Lyapunov Stability Analyses of Digital Phase-Locked Loops *

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I. Introduction

Phase-locked loops (PLLs) are used to track the phase and frequency of the carrier component of an incoming signal. The development of digital phase-locked loops (DPLLs) began in the late 1960s, and reached a reasonable state of maturity by the early 1980s (Lindsey and Chie, 1981).

Several DPLL architectures have been proposed over the years. Perhaps the most popular implementation is the zero-crossing DPLL, in which the loop attempts to sample the incoming waveform at the zero crossings. This type of loop dominates the literature because it is simple to implement and model. This paper will concentrate on zero-crossing DPLLs with one of two phase detector characteristics: sinusoidal or inverse tangent.

Zero-crossing DPLLs with sinusoidal phase detector characteristics were first introduced by Gill (1971). Their stability properties were studied by Weinburg and Liu (1974) and further developed by Osborne (1980a, 1980b). Zero-crossing DPLLs with inverse tangent phase detector characterics, known as digital tanlock loops (DTLs), were introduced and analyzed by Lee and Un (1982).

This paper uses Lyapunov stability theory to analyze the stability properties of DPLLs. As is the case with most stability analyses, this paper will deal with the noise-free case. While practical analysis and simulation of a real system must include some noise model, a prerequisite for such analysis is the knowledge or assumption of the system's stability. Lyapunov stability theory has been previously applied to analog PLLs (Abramovitch, 1990) but this present paper is the first known application of Lyapunov theory to DPLLs. The stability conditions derived in this paper are the same as those obtained elsewhere. The purpose of this paper is to expose Lyapunov theory as a viable option for DPLL design and analysis.

Lyapunov's stability theorem for digital systems can be stated as follows (Slotine and Li, 1991).

Theorem 1 *Consider a digital system described by* $x_{k+1} = f(x_k)$ *where* $x_k \in R^n$ *and* $f(0) = 0$. *Suppose that there exists a continuous scalar function* $V_k \equiv V(x_k)$ *such that*

1. $V(0) = 0$;
2. $V(x) > 0 \quad$ *for all* $\quad x \neq 0$;
3. $\lim_{\|x\| \to \infty} V(x) = \infty$; *and*
4. $\Delta V_k \equiv V(x_{k+1}) - V(x_k) < 0 \quad$ *for all* $\quad x_k \neq 0$.

Then the equilibrium state $x = 0$ *is asymptotically stable in the large.*

If the equilibrium state of the system $x_{k+1} = f(x_k)$ is at some nonzero state, then a state transformation $x' = g(x)$ must be made so that the equilibrium state of the equivalent system $x'_{k+1} = h(x'_k)$ is at $x' = 0$. This will be demonstrated in Section II-A.

II. Sinusoidal DPLL Stability Analyses

Consider a zero-crossing DPLL with the incoming signal denoted by

$$s(t) = A\sin(\omega_0 t + \theta(t)). \tag{1}$$

This signal is sampled by a digital clock at time instants t_k. The sampling times t_k are determined as

$$t_k = kT - D(z)\sum_{i=0}^{k-1} s(t_i) \tag{2}$$

where $T = 2\pi/\omega_0$ and $D(z)$ is the loop filter. The DPLL attempts to sample $s(t)$ at the zero crossings. So the phase error is defined as

$$\phi_k \equiv \omega_0 t_k + \theta_k \tag{3}$$

where $\theta_k \equiv \theta(t_k)$. It has been shown in (Osborne, 1980a) and (Weinburg and Liu, 1974) that the dynamics of such

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a DPLL can be described by the following difference equation in the phase error ϕ.

$$\phi_{k+1} = \phi_k + \theta_{k+1} - \theta_k - A\omega_0 D(z)\sin\phi_k. \tag{4}$$

In this paper, we will consider the input phase $\theta(t)$ to be a ramp; that is,

$$\theta(t) = \theta_0 + (\omega_1 - \omega_0)t \tag{5}$$

(recall that ω_0 is the nominal input frequency). Then (4) becomes

$$\phi_{k+1} = \phi_k + \Lambda - A\omega_1 D(z)\sin\phi_k \tag{6}$$

where $\Lambda \equiv (\omega_1 - \omega_0)T$.

A. <u>First Order Sinusoidal DPLL</u>

Now consider a first order loop filter; that is, $D(z) = G_1$. Then (6) becomes

$$\begin{aligned} \phi_{k+1} &= \phi_k + \Lambda - A\omega_1 G_1 \sin\phi_k & (7)\\ &= \phi_k + \Lambda - K_1 \sin\phi_k & (8) \end{aligned}$$

where $K_1 \equiv A\omega_1 G_1$. It can be easily shown (Osborne, 1980a) that in order for this system to have an equilibrium state, we must have $|K_1| > |\Lambda|$. But the equilibrium state of this system is, in general, *not* at $\phi = 0$. We will therefore make the state transformation

$$y_k = \phi_k - \sin^{-1}(\Lambda/K_1). \tag{9}$$

Then the dynamics equation (8) of the DPLL becomes

$$y_{k+1} = y_k - K_2 \sin y_k - \Lambda\cos y_k + \Lambda \tag{10}$$

where K_2 is given by

$$K_2 \equiv \sqrt{K_1^2 - \Lambda^2}. \tag{11}$$

Note that the equilibrium state of the system occurs at $y = 0$. We will choose our Lyapunov function as $V_k = y_k^2$. Then ΔV_k can be computed as

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k & (12)\\ &= y_{k+1}^2 - y_k^2 & (13)\\ &= (y_k - K_2\sin y_k - \Lambda\cos y_k + \Lambda)^2 - y_k^2 & (14) \end{aligned}$$

Linearizing (14) about the equilibrium state gives

$$\begin{aligned} \Delta V_k &= y_k^2(1 - K_2)^2 - y_k^2 & (15)\\ &= K_2(K_2 - 2)y_k^2 & (16) \end{aligned}$$

Lyapunov stability theory requires that $\Delta V_k < 0\ \forall y_k \neq 0$. This implies that $0 < K_2 < 2$. Recalling the definition (11) of K_2, we see that for Lyapunov stability we require that

$$K_1^2 - \Lambda^2 < 4 \tag{17}$$

which agrees with previously published results (Osborne, 1980a). For the case where the frequency offset is zero (i.e. $\Lambda = 0$) we obtain

$$|K_1| < 2 \tag{18}$$

which also agrees with previously published results (Osborne, 1980a). Since we linearized the equation for ΔV_k about the equilibrium state, these results are local stability conditions.

B. <u>Second Order Sinusoidal DPLL</u>

Now consider a second order DPLL where $D(z) = G_1 + G_2/(1 - z^{-1})$. Then (6) becomes

$$\phi_{k+2} = 2\phi_{k+1} - \phi_k + K_1\sin\phi_k - rK_1\sin\phi_{k+1} \tag{19}$$

where $K_1 \equiv AG_1\omega_1$ (as before) and $r \equiv 1 + G_2/G_1$. We define the state vector as

$$x_k = \begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \begin{pmatrix} \phi_k \\ \phi_{k+1} \end{pmatrix}. \tag{20}$$

So we can see that

$$\begin{aligned} x_{k+1} &= \begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{pmatrix} & (21)\\ &= \begin{pmatrix} x_k^2 \\ 2x_k^2 - x_k^1 + K_1\sin x_k^1 - rK_1\sin x_k^2 \end{pmatrix}. \end{aligned}$$

At this point we need to linearise the system equation around the equilibrium state $x_k = 0$ so that $\sin x_k^1 \approx x_k^1$ and $\sin x_k^2 \approx x_k^2$. Then (21) becomes

$$\begin{aligned} \begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{pmatrix} &= \begin{pmatrix} x_k^2 \\ (2 - rK_1)x_k^2 + (K_1 - 1)x_k^1 \end{pmatrix} & (22)\\ &= \begin{pmatrix} 0 & 1 \\ K_1 - 1 & 2 - rK_1 \end{pmatrix}\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} & (23) \end{aligned}$$

which can be written more compactly as

$$x_{k+1} = Bx_k. \tag{24}$$

Using the Lyapunov function $V_k = x_k^T x_k$, the change in the energy of the system is given by

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k & (25)\\ &= x_k^T B^T B x_k - x_k^T x_k & (26)\\ &= x_k^T (B^T B - I)x_k. & (27) \end{aligned}$$

In order for the system to satisfy the Lyapunov stability criterion, we must have $\Delta V_k < 0\ \forall x_k \neq 0$. Therefore the matrix $(B^T B - I)$ must be negative definite. This implies that the eigenvalues of $B^T B$ must be less than 1. This in turn implies that

$$0 < K_1 < \frac{4}{r+1} \quad \text{and} \quad r > 1. \tag{28}$$

This is the same local stability result as that obtained by Osborne (1980b).

C. Third Order Sinusoidal DPLL

Now consider a third order DPLL where $D(z) = G_1 + G_2/(1-z^{-1}) + G_3/(1-z^{-1})^2$. Then (6) becomes

$$\begin{aligned}\phi_{k+3} &= 3\phi_{k+2} - 3\phi_{k+1} + \phi_k - pK_1 \sin\phi_{k+2} \qquad (29)\\ &\quad + K_1(1+r)\sin\phi_{k+1} - K_1 \sin\phi_k\end{aligned}$$

where K_1 and r are defined as before, and $p \equiv r + G_3/G_1$. We define the state vector as

$$x_k = \begin{pmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{pmatrix} = \begin{pmatrix} \phi_k \\ \phi_{k+1} \\ \phi_{k+2} \end{pmatrix}. \qquad (30)$$

Now we can linearize the system equation as in Section B. and use the Lyapunov function $V_k = x_k^T x_k$. The details of the analysis are omitted because they are exactly analagous to the development in Section B. We obtain the same local stability requirements as those given by Osborne (1980b).

$$\begin{aligned} p - r &> 0 \qquad (31)\\ p + r - 2 &> 0 \qquad (32)\\ K_1(p-1) - p + r &> 0 \qquad (33)\\ 8 - K_1(p+r+2) &> 0. \qquad (34)\end{aligned}$$

III. Digital Tanlock Loop Stability Analyses

Consider a digital tanlock loop (DTL) with the incoming signal denoted by $x(t)$. The incoming signal is shifted by 90° to obtain $y(t)$. These signals can be expressed as

$$\begin{aligned} x(t) &= A\sin(\omega_0 t + \theta(t)) \qquad (35)\\ y(t) &= A\cos(\omega_0 t + \theta(t)). \qquad (36)\end{aligned}$$

Both $x(t)$ and $y(t)$ are sampled by a digital clock at time instants t_k. The sampling times t_k are determined as

$$t_k = kT - D(z)\sum_{i=0}^{k-1} e_i \qquad (37)$$

where $T = 2\pi/\omega_0$, $D(z)$ is the loop filter, and

$$e_k = \tan^{-1}(x_k/y_k). \qquad (38)$$

The DTL attempts to sample $x(t)$ and $y(t)$ at the zero crossings of $x(t)$. So the phase error is defined as

$$\phi_k \equiv \omega_0 t_k + \theta_k \qquad (39)$$

where $\theta_k \equiv \theta(t_k)$. Lee and Un (1982) showed that the dynamics of the DTL can be described by the following difference equation in the phase error ϕ.

$$\phi_{k+1} = \phi_k + \theta_{k+1} - \theta_k - \omega_0 D(z) e_k. \qquad (40)$$

As before, we will consider the input phase $\theta(t)$ to be a phase ramp; that is,

$$\theta(t) = \theta_0 + (\omega_1 - \omega_0)t \qquad (41)$$

(recall that ω_0 is the nominal input frequency). We will also assume that $\phi_k \in (-\pi, \pi)$. Then $e_k = \phi_k$ and (40) becomes

$$\phi_{k+1} = \phi_k + \Lambda - \omega_1 D(z)\phi_k \qquad (42)$$

where $\Lambda \equiv (\omega_1 - \omega_0)T$. Now we can apply Lyapunov theory to (42) where $D(z)$ is either first, second, or third order. Doing so results in the same DTL stability requirements as those reported by Lee and Un (1982). The procedure is ommitted from this paper since it is exactly analagous to the procedure used in the preceding section.

IV. Conclusion

Lyapunov's second method has been used to analyze the stability of DPLLs. This method is not new to control theory, but its application to DPLLs appears to be novel. The results derived in this paper reinforce results which have been derived in the past and observed in simulations over many years. The specific systems analyzed in this paper were first, second, and third order DPLLs with both sinusoidal and inverse tangent phase characteristics. Lyapunov's method appears to be quite applicable and straightforward as a design and analysis tool for DPLLs.

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