

### 1. Functions and Operators

$$(1.1) \operatorname{sinc}(\alpha) = \begin{cases} 1, & \alpha = 0 \\ \frac{\sin(\pi\alpha)}{\pi\alpha}, & \alpha \neq 0 \end{cases} \quad (1.2) \operatorname{Tan}^{-1}(\beta, \alpha) = \begin{cases} \arctan(\beta/\alpha), & \alpha \geq 0 \\ \pi + \arctan(\beta/\alpha), & \alpha < 0 \end{cases}$$

$$(1.3) \angle\alpha = \operatorname{Tan}^{-1}(\operatorname{Im}\{\alpha\}, \operatorname{Re}\{\alpha\}) \quad (1.4) \delta(\alpha) = 0, \alpha \neq 0; \quad \int_{-\infty}^{\infty} \delta(\alpha) d\alpha = 1$$

$$(1.5) u(\alpha) = \begin{cases} 0, & \alpha < 0 \\ 0.5, & \alpha = 0 \\ 1, & \alpha > 0 \end{cases} \quad (1.6) \operatorname{rect}(\alpha) = u(\alpha + 0.5) - u(\alpha - 0.5)$$

### 2. Trigonometric Identities:

$$(2.1) \cos(-\alpha) = \cos(\alpha) \quad (2.2) \sin(-\alpha) = -\sin(\alpha) \quad (2.3) e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

$$(2.4) \cos\left(\alpha \pm \frac{\pi}{2}\right) = \mp \sin(\alpha) \quad (2.5) \sin\left(\alpha \pm \frac{\pi}{2}\right) = \pm \cos(\alpha) \quad (2.6) \cos(\alpha) = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha})$$

$$(2.7) \cos(\alpha \pm \pi) = -\cos(\alpha) \quad (2.8) \sin(\alpha \pm \pi) = -\sin(\alpha) \quad (2.9) \sin(\alpha) = \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha})$$

$$(2.10) \cos^2(\alpha) + \sin^2(\alpha) = 1 \quad (2.11) \cos^2(\alpha) - \sin^2(\alpha) = \cos(2\alpha)$$

$$(2.12) \cos^2(\alpha) = \frac{1}{2}[1 + \cos(2\alpha)] \quad (2.13) \sin^2(\alpha) = \frac{1}{2}[1 - \cos(2\alpha)]$$

$$(2.14) \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \quad (2.15) \sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$(2.16) \cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \quad (2.17) \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$(2.18) \sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$(2.19) a\cos(\alpha) + b\sin(\alpha) = \sqrt{a^2 + b^2} \cos[\alpha - \operatorname{Tan}^{-1}(b, a)]$$

### 3. Derivatives

$$(3.1) \frac{d[\cos(\beta)]}{dt} = -\sin(\beta) \frac{d\beta}{dt} \quad (3.2) \frac{d[\sin(\beta)]}{dt} = \cos(\beta) \frac{d\beta}{dt} \quad (3.3) \frac{d(e^\beta)}{dt} = e^\beta \frac{d\beta}{dt}$$

$$(3.4) \frac{d}{dt} \left[ \int_c^t x(\beta) d\beta \right] = x(t) \quad (3.5) \frac{d}{dt} [x(t)u(t)] = \frac{dx(t)}{dt} u(t) + x(0)\delta(t)$$

### 4. Anti-derivatives

$$(4.1) \int \cos(at) dt = \frac{1}{a} \sin(at) \quad (4.2) \int \sin(at) dt = -\frac{1}{a} \cos(at)$$

$$(4.3) \int t \cos(at) dt = \frac{1}{a^2} \cos(at) + \frac{t}{a} \sin(at) \quad (4.4) \int t \sin(at) dt = \frac{1}{a^2} \sin(at) - \frac{t}{a} \cos(at)$$

$$(4.5) \int e^{at} dt = \frac{1}{a} e^{at} \quad (4.6) \int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2 + b^2} [a \cos(bt) + b \sin(bt)]$$

$$(4.7) \int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2 + b^2} [a \sin(bt) - b \cos(bt)]$$

## 5. Integrals

$$(5.1) \int_0^\infty \text{sinc}(\alpha) d\alpha = \frac{1}{2} \quad (5.2) \int_0^\infty \text{sinc}^2(\alpha) d\alpha = \frac{1}{2} \quad (5.3) \int_{-b}^b e^{\pm j2\pi\alpha} d\alpha = 2b \text{sinc}(2b)$$

$$(5.4) \int_a^b x(t) \delta(t - c) dt = x(c) [u(b - c^+) - u(a^+ - c)]$$

$$(5.5) \int_{-\infty}^\infty x(t) u(b - t) dt = \int_{-\infty}^b x(t) dt \quad (5.6) \int_{-\infty}^a x(t) u(t) dt = u(a) \int_0^a x(t) dt$$

$$(5.7) \int_{-\infty}^\infty x(t) u(t) u(b - t) dt = u(b) \int_0^b x(t) dt \quad (5.8) \int_{-\infty}^\infty x(t) \delta(t - b) dt = x(b)$$

$$(5.9) \int_{-\infty}^a x(t) u(t) u(t - b) dt = u(a) u(a - b) \int_0^a x(t) dt$$

## 6. Summation Formulas

$$(6.1) \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha} \quad (|\alpha| < 1) \quad (6.2) \sum_{n=0}^{N-1} \cos(\lambda n) = \frac{\sin(\lambda N / 2)}{\sin(\lambda / 2)} \cos\left[\frac{\lambda(N - 1)}{2}\right]$$

$$(6.3) \sum_{n=-\infty}^{\infty} x[n] u[m - n] = \sum_{n=-\infty}^m x[n] \quad (6.4) \sum_{n=-\infty}^m x[n] u[n] = u[m] \sum_{n=0}^m x[n]$$

$$(6.5) \sum_{n=0}^{N-1} \sin(\lambda n) = \frac{\sin(\lambda N / 2)}{\sin(\lambda / 2)} \sin\left[\frac{\lambda(N - 1)}{2}\right] \quad (6.6) \sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1 - \alpha^N}{1 - \alpha}, & \alpha \neq 1 \end{cases}$$

$$(6.7) \sum_{n=-\infty}^{\infty} x[n] u[n - m] = \sum_{n=m}^{\infty} x[n] \quad (6.8) \sum_{n=-\infty}^{\infty} x[n] u[n] u[k - n] = u[k] \sum_{n=0}^k x[n]$$

## 7. Miscellaneous Relations

$$(7.1) \left| \int_a^b x(\alpha) d\alpha \right| \leq \int_a^b |x(\alpha)| d\alpha \quad (7.2) \left| \sum_{k=k_1}^{k_2} x[k] \right| \leq \sum_{k=k_1}^{k_2} |x[k]|$$

$$(7.3) \int_{-\infty}^{\infty} x(t') h(t - t') dt' = \int_{-\infty}^{\infty} x(t - t') h(t') dt'$$

$$(7.4) \sum_{k=-\infty}^{\infty} x[k] h[n - k] = \sum_{k=-\infty}^{\infty} x[n - k] h[k]$$

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**8. Series Expansions**


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$$(8.1) e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \dots \quad (8.2) \ln(1 + \alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \frac{\alpha^4}{4} + \dots$$

$$(8.3) \cos(\alpha) = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \dots \quad (8.4) \sin(\alpha) = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \dots$$

$$(8.5) \tan(\alpha) = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15} + \frac{17\alpha^7}{315} + \dots \quad (8.6) \tan^{-1}(\alpha) = \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{5} - \dots$$


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**9. Fourier Series**

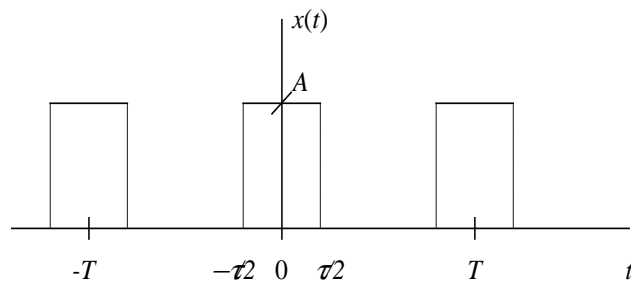

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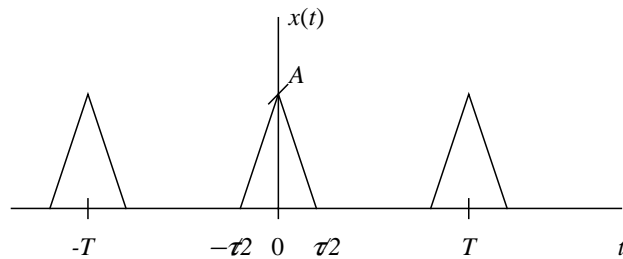
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**(9.1) Rectangular Pulse Train**

$$X_n = \frac{A\tau}{T} \operatorname{sinc}\left(\frac{n\tau}{T}\right)$$

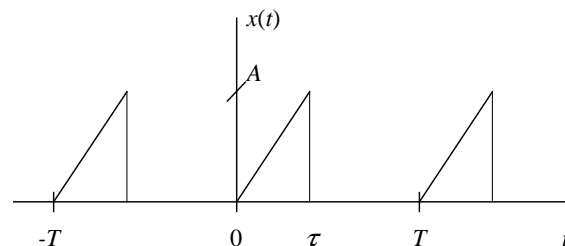

**(9.2) Triangular Pulse Train**

$$X_n = \frac{A\tau}{2T} \operatorname{sinc}^2\left(\frac{n\tau}{2T}\right)$$


**(9.3) Sawtooth Pulse Train**

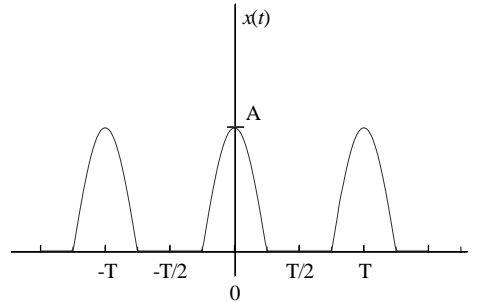
$$X_0 = \frac{A\tau}{2T}$$

$$X_n = \frac{AT}{4\pi^2 n^2 \tau} \left[ \left( \frac{j2\pi n \tau}{T} + 1 \right) e^{-j2\pi n \tau / T} - 1 \right]$$

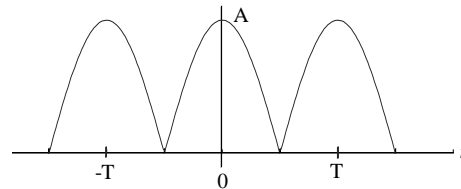


**(9.4) Half-Wave Rectified Sinusoid**

$$X_n = \frac{A}{4} \left\{ \text{sinc} \left[ \frac{(n+1)}{2} \right] + \text{sinc} \left[ \frac{(n-1)}{2} \right] \right\}$$

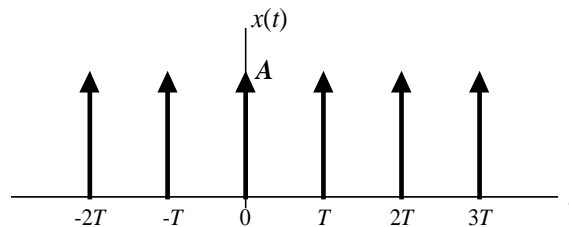
**(9.5) Full-Wave Rectified Sinusoid**

$$X_n = \frac{A[1 + (-1)^n]}{4} \left\{ \text{sinc} \left[ \frac{(n+1)}{2} \right] + \text{sinc} \left[ \frac{(n-1)}{2} \right] \right\}$$

**(9.6) Impulse Train**

$$X_n = \frac{A}{T}$$

$A$  = strength (area) of each impulse

**10. Operational Properties of Fourier Coefficients**

$$(10.0) \quad x_i(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \quad X_i[k] = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j2\pi kt/T} dt$$

$$(10.1) \quad a_1 x_1(t) + a_2 x_2(t) + \dots \quad a_1 X_1[k] + a_2 X_2[k] + \dots$$

$$(10.2) \quad x(t - t_0) \quad e^{-j2\pi kt_0/T} X[k]$$

$$(10.3) \quad e^{j2\pi f_0 t} \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \quad \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi \left(f_0 + \frac{k}{T}\right) t} \quad \text{Periodic only if } Tf_0 \text{ is an integer}$$

$$(10.4) \quad \cos(2\pi f_0 t) \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \quad \frac{1}{2} \sum_{k=-\infty}^{\infty} X[k] \left( e^{j2\pi \left(f_0 + \frac{k}{T}\right) t} + e^{-j2\pi \left(f_0 - \frac{k}{T}\right) t} \right)$$

Periodic only if  $Tf_0$  is an integer

$$(10.5) \quad x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \quad x(at) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T_a} \quad a > 0, T_a = T/a$$

$$(10.6) \quad x(-t) \quad X[-k] = X^*[k]$$

(10.7)	$\frac{dx(t)}{dt}$	$\frac{j2\pi k}{T} X[k]$	
(10.8)	$\int_{-\infty}^t x(t') dt'$	$\frac{T}{j2\pi k} X[k]$	see note below
The integral exists only if $x(t)$ has no dc component; i.e., only if $X[0] = 0$ .			
(10.9)	$\frac{1}{T} \int_{t_0}^{t_0+T}  x(t) ^2 dt = \sum_{k=-\infty}^{\infty}  X[k] ^2$		Parseval's Identity

## 11. Fourier Transforms

	Function	Fourier Transform	
(11.0)	$\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$	$\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$	
(11.1)	1	$\delta(f)$	
(11.2)	$\delta(t)$	1	
(11.3)	$u(t)$	$\frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$	
(11.4)	$e^{-t/\tau} u(t) \quad (\tau > 0)$	$\frac{\tau}{1 + j2\pi f\tau}$	
(11.5)	$e^{j2\pi f_0 t}$	$\delta(f - f_0)$	
(11.6)	$\cos(2\pi f_0 t)$	$\frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0)$	
(11.7)	$\sin(2\pi f_0 t)$	$\frac{1}{2j} \delta(f - f_0) - \frac{1}{2j} \delta(f + f_0)$	
(11.8)	$r\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}(f\tau)$	See (1.1) and (1.6)
(11.9)	$2F \operatorname{sinc}(2Ft)$	$r\left(\frac{f}{2F}\right)$	See (1.1) and (1.6)
(11.10)	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$	
(11.11)	$\sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t}$	$\sum_{k=-\infty}^{\infty} X_k \delta(f - k f_0)$	

## 12. Operational Properties of the Fourier Transform

(12.1)	$\int_{-\infty}^{\infty}  x(t) ^2 dt = \int_{-\infty}^{\infty}  X(f) ^2 df$	Rayleigh's Identity
(12.2)	$a_1 x_1(t) + a_2 x_2(t) + \dots$	$a_1 X_1(f) + a_2 X_2(f) + \dots$
(12.3)	$x_1(t) * x_2(t)$	$X_1(f) X_2(f)$
(12.4)	$x_1(t) x_2(t)$	$X_1(f) * X_2(f)$

(12.5)	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
(12.6)	$\int_{-\infty}^t x(t) dt$	$\frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f)$
(12.7)	$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
(12.8)	$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
(12.9)	$x(at)$	$\frac{1}{ a } X\left(\frac{f}{a}\right)$

### 13. Bilateral Laplace Transforms

Function	Transform
(13.0) $x(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} X(s)e^{-st} ds$	$X(s) = \int_{-\infty}^{\infty} x(t)e^{st} dt$
(13.1) $\delta(t)$	1
(13.2) $u(t)$	$\frac{1}{s}$
(13.3) $e^{-t/\tau} u(t) \quad (\tau > 0)$	$\frac{\tau}{1 + s\tau}$
(13.4) $\cos(2\pi f_0 t) u(t)$	$\frac{s}{s^2 + (2\pi f_0)^2}$
(13.5) $\sin(2\pi f_0 t) u(t)$	$\frac{2\pi f_0}{s^2 + (2\pi f_0)^2}$

### 14. Operational properties of the Laplace Transformation

Function	Transform
(14.1) $x_i(t)$	$X_i(s)$
(14.2) $a_1 x_1(t) + a_2 x_2(t) + \dots$	$a_1 X_1(s) + a_2 X_2(s) + \dots$
(14.3) $x_1(t) * x_2(t)$	$X_1(s) X_2(s)$
(14.4) $\frac{dx(t)}{dt}$	$sX(s)$
(14.5) $\int_{-\infty}^t x(t) dt$	$\frac{1}{s} X(s)$
(14.6) $e^{-at} x(t)$	$X(s + a)$
(14.7) $x(t - t_0)$	$e^{-st_0} X(s)$
(14.8) $x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$

**15. Bilateral z Transforms**

Function	Transform
(15.0) $x[n] = \frac{1}{j2\pi} \oint_C X(z)z^{n-1} dz$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
(15.1) $\delta[n]$	1
(15.2) $u[n]$	$\frac{z}{z-1}$
(15.3) $\alpha^n u[n]$	$\frac{z}{z-\alpha}$
(15.4) $n\alpha^n u[n]$	$\frac{\alpha z}{(z-\alpha)^2}$
(15.5) $\cos(\lambda n)u[n]$	$\frac{z^2 - \cos(\lambda)z}{z^2 - 2\cos(\lambda)z + 1}$
(15.6) $\sin(\lambda n)u[n]$	$\frac{\sin(\lambda)z}{z^2 - 2\cos(\lambda)z + 1}$
(15.7) $\alpha^n \cos(\lambda n)u[n]$	$\frac{z^2 - \alpha \cos(\lambda)z}{z^2 - 2\alpha \cos(\lambda)z + \alpha^2}$
(15.8) $\alpha^n \sin(\lambda n)u[n]$	$\frac{\alpha \sin(\lambda)z}{z^2 - 2\alpha \cos(\lambda)z + \alpha^2}$

**16. Operational Properties of the z Transform**

Function	Transform
$x_i[n]$	$X_i(z)$
(16.1) $a_1 x_1[n] + a_2 x_2[n] + \dots$	$a_1 X_1(z) + a_2 X_2(z) + \dots$
(16.2) $x_1[n] * x_2[n]$	$X_1(z) X_2(z)$
(16.3) $\sum_{m=-\infty}^n x[m]$	$\frac{z}{z-1} X(z)$
(16.4) $e^{j\lambda n} x[n]$	$X(e^{-j\lambda} z)$
(16.5) $x[n - n_0]$	$z^{-n_0} X(z)$

## 17. Ordinary Linear Constant-Coefficient Differential Equations

Solutions are given for input  $x(t) = A \cos(\omega_0 t)u(t)$ . To obtain solutions for  $x(t) = Au(t)$ , set  $\omega_0 = 0$ .

### 17.1 First Order:

$$\tau \frac{dy}{dt} + y = x, \quad x(t) = A \cos(\omega_0 t)u(t)$$

The initial value of the solution is  $y(0^+)$ . Calculate

$$H(j\omega_0) = \frac{1}{1 + j\omega_0\tau}, \quad B = |H(j\omega_0)|A, \quad \theta = \angle H(j\omega_0)$$

$$y_f(t) = B \cos(\omega_0 t + \theta), \quad y_f(0) = B \cos(\theta), \quad C = y(0^+) - y_f(0)$$

$$y(t) = \{C e^{-t/\tau} + y_f(t)\}u(t)$$

### 17.2 Second Order:

$$\frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = x; \quad x(t) = A \cos(\omega_0 t)u(t) \quad (b \neq 0)$$

The initial conditions are  $y(0^+)$ ,  $\left. \frac{dy}{dt} \right|_{t=0^+} = \dot{y}(0^+)$ ,

The *complete response* is

$$y(t) = [y_u(t) + y_f(t)]u(t)$$

where  $y_f(t)$  is the forced response and  $y_u(t)$  is the unforced response, as determined below.

(a) *Forced Response*: Calculate

$$H(j\omega_0) = \frac{1}{(j\omega_0)^2 + j\omega_0 b + c}, \quad B = |H(j\omega_0)|A, \quad \theta = \angle H(j\omega_0)$$

and

$$y_f(t) = B \cos[\omega_0 t + \theta]$$

*Unforced Response*: Calculate

$$D = \frac{b^2}{4} - c, \quad y_f(0) = B \cos(\theta), \quad \dot{y}_f(0) = -\omega_0 B \sin(\theta),$$

$$K_1 = y(0) - y_f(0), \quad K_2 = \dot{y}(0) - \dot{y}_f(0)$$

**17.2.1**  $D = 0$  (critically-damped case): Calculate

$$s = -\frac{b}{2}, \quad C_1 = K_1, \quad C_2 = K_2 - sC_1$$

The complete response is

$$y(t) = \{(C_1 + C_2 t)e^{st} + y_f(t)\}u(t)$$

**17.2.2**  $D > 0$  (over-damped case): Calculate

$$s_1 = -\frac{b}{2} + \sqrt{D}, \quad s_2 = -\frac{b}{2} - \sqrt{D}$$

$$C_1 = \frac{K_1 s_2 - K_2}{s_2 - s_1}, \quad C_2 = \frac{K_2 - s_1 C_1}{s_2}$$

The complete response is

$$y(t) = \{C_1 e^{s_1 t} + C_2 e^{s_2 t} + y_f(t)\}u(t)$$

**17.2.3**  $D < 0$  (under-damped case): Calculate

$$\sigma = -\frac{b}{2}, \quad \omega = \sqrt{-D},$$

$$E = \frac{K_2 - \sigma K_1 + j\omega K_1}{2j\omega}, \quad C = 2|E|, \quad \phi = \angle E$$

The complete response is

$$y(t) = \{C e^{\sigma t} \cos(\omega t + \phi) + y_f(t)\}u(t)$$

## 18. Ordinary Linear Constant-Coefficient Difference Equations

Solutions are given for input  $x[n] = A \cos[n\lambda]u[n]$ , with  $A > 0$ . To obtain solutions for  $x[n] = Au[n]$ , set  $\lambda = 0$ .

**18.1 First Order:** The initial condition is  $y[-1]$

$$y[n] + ay[n-1] = x[n], \quad x[n] = A \cos[n\lambda]u[n]$$

Calculate

$$H(e^{j\lambda}) = \frac{1}{1 + ae^{-j\lambda}}, \quad B = |H(e^{j\lambda})|A, \quad \theta = \angle H(e^{j\lambda})$$

The forced response is

$$y_f[n] = B \cos[n\lambda + \theta]$$

Calculate

$$y_f[-1] = B \cos[\theta - \lambda], \quad C = y[-1] - y_f[-1]$$

The complete response is

$$y[n] = \{C(-a)^{n+1} + y_f[n]\}u[n]$$

**18.2 Second Order:** Initial conditions are  $y[-1]$ ,  $y[-2]$ .

$$y[n] + by[n-1] + cy[n-2] = x[n], \quad x[n] = A \cos(n\lambda)u[n]$$

Calculate

$$H(e^{j\lambda}) = \frac{1}{1 + be^{-j\lambda} + ce^{-2j\lambda}}, \quad B = |H(e^{j\lambda})|A, \quad \theta = \angle H(e^{j\lambda})$$

The complete response is the sum of the forced and unforced responses:

$$y[n] = (y_f[n] + y_u[n])u[n]$$

The forced Response is

$$y_f[n] = B \cos[\lambda n + \theta]$$

Calculate

$$D = \frac{b^2}{4} - c, \quad y_f[-1] = B \cos(\theta - \lambda), \quad y_f[-2] = B \cos(\theta - 2\lambda)$$

$$K_1 = y[-1] - y_f[-1], \quad K_2 = y[-2] - y_f[-2]$$

**18.2.1**  $D = 0$  (critically-damped case): Calculate

$$\alpha = -\frac{b}{2} + \sqrt{D}, \quad C_1 = \alpha(2K_1 - \alpha K_2), \quad C_2 = C_1 - \alpha K_1$$

The complete response is

$$y[n] = \{(C_1 + C_2 n)\alpha^n + y_f[n]\}u[n]$$

**18.2.2**  $D > 0$  (over-damped case): Calculate

$$\alpha = -\frac{b}{2} + \sqrt{D}, \quad \beta = -\frac{b}{2} - \sqrt{D}$$

$$C_1 = \frac{\alpha^2(\beta K_2 - K_1)}{\beta - \alpha}, \quad C_2 = \frac{\beta(\alpha K_1 - C_1)}{\alpha}$$

The complete response is

$$y[n] = [C_1 \alpha^n + C_2 \beta^n + y_f[n]]u[n]$$

**18.2.3**  $D < 0$  (under-damped case): Calculate

$$\alpha = -\frac{b}{2} + j\sqrt{-D}, \quad \sigma = |\alpha|, \quad \gamma = \angle \alpha$$

$$E = \frac{\alpha^2(K_1 - \alpha^* K_2)}{\alpha - \alpha^*}, \quad C = 2|E|, \quad \phi = \angle E,$$

The complete response is

$$y[n] = [C \sigma^n \cos(n\gamma + \phi) + y_f[n]]u[n]$$