if the elitism parameter is too high then convergence becomes worse.

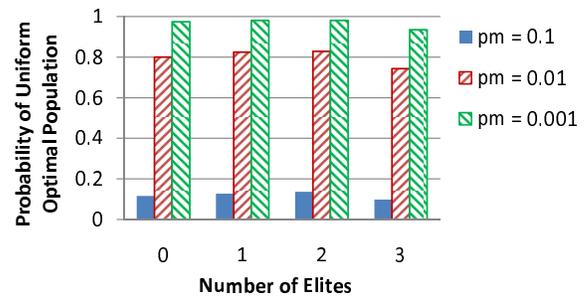


Fig. 6. Probability of BBO achieving a uniform optimal population for problem F_2 , where p_m is the mutation probability per bit. An elitism parameter of one slightly improves convergence to the optimum.

problem F_1 , for various mutation rates, and for various values of the elitism parameter. Figures 9 and 10 show corresponding results for problem F_2 and problem F_3 . We see that the addition of elitism significantly improves the probability that the population has at least one optima. In fact, with a low enough mutation rate, the use of elitism reduces the probability of no optima to essentially zero. But as noted above, if we use too many elites in the population, the performance worsens.

The MATLAB[®] code that was used to generate the results in this paper is available at <http://academic.csuohio.edu/simond/bbo/markov>.

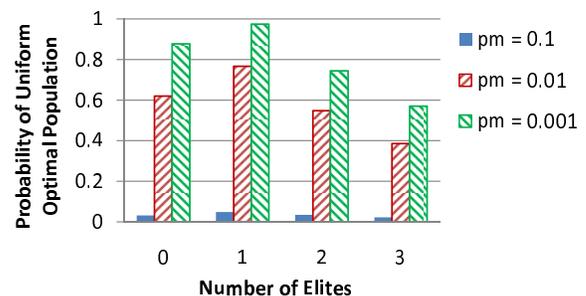


Fig. 7. Probability of BBO achieving a uniform optimal population for problem F_3 , where p_m is the mutation probability per bit. An elitism parameter of 1 significantly improves convergence to the optimum, but if the elitism parameter is too high then convergence becomes worse.

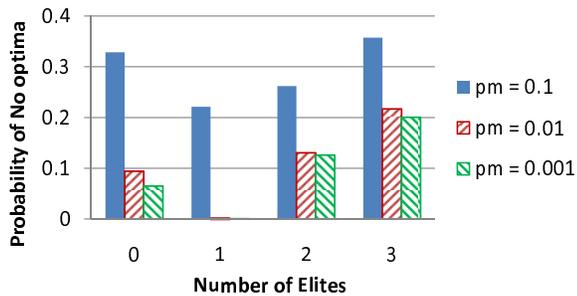


Fig. 8. Probability that BBO does not find any optima for problem F_1 , where p_m is the mutation probability per bit. An elitism parameter of 1 significantly improves the probability of finding an optimum, but if the elitism parameter is too high then the performance worsens.

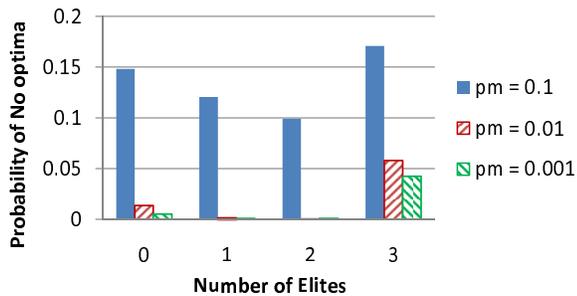


Fig. 9. Probability that BBO does not find any optima for problem F_2 , where p_m is the mutation probability per bit. An elitism parameter of one slightly improves the probability of finding an optimum.

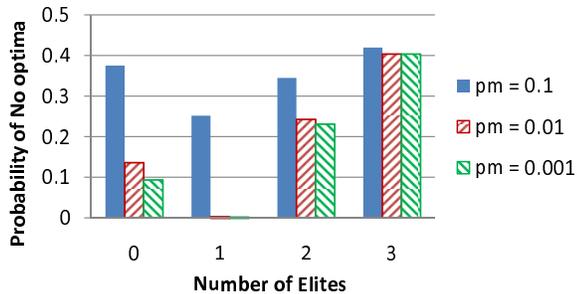


Fig. 10. Probability that BBO does not find any optima for problem F_3 , where p_m is the mutation probability per bit. An elitism parameter of 1 significantly improves the probability of finding an optimum, but if the elitism parameter is too high then the performance worsens.

V. CONCLUSION

We have presented a Markov analysis of biogeography-based optimization. This analysis shows the theoretical probability of each possible population for a BBO problem. Three small representative problems were investigated. The results show that GAs outperform BBO slightly if the mutation rate is large, but BBO far outperforms GAs if the mutation rate is small. Elitism improves performance, but if too many elites are used, then performance worsens.

The analysis in this paper is computationally expensive because the size of the Markov transition matrix increases combinatorially with the problem size. Computational savings

can be obtained by grouping Markov states together and computing the probability that the population transitions from one group of populations to another [15], but this is left for further research. Computational savings could also be obtained by not allowing duplicate individuals in the population. This would change the Markov analysis and reduce the size of the transition matrix, but only by a small amount.

Other future work includes extending this analysis to other BBO variations. This paper investigated the original BBO algorithm with linear migration curves, which is called partial immigration-based BBO. An extension of our Markov analysis to other BBO variations would analytically show their advantages or disadvantages.

Finally, the Markov analysis developed here forms a foundation that can be used to develop a dynamic systems analysis of BBO. Dynamic systems analysis of EAs is used to find the proportion of each possible individual in a population as the population size tends to infinity. This is exemplified by the extension of Markov analysis for GAs to dynamic systems analysis [15].

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