

# A Game Theory Approach to Constrained Minimax State Estimation

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## Abstract

This paper presents a game theory approach to the constrained state estimation of linear discrete time dynamic systems. In the application of state estimators there is often known model or signal information that is either ignored or dealt with heuristically. For example, constraints on the state values (which may be based on physical considerations) are often neglected because they do not easily fit into the structure of the state estimator. This paper develops a method for incorporating state equality constraints into a minimax state estimator. The algorithm is demonstrated on a simple vehicle tracking simulation.

*Key Words* –  $H_\infty$  Filter, Minimax Filter, State Constraints, State Estimation, Game Theory.

## 1 Introduction

In the application of state estimators there is often known model or signal information that is either ignored or dealt with heuristically [11]. This paper presents a way to generalize a minimax state estimator in such a way that known relations among the state variables (i.e., state constraints) are satisfied by the state estimate. Constrained state estimation has not, to our knowledge, been studied from a game theory or minimax point of view.

Interest in minimax estimation (also called  $H_\infty$  estimation) began in 1981 [31], when it was noted that in dealing with noise with unknown statistics, the noise could be modeled as a deterministic signal. This replaces the Kalman filtering method of modeling the noise as a random process. This results in estimators that are more robust to unmodeled noise and uncertainty, as will be illustrated in Section 5.

Although state constraints have not yet been incorporated into minimax filters, they have been incorporated into Kalman filters using a variety of different approaches.

Sometimes state constraints are enforced heuristically in Kalman filters [11]. Some researchers have treated state constraints by reducing the system model parameterization [27], but this approach is not always desirable or even possible [28]. Other researchers treat state constraints as perfect measurements [7, 16]. This results in a singular covariance matrix but does not present any theoretical problems [4]. In fact, Kalman's original paper [10] presents an example that uses perfect measurements (i.e., no measurement noise). But there are several considerations that indicate against the use of perfect measurements in a Kalman filter implementation. Although the Kalman filter does not formally require a nonsingular covariance matrix, in practice a singular covariance increases the possibility of numerical problems [12, p. 249], [24, p. 365]. Also, the incorporation of state constraints as perfect measurements increases the dimension of the problem, which in turn increases the size of the matrix that needs to be inverted in the Kalman gain computation. These issues are addressed in [19], which develops a constrained Kalman filter by projecting the standard Kalman filter estimate onto the constraint surface.

Numerous efforts have been pursued to incorporate constraints into  $H_\infty$  control problems. For instance,  $H_\infty$  control can be achieved subject to constraints on the system time response [8, 17, 18], state variables [14], controller poles [25], state integrals [13], and control variables [1, 33].

Fewer attempts have been made to incorporate constraints into  $H_\infty$  filtering problems. One example is  $H_\infty$  filter design with poles that are constrained to a specific region [15]. FIR and IIR filters can be designed such that the  $H_\infty$  norm of the error transfer function is minimized while constraining the filter output to lie within a prescribed envelope [26, 32]. However, to our knowledge, there have not been any efforts to incorporate state equality constraints into  $H_\infty$  filtering problems.

This paper generalizes the results of [30] so that minimax state estimation can be performed while satisfying equality constraints on the state estimate. The major

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contribution of this paper is the development of a minimax state estimator for linear systems that enforces equality constraints on the state estimate. We formulate the problem as a particular game which was shown in [30] to be equivalent to an  $H_\infty$  state estimation problem. We then derive the estimator gain and adversary gain that yields a saddle point for the constrained estimation problem.

Constrained estimators other than  $H_\infty$  filters can be implemented on constrained problems. The most notable alternative to constrained  $H_\infty$  filtering is constrained Kalman filtering [19]. The choice of whether to use a constrained Kalman or constrained  $H_\infty$  filter is problem dependent, but the general advantages of  $H_\infty$  estimation can be summarized as follows [6].

- $H_\infty$  filtering provides a rigorous method for dealing with systems that have model uncertainty.
- Continuous time  $H_\infty$  filtering provides a natural way to limit the frequency response of the estimator. (Although this paper deals strictly with discrete time filtering, the methods herein can also be used to extend existing continuous time  $H_\infty$  filtering results to constrained filtering. This is an area for further research.)
- $H_\infty$  filtering can be used to guarantee stability margins or minimize worst case estimation error.
- $H_\infty$  filtering may more appropriate for systems where the model changes unpredictably, and model identification and gain scheduling are too complex or time consuming.

Section 2 of this paper formulates the problem, and Section 3 develops the solution through a series of preliminary lemmas and the main saddle point theorem of this paper. As expected, it turns out that the unconstrained minimax estimator is a special case of the constrained minimax estimator. Section 4 discuss how the methods of this paper can be extended to inequality constraints. Section 5 presents some simulation results, and Section 6 offers some concluding remarks. Some lemma proofs are presented in the appendix.

## 2 Problem Statement

Consider the discrete linear time-invariant system given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k + \delta_k \\ y_k &= Cx_k + m_k \end{aligned} \quad (1)$$

where  $k$  is the time index,  $x$  is the state vector,  $y$  is the measurement,  $\{w_k\}$  and  $\{m_k\}$  are white noise sequences, and  $\{\delta_k\}$  is a noise sequence generated by an adversary. We assume that  $\{w_k\}$  and  $\{m_k\}$  are mutually uncorrelated unity-variance white noise sequences. In general,  $A$ ,  $B$ , and  $C$  can be time-varying matrices, but we will omit the time subscript on these matrices for ease of notation. In addition to the state equation we know (on the basis

of physical considerations or other *a priori* information) that the states satisfy the following constraint.

$$D_k x_k = d_k \quad (2)$$

We assume that the  $D_k$  matrix is full rank and normalized so that  $D_k D_k^T = I$ . In general,  $D_k$  is an  $s \times n$  matrix, where  $s$  is the number of constraints,  $n$  is the number of states, and  $s < n$ . If  $s = n$  then (2) completely defines  $x_k$ , which makes the estimation problem trivial. For  $s < n$ , which is the case in this paper, there are fewer constraints than states, which makes the estimation problem nontrivial. Assuming that  $D_k$  is full rank is the same as the assumption made in the constrained Kalman filtering problem [19]. We define the following matrix for notational convenience.

$$V_k = D_k^T D_k \quad (3)$$

We will assume that both the noisy system and the noise-free system satisfy the above state constraint. The problem is to find an estimate  $\hat{x}_{k+1}$  of  $x_{k+1}$  given the measurements  $\{y_0, y_1, \dots, y_k\}$ . The estimate should satisfy the state constraint. We will restrict the state estimator to have an observer structure so that it results in an unbiased estimate [2].

$$\begin{aligned} \hat{x}_0 &= 0 \\ \hat{x}_{k+1} &= A\hat{x}_k + K_k(y_k - C\hat{x}_k) \end{aligned} \quad (4)$$

The main advantage of unbiased estimators over biased estimators is that unbiased estimators make it easier to quantify the estimation error. With biased estimators we must quantify the error using both the bias and some other measure (e.g., mean square error or worst case error). In general, unbiased estimators are preferred over biased estimators because of their greater mathematical tractability.

**Lemma 1** *If we have an estimator gain of the form*

$$K_k = (I - V_{k+1})\tilde{K}_k \quad (5)$$

where  $\tilde{K}_k$  is any dimensionally appropriate matrix, then the state estimate (4) satisfies the state constraint (2).

**Proof:** See the appendix.

The noise  $\delta_k$  in (1) is introduced by an adversary that has the goal of maximizing the estimation error. Similar to [30], we will assume that our adversary's input to the system is given by

$$\delta_k = L_k(G_k(x_k - \hat{x}_k) + n_k) \quad (6)$$

where  $L_k$  is a gain to be determined,  $G_k$  is a given matrix, and  $\{n_k\}$  is a noise sequence. We will assume that  $\{w_k\}$ ,  $\{m_k\}$ , and  $\{n_k\}$  are mutually uncorrelated unity-variance white noise sequences that are uncorrelated with  $x_0$ . This form of the adversary's input is not intuitive because it uses the state estimation error, but this form is taken because the solution of the resulting problem results in a state estimator that bounds the infinity norm

of the transfer function from the random noise terms to the state estimation error [30]. (This is discussed further following (15).)  $G_k$  can be considered by the designer as a tuning parameter or weighting matrix that can be adjusted on the basis of our *a priori* knowledge about the adversary's noise input. Suppose, for example, that we know ahead of time that the first component of the adversary's noise input to the system is twice the magnitude of the second component, the third component is zero, etc.; then that information can be reflected in the designer's choice of  $G_k$ . We do not need to make any assumptions about the form of  $G_k$  (e.g., it does not need to be positive definite or square).

From (6) we can see that as  $G_k$  approaches the zero matrix, the adversary's input becomes purely a random process without any deterministic component. This causes the optimal minimax filter to approach the Kalman filter; that is, we obtain better RMS error performance but not as good worst-case error performance. As  $G_k$  becomes large, the minimax filter places more emphasis on minimizing the estimation error due to the deterministic component of the adversary's input. That is, the minimax filter assumes less about the adversary's input, and we obtain better worst-case error performance but worse RMS error performance.

**Lemma 2** *In order for the noise-free system (1) to satisfy the state constraint (2), the adversary gain  $L_k$  must satisfy the following equality.*

$$D_{k+1}L_k = 0 \quad (7)$$

One way to satisfy this equality is for  $L_k$  to be of the form

$$L_k = (I - V_{k+1})\tilde{L}_k \quad (8)$$

where  $\tilde{L}_k$  is any dimensionally appropriate matrix.

**Proof:** See the appendix.

The estimation error is defined as follows.

$$e_k = x_k - \hat{x}_k \quad (9)$$

It can be shown from the preceding equations that the dynamic system describing the evolution of the estimation error is given as follows.

$$\begin{aligned} e_0 &= x_0 \\ e_{k+1} &= (A - K_k C + L_k G_k)e_k + \\ &\quad Bw_k + L_k n_k - K_k m_k \end{aligned} \quad (10)$$

Since  $D_k x_k = D_k \hat{x}_k = d_k$ , we see that  $D_k e_k = 0$ . But we also know by following a procedure similar to the proof of Lemma 1 that  $D_{k+1} A e_k = 0$ . Therefore we can subtract the zero term  $D_{k+1}^T D_{k+1} A e_k = V_{k+1} A e_k$  from the error equation (10) to obtain the following.

$$\begin{aligned} e_0 &= x_0 \\ e_{k+1} &= ((I - V_{k+1})A - K_k C + L_k G_k)e_k + \\ &\quad Bw_k + L_k n_k - K_k m_k \end{aligned} \quad (11)$$

However, this is an inappropriate term for a minimax problem because the adversary can arbitrarily increase  $e_k$  by arbitrarily increasing  $L_k$ . To prevent this, we decompose  $e_k$  as follows.

$$e_k = e_{1,k} + e_{2,k} \quad (12)$$

where  $e_{1,k}$  and  $e_{2,k}$  evolve as follows.

$$\begin{aligned} e_{1,0} &= x_0 \\ e_{1,k} &= ((I - V_{k+1})A - K_k C + L_k G_k)e_{1,k} + \\ &\quad Bw_k - K_k m_k \\ e_{2,0} &= 0 \\ e_{2,k} &= ((I - V_{k+1})A - K_k C + L_k G_k)e_{2,k} + L_k n_k \end{aligned} \quad (13)$$

Motivated by [30], we define the objective function as

$$J(K, L) = \text{trace} \sum_{k=0}^N W_k E(e_{1,k} e_{1,k}^T - e_{2,k} e_{2,k}^T) \quad (15)$$

where  $W_k$  is any positive definite weighting matrix. The differential game is for the filter designer to find a gain sequence  $\{K_k\}$  that minimizes  $J$ , and for the adversary to find a gain sequence  $\{L_k\}$  that maximizes  $J$ . As such,  $J$  is considered a function of  $\{K_k\}$  and  $\{L_k\}$ , which we denote in shorthand notation as  $K$  and  $L$ . This objective function is not intuitive, but is used here because the solution of the problem results in a state estimator that bounds the infinity norm of the transfer function from the random noise terms to the state estimation error [30]. This is expressed more completely in the next section in Lemma 3.

### 3 Problem Solution

The solution is obtained by finding optimal gain sequences  $\{K_k^*\}$  and  $\{L_k^*\}$  that satisfy the following saddle point.

$$J(K^*, L) \leq J(K^*, L^*) \leq J(K, L^*) \text{ for all } K, L \quad (16)$$

To solve this problem we will write the cost function (15) in a more convenient form. Define the matrix  $F_k$  as follows.

$$F_k = (I - V_{k+1})A - K_k C + L_k G_k \quad (17)$$

Define the following matrix difference equation.

$$\begin{aligned} Q_0 &= E(x_0 x_0^T) \\ Q_{k+1} &= F_k Q_k F_k^T + B B^T + K_k K_k^T - L_k L_k^T \end{aligned} \quad (18)$$

Then we have the following lemma.

**Lemma 3** *The cost function (15) is given as follows.*

$$J(K, L) = \text{trace} \sum_{k=0}^N W_k Q_k \quad (19)$$

Also, the minimization of  $J(K, L^*)$  with respect to  $K$  results in an estimator with the following bound for the square of the induced  $l_2$  norm of the system.

$$\sup_{w_k, m_k} \frac{\sum_{k=0}^N \|G_k e_k\|_2^2}{\sum_{k=0}^N (\|w_k\|_2^2 + \|m_k\|_2^2)} < \gamma \quad (20)$$

Note that the induced  $l_2$  norm reduces to the system  $H_\infty$  norm when the system is time invariant.

**Proof:** The proof follows in a straightforward way similar to Lemma 1 in [30].

The above lemma justifies the use of the term ‘‘minimax’’ for the state estimator. Regardless of the disturbances that enter the system via the noise sequences  $\{w_k\}$  and  $\{m_k\}$ , the gain from the noise to the weighted estimation error will always be less than the bound given in the above lemma.

Now define  $\tilde{Q}_k$  and  $\Sigma_k$  as the nonsingular solutions to the following set of equations.

$$\begin{aligned}\tilde{Q}_0 &= E(x_0 x_0^T) \\ \tilde{Q}_k(I - C^T C \Sigma_k) &= (I - \tilde{Q}_k G_k^T G_k) \Sigma_k \\ \tilde{Q}_{k+1} &= (I - V_{k+1}) A \Sigma_k A^T (I - V_{k+1}) + B B^T\end{aligned}\quad (21)$$

Nonsingular solutions to these equations are not always guaranteed to exist, in which case a solution to the minimax state estimation problem may not exist. However, if solutions to these equations do exist, then we see that  $\Sigma_k$  can be computed as

$$\Sigma_k = (\tilde{Q}_k C^T C - \tilde{Q}_k G_k^T G_k + I)^{-1} \tilde{Q}_k \quad (22)$$

so that we have explicit formulas to iteratively compute  $\tilde{Q}_k$  and  $\Sigma_k$ . Also note that we have to assume that  $\tilde{Q}_k$  and  $\Sigma_k$  are nonsingular. This assumption will be necessary for the proof of Lemma 4. We propose the following gain matrices for our estimator and adversary.

$$\begin{aligned}K_k^* &= (I - V_{k+1}) A \Sigma_k C^T \\ L_k^* &= (I - V_{k+1}) A \Sigma_k G_k^T\end{aligned}\quad (23)$$

Note that  $K_k^*$  and  $L_k^*$  satisfy the gain forms in (5) and (8), which guarantees that the state estimate and the noise-free system satisfy the constraint (2).

**Lemma 4** Denote by  $F_k^*$  the matrix of (17) when the  $K_k^*$  and  $L_k^*$  gains from (23) are substituted for  $K_k$  and  $L_k$ . Then we obtain the following for  $F_k^*$ .

$$F_k^* = (I - V_{k+1}) A \Sigma_k \tilde{Q}_k^{-1} \quad (24)$$

**Proof:** The proof closely follows the proof of Lemma A.1 in [30] and is also available at [23].

**Lemma 5** If in (18) we substitute  $K_k^*$  and  $L_k^*$  from (23) for  $K_k$  and  $L_k$ , then we obtain the  $\tilde{Q}_k$  given by (21). That is,  $Q_k^* = \tilde{Q}_k$ .

**Proof:** The proof closely follows the proof of Lemma A.2 in [30] and is also available at [23].

Now we define the matrices  $\rho_k$  and  $\Delta_k$  as follows.

$$\begin{aligned}\rho_k &= (K_k - K_k^*)(K_k - K_k^*)^T - \\ &\quad (L_k - L_k^*)(L_k - L_k^*)^T \\ \Delta_k &= Q_k - \tilde{Q}_k\end{aligned}\quad (25)$$

Then we obtain the following result.

**Lemma 6**  $\Delta_k$  satisfies the following difference equation.

$$\Delta_{k+1} = F_k \Delta_k F_k^T + (F_k - F_k^*) \tilde{Q}_k (F_k - F_k^*)^T + \rho_k \quad (26)$$

**Proof:** The proof closely follows the proof of Lemma A.3 in [30] and is also available at [23].

**Lemma 7** Suppose some matrix sequence  $\{\hat{\Delta}_k\}$  satisfies the equation

$$\begin{aligned}\hat{\Delta}_0 &\geq 0 \\ \hat{\Delta}_{k+1} &= \hat{F}_k \hat{\Delta}_k \hat{F}_k^T + R_k\end{aligned}\quad (27)$$

where  $R_k \geq 0$ . Then  $\hat{\Delta}_k \geq 0$  for all  $k$ . Similarly, if the matrix sequence satisfies the above difference equation with the initial condition  $\hat{\Delta}_0 \leq 0$  and  $R_k \leq 0$ , then  $\hat{\Delta}_k \leq 0$  for all  $k$ .

**Proof:** The proof is easily obtained by induction [30].

Now consider the  $Q_k$  matrix of (18). We see that  $Q_k$  is a function of  $K_k$  and  $L_k$ . Therefore we can write  $Q_k$  as  $Q(K_k, L_k)$ . With this notation we obtain the following two lemmas.

**Lemma 8** If  $(I + C \tilde{Q}_k C^T) \geq 0$  then

$$Q(K_k^*, L_k^*) \leq Q(K_k, L_k^*) \quad (28)$$

**Proof:** The proof is obtained by substituting  $L_k = L_k^*$  in (26) and noting by Lemma 7 that  $\Delta_k \geq 0$ . The proof closely follows the proof of case (a) of Theorem 1 in [30] and is also available at [23].

**Lemma 9** If  $(I - G_k \tilde{Q}_k G_k^T) \geq 0$  then

$$Q(K_k^*, L_k^*) \geq Q(K_k^*, L_k) \quad (29)$$

**Proof:** The proof is obtained by substituting  $K_k = K_k^*$  in (26) and noting by Lemma 7 that  $\Delta_k \geq 0$ . The proof closely follows the proof of case (b) of Theorem 1 in [30] and is also available at [23].

**Theorem 1** If  $(I + C \tilde{Q}_k C^T) \geq 0$  and  $(I - G_k \tilde{Q}_k G_k^T) \geq 0$ , then the estimator and adversary gains defined by (23) satisfy the saddle point equilibrium (16).

**Proof:** From the preceding two lemmas we see that

$$Q(K_k^*, L_k) \leq Q(K_k^*, L_k^*) \leq Q(K_k, L_k^*) \quad (30)$$

Combining this with (19) and the positive definiteness of  $W_k$  we obtain the desired saddle point of (16).

**QED**

Note that as  $G_k$  becomes larger, we will be less likely to satisfy the  $(I - G_k \tilde{Q}_k G_k^T) \geq 0$  condition. From (6) we see that a larger  $G_k$  gives the adversary more latitude in choosing a disturbance. This makes it less likely that the designer can minimize the cost function.

The mean square estimation error with the optimal gain cannot be specified because it depends on the adversary’s input  $\delta_k$ . However, we can state an upper bound for the mean square estimation error as follows.

**Lemma 10** *If the estimator gain defined by (23) then the mean square estimation error is bounded as follows.*

$$E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \leq \tilde{Q}_k \quad (31)$$

*This provides additional motivation for using the game theory approach presented in this paper. The estimator not only bounds the worst case estimation error, but also bounds the mean square estimation error.*

**Proof:** The proof closely follows a proof presented in [30].

Now consider the special case that there are no state constraints. Then in (2) we can set the  $D_k$  matrix equal to a zero row vector and the  $d_k$  vector equal to a zero column vector. In this case  $V_{k+1} = 0$  and we obtain from (21) and (23) the following estimator and adversary strategies.

$$\begin{aligned} \tilde{Q}_0 &= E(x_0 x_0^T) & (32) \\ \tilde{Q}_k(I - C^T C \Sigma_k) &= (I - \tilde{Q}_k G_k^T G_k) \Sigma_k \\ \tilde{Q}_{k+1} &= A \Sigma_k A^T + B B^T \\ K_k^* &= A \Sigma_k C^T \\ L_k^* &= A \Sigma_k G_k^T \end{aligned}$$

This is identical to the unconstrained minimax estimator [30], which was shown to be equivalent to the  $H_\infty$  estimator.

The constrained  $H_\infty$  estimator can be summarized as follows.

### Algorithm 1 – $H_\infty$ filtering with equality constraints

1. We have a linear system given as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k + \delta_k & (33) \\ y_k &= C_k x_k + m_k \\ D_k x_k &= d_k \end{aligned}$$

where  $\{w_k\}$  and  $\{m_k\}$  are uncorrelated unity variance white noise sequences, and  $\{\delta_k\}$  is a noise sequence generated by an adversary. We assume that the constraints are normalized so  $D_k D_k^T = I$ .

2. Initialize the filter as follows.

$$\begin{aligned} \hat{x}_0 &= 0 & (34) \\ \tilde{Q}_0 &= E(x_0 x_0^T) \end{aligned}$$

3. At each time step  $k = 0, 1, \dots$ , do the following.

- (a) Choose the tuning parameter matrix  $G_k$  to weight the deterministic, biased component of the process noise. If  $G_k = 0$  then we are assuming that the process noise is zero mean and purely random, and we get Kalman filter performance. As  $G_k$  increases we are assuming that there is more of a deterministic, biased component to the process noise. This gives us better worst case error performance but worse RMS error performance.

- (b) Compute the next state estimate as follows.

$$\begin{aligned} V_{k+1} &= D_{k+1}^T D_{k+1} & (35) \\ \Sigma_k &= (\tilde{Q}_k C_k^T C_k - \tilde{Q}_k G_k^T G_k + I)^{-1} \tilde{Q}_k \\ \tilde{Q}_{k+1} &= (I - V_{k+1}) A_k \Sigma_k A_k^T (I - V_{k+1}) + B_k B_k^T \\ K_k &= (I - V_{k+1}) A_k \Sigma_k C_k^T \\ \hat{x}_{k+1} &= A_k \hat{x}_k + K_k (y_k - C_k \hat{x}_k) \end{aligned}$$

- (c) Verify that  $(I - G_k \tilde{Q}_k G_k^T) \geq 0$ . If it is not then the filter solution is invalid, so we can decrease  $G_k$  and try again.

## 4 Inequality Constraints

Constrained state estimation problems can always be solved by reducing the system model parameterization [27], or by introducing artificial perfect measurements into the problem [7, 16]. However, those methods cannot be extended to inequality constraints, while the method discussed in this paper can be extended to equality constraints. In the case of state inequality constraints (i.e., constraints of the form  $D_k x_k \leq d_k$ ), a standard active set method [3, 5] can be used to solve the minimax filtering problem. An active set method uses the fact that it is only those constraints that are active at the solution of the problem that affect the optimality conditions; the inactive constraints can be ignored. Therefore, an inequality constrained problem is equivalent to an equality constrained problem. An active set method determines which constraints are active at the solution of the problem and then solves the problem using the active constraints as equality constraints. Inequality constraints will significantly increase the computational effort required for a problem solution because the active constraints need to be determined, but conceptually this poses no difficulty. This method has been used to extend the equality constrained Kalman filter to an inequality constrained Kalman filter [22].

In case we have inequality constraints  $D_k x_k \leq d_k$  instead of equality constraints, Algorithm 1 of the previous section can be modified as follows.

### Algorithm 2 – $H_\infty$ filtering with inequality constraints

1. Same as Step 1 in Algorithm 1, except that the constraints are of the form  $D_k x_k \leq d_k$ .
2. Same as Step 2 in Algorithm 1, except we also initialize the cost function  $Q_0 = E(x_0 x_0^T)$  as shown in (18).
3. At each time step  $k = 0, 1, \dots$ , do the following.
  - (a) Same as Step 3a in Algorithm 1.
  - (b) Use the `fmincon` function in Matlab's Optimization Toolbox to find the state estimate

$\hat{x}_{k+1}$  and the set of active constraints that minimizes the cost function  $Q_{k+1}$ .

$$\hat{x}_{k+1} = \text{fmincon}(\text{fun}, \hat{x}_k, D_k, d_k) \quad (36)$$

In the `fmincon` call above,  $\hat{x}_k$  is the starting point for the optimization algorithm. The cost function routine `fun` is a user written function that takes a state estimate  $\hat{x}$  as an input and returns the cost function  $Q_{k+1}$ . Function `fun` is given as follows.

```
function Qk+1 = fun( $\hat{x}$ )
DActive = []; % Active rows of Dk+1
for i = 1 to n
    if ||Dk+1(i,:)· $\hat{x}$  - dk+1(i)|| <  $\epsilon$ 
        DActive = [DActive ; Dk+1(i,:)]
    end
end
Vk+1 = (DActive)T(DActive)
 $\Sigma_k$  = ( $\tilde{Q}_k C_k^T C_k - \tilde{Q}_k G_k^T G_k + I$ )-1  $\tilde{Q}_k$ 
 $\tilde{Q}_{k+1}$  = ( $I - V_{k+1}$ )Ak $\Sigma_k$ AkT( $I - V_{k+1}$ ) + BkBkT
Kk = ( $I - V_{k+1}$ )Ak $\Sigma_k$ CkT
Lk = ( $I - V_{k+1}$ )Ak $\Sigma_k$ GkT
Qk+1 = AkQkAkT + BkBkT + KkKkT - LkLkT
```

(c) Same as Step 3c in Algorithm 1.

In the `fun` routine above,  $\epsilon$  is a user defined parameter that marks the dividing line between constraints that lie on the constraint boundary (equality constraints), and constraints that do not (inequality constraints).

Note that Matlab's `fmincon` function is flexible enough to accommodate variations in this approach, for example, if some of the constraints are equality constraints while others are inequality constraints, or if some of the constraints are nonlinear. See the Matlab documentation at [www.mathworks.com](http://www.mathworks.com) for details.

## 5 Simulation Results

In this section we present a simple example to illustrate the usefulness of the constrained minimax filter. Consider a land-based vehicle that is equipped to measure its latitude and longitude (e.g., through the use of a GPS receiver). This is the same example as that considered in [19]. The vehicle dynamics and measurements can be approximated by the following equations.

$$x_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ T \sin \theta \\ T \cos \theta \end{bmatrix} u_k + Bw_k + \delta_k$$

$$y'_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k + m'_k$$

The first two components of  $x_k$  are the latitude and longitude positions, the last two components of  $x_k$  are the latitude and longitude velocities,  $w_k$  represents a unity-variance process disturbance due to potholes and the like,

$\delta_k$  is some unknown process noise (due to an adversary), and  $u_k$  is the commanded acceleration.  $T$  is the sample period of the estimator, and  $\theta$  is the heading angle (measured counterclockwise from due east). The measurement  $y'_k$  consists of latitude and longitude, and  $m'_k$  is the measurement noise. Suppose the one-sigma measurement noises are known to be  $\sigma_1$  and  $\sigma_2$ . Then we must normalize our measurement equation to enforce the condition that the measurement noise is unity variance. We therefore define the normalized measurement  $y_k$  as

$$y_k = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}^{-1} y'_k$$

In our simulation we set the covariances of the process and measurement noise as follows.

$$Q = \text{Diag}(4 \text{ m/s}, 4 \text{ m/s}, 1 \text{ m/s}^2, 1 \text{ m/s}^2)$$

$$R = \text{Diag}(\sigma_1^2, \sigma_2^2) = \text{Diag}(900 \text{ m}^2, 900 \text{ m}^2)$$

We can use a minimax filter to estimate the position of the vehicle. There may be times when the vehicle is travelling off-road, or on an unknown road, in which case the problem is unconstrained. At other times it may be known that the vehicle is travelling on a given road, in which case the state estimation problem is constrained. For instance, if it is known that the vehicle is travelling on a straight road with a heading of  $\theta$  then the matrix  $D_k$  and the vector  $d_k$  of (2) can be given as follows.

$$D_k = \begin{bmatrix} 1 & -\tan \theta & 0 & 0 \\ 0 & 0 & 1 & -\tan \theta \end{bmatrix}$$

$$d_k = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

We can enforce the condition  $D_k D_k^T = I$  by dividing  $D_k$  by  $\sqrt{1 + \tan^2 \theta}$ . The sample period  $T$  is 1 s and the heading  $\theta$  is set to a constant 60 degrees. The commanded acceleration is alternately set to  $\pm 1 \text{ m/s}^2$ , as if the vehicle was alternately accelerating and decelerating in traffic. The initial conditions are set to

$$x_0 = \begin{bmatrix} 0 & 0 & 173 & 100 \end{bmatrix}^T$$

We found via tuning that a  $G_k$  matrix of  $I/\gamma$ , with  $\gamma = 40$ , gave good filter performance. Larger values of  $\gamma$  make the minimax filter perform like a Kalman filter. Smaller values of  $\gamma$  prevent the minimax filter from finding a solution as the positive definite conditions in Theorem 1 are not satisfied.

The unconstrained and constrained minimax filters were simulated using MATLAB for 120 seconds. 100 Monte Carlo simulation runs were performed, and the average RMS position and estimation errors at each time step are plotted in Figures 1 and 2. It can be seen that the constrained filter results in more accurate estimates. The unconstrained estimator results in position errors that average around 26 m, while the constrained estimator gives position errors that average about 19 m. The unconstrained velocity error averages around 3.5 m/s, while the constrained velocity error averages about 3.1 m/s. A

MATLAB m-file that implements the algorithms in this paper and that was used to produce these simulation results can be downloaded from [23].

The Kalman filter performs better than the minimax filter when the noise statistics are nominal. Table 1 shows a comparison of the unconstrained and constrained Kalman and minimax filters in this case. However, if the noise statistics are not known, then the minimax filter may perform better than the Kalman filter. Table 2 shows a comparison of the unconstrained and constrained Kalman and minimax filters when the acceleration noise on the system has a bias of 1 meter/s<sup>2</sup> in both the north and east directions.

Table 1: RMS estimation errors (averaged over 100 Monte Carlo simulations) of the unconstrained and constrained Kalman and minimax filters with nominal noise statistics. The Kalman filters perform better than the minimax filters in this case. Position errors are in units of meters, and velocity errors are in units of meters/s.

	Kalman		Minimax	
	Pos.	Vel.	Pos.	Vel.
Unconstrained	21.5	3.31	26.3	3.52
Constrained	16.6	3.04	19.0	3.12

Table 2: RMS estimation errors (averaged over 100 Monte Carlo simulations) of the unconstrained and constrained Kalman and minimax filters with off-nominal noise statistics. The minimax filters estimate position better than the Kalman filters in this case, although the Kalman filters still estimate velocity better than the minimax filters. Position errors are in units of meters, and velocity errors are in units of meters/s.

	Kalman		Minimax	
	Pos.	Vel.	Pos.	Vel.
Unconstrained	48.6	12.3	37.7	14.0
Constrained	46.5	12.2	33.0	13.9

## 6 Conclusion and Future Work

We have presented a method based on game theory for incorporating linear state equality constraints in a minimax filter. Simulation results demonstrate the effectiveness of this method. If the state constraints are nonlinear they can be linearized at each time point, just as nonlinear state equations can be linearized at each time point. Stability analysis has not been discussed in this paper,

but is left for future work. Present efforts are focused on applying these results to fault-tolerant neural network training [20] and Mamdani fuzzy membership function optimization with sum normal constraints [21]. We are also interested in extending this work to inequality constraints for the application of turbofan health parameter estimation [22].

In the case of parameter uncertainties in the system model or measurement matrix, the methods of this paper do not apply. A number of schemes have been proposed for optimal filtering for uncertain systems, but none incorporate state constraints. For example, [9] discusses  $H_\infty$  filtering with error variance constraints for systems with parameter uncertainties, and [29] discusses Kalman filtering for systems with parameter uncertainties. Future work could focus on reworking the methods presented in those papers to incorporate state constraints.

## Appendix

**Proof of Lemma 1:** We assume that the noise-free system dynamics satisfy the state constraint. That is, if  $D_k x_k = d_k$ , then  $D_{k+1} A x_k = D_{k+1} x_{k+1} = d_{k+1}$ . Therefore, if  $\hat{x}_k$  satisfies the state constraint at time  $k$ , we know that  $D_{k+1} A \hat{x}_k = d_{k+1}$ . We can therefore write from (4)

$$D_{k+1} \hat{x}_{k+1} = D_{k+1} A \hat{x}_k + D_{k+1} K_k (y_k - C \hat{x}_k)$$

But if  $\hat{x}_k$  satisfied the state constraint, then we know that the first term on the right side of the above equation is equal to  $d_{k+1}$ . Making this substitution, and substituting (5) for the gain  $K_k$ , we obtain

$$\begin{aligned} D_{k+1} \hat{x}_{k+1} &= d_{k+1} + \\ &D_{k+1} (I - D_{k+1}^T D_{k+1}) \tilde{K}_k (y_k - C \hat{x}_k) \\ &= d_{k+1} \end{aligned}$$

where the last equality follows from the fact that  $D_{k+1} D_{k+1}^T = I$ . So if  $\hat{x}_0$  satisfies the constraint at the initial time, then the proof of the lemma follows by induction.

**QED**

**Proof of Lemma 2:** From (1) and (6) we obtain

$$\begin{aligned} x_{k+1} &= A x_k + B w_k + L_k (G_k x_k - G_k \hat{x}_k + n_k) \\ &= (A + L_k G_k) x_k + B w_k + L_k n_k - L_k G_k \hat{x}_k \end{aligned}$$

Premultiplying both sides by  $D_{k+1}$  we obtain

$$\begin{aligned} D_{k+1} x_{k+1} &= D_{k+1} (A + L_k G_k) x_k + D_{k+1} B w_k + \\ &D_{k+1} L_k n_k - D_{k+1} L_k G_k \hat{x}_k \end{aligned}$$

If we set  $w_k = n_k = 0$  in the above equation we obtain the following.

$$\begin{aligned} D_{k+1} x_{k+1} &= D_{k+1} (A + L_k G_k) x_k - D_{k+1} L_k G_k \hat{x}_k \\ &= d_{k+1} + D_{k+1} L_k G_k x_k - D_{k+1} L_k G_k \hat{x}_k \end{aligned}$$

Since  $x_{k+1}$  satisfied the state constraint, the above equation is equal to  $d_{k+1}$ . We therefore obtain the following.

$$D_{k+1} L_k G_k (x_k - \hat{x}_k) = 0$$

This equation does not constrain  $G_k$ ; however, it must hold for all  $G_k$  and  $(x_k - \hat{x}_k)$ . This means that  $D_{k+1}L_k = 0$ . Next we see that if  $L_k$  is of the form

$$L_k = (I - V_{k+1})\tilde{L}_k$$

then  $D_{k+1}L_k = 0$  (using the fact that  $D_{k+1}D_{k+1}^T = I$ ).

**QED**

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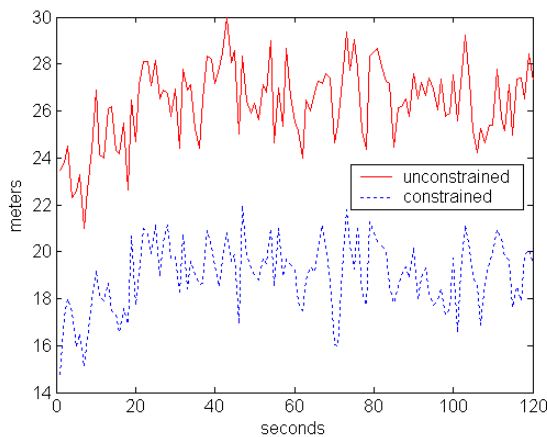


Figure 1: Unconstrained and constrained minimax filter position estimation errors. The plot shows the average RMS estimation errors of 100 Monte Carlo simulations.

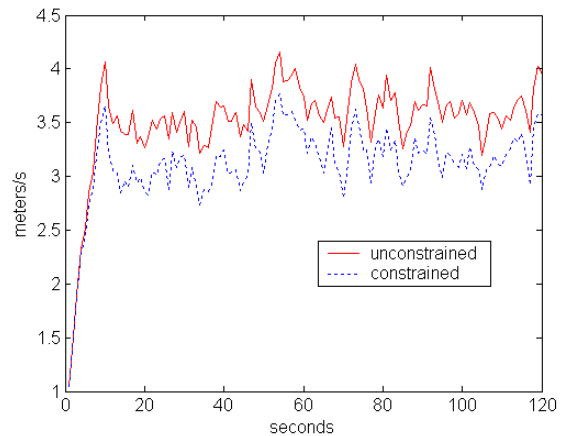


Figure 2: Unconstrained and constrained minimax filter velocity estimation errors. The plot shows the average RMS estimation errors of 100 Monte Carlo simulations.