

# Chapter 25 Analysis of Simulation Results

- Four issues in Simulation

- (1) Model Verification

- Build the model right
    - Is the model correctly implemented ?
    - Input parameters are correctly represented ?

- (2) Model Validation

- Build the right model
    - Is the model accurate representative of the real system ?

- (3) Transient Removal

- How many of the initial observations should be removed ?

- (4) Stopping Criterion

- How long to run the simulation ?
    - Long run vs replications ?

# Verification and Validation

- Valid and verified
- Invalid and verified
  - the model correctly implements the assumptions
  - but the assumptions are far from reality
- Valid and unverified
  - the modeling assumptions are reasonable
  - but they are not correctly implemented (programmed)
- Invalid and unverified

# (1) Model Verification Techniques

- Design
  - Top Down Modular Design
  - Antibugging : self-checks, assert(N=10)
  - Deterministic Models : no randomness
  - Structured Walk-Through : explain code
- Runtime support
  - Trace : events trace , procedure trace , variables trace
  - On-Line Graphic Displays : more comprehensive than trace
- Experiment
  - Run Simplified Cases : compare with expected result
  - Seed Independence : Similar results for different seeds for random-number generation
  - *Continuity Test : slight change in input should cause only slight change in output*
  - *Degeneracy Tests : boundary condition (zero customer case)*
  - *Consistency Tests : 2 servers, 10 customers/sec ~ 4 servers, 20 customers/sec*

## (2) Model Validation Techniques

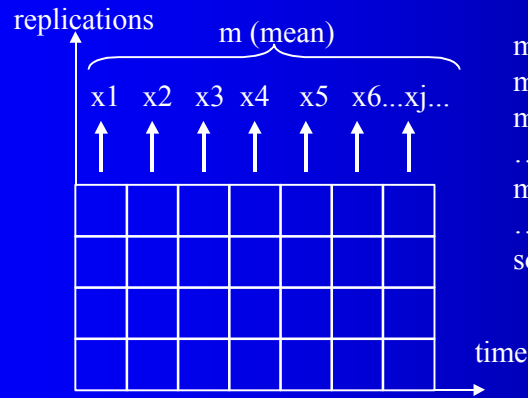
- One should Validate:
  - Assumptions
  - Input parameter values and distributions
  - Output values and conclusions
- By
  - Expert intuition
  - Real system measurements
  - Theoretical results
- Example
  - Linear regression model & check  $R^2$

## (3) Transient Removal

- Generally steady state performance is interesting
- Remove the initial part
- Heuristics:
  1. Long Runs
    - may waste resources
  2. Proper Initialization
    - already some customers in the system
  3. Truncation
    - truncate the initial part : the first of the remaining is not the min/max
  4. Initial Data Deletion
  5. Moving Average of Independent Replications
  6. Batch Means

# Initial Data Deletion (Ensemble)

- Use several replications to smoothen the average (Fig.25.8)



$$m_1 = (x_1 + x_2 + x_3 + \dots + x_n) / n$$

$$m_2 = (x_2 + x_3 + \dots + x_n) / (n-1)$$

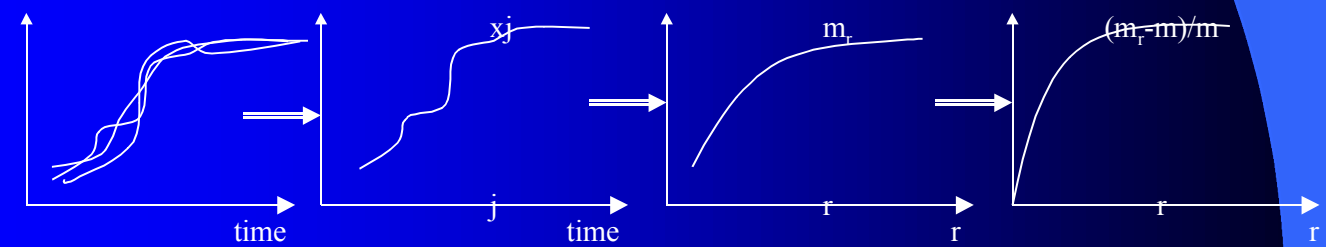
$$m_3 = (x_3 + \dots + x_n) / (n-2)$$

...

$$m_r = (x_r + \dots + x_n) / (n-r+1)$$

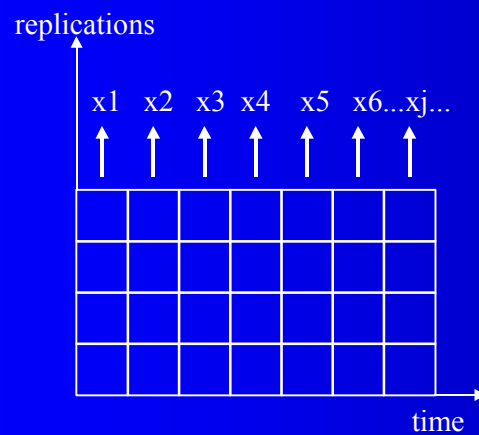
...

some point becomes a knee, where the transient interval ends



# Moving Average

- Mean over a moving time interval window instead of overall mean (Fig.25.9)



$$k=1: (2k+1=3)$$

$$m_2 = (x_1+x_2+x_3)/3$$

$$m_3 = (x_2+x_3+x_4)/3 \dots$$

$$m_{n-1} = (x_{n-2}+x_{n-1}+x_n)/3$$

$$k=2: (2k+1=5)$$

$$m_3 = (x_1+x_2+x_3+x_4+x_5)/5$$

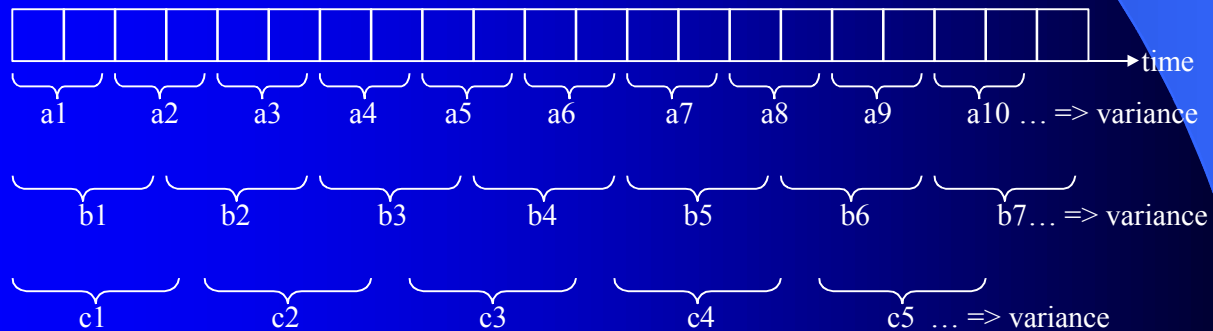
$$m_4 = (x_2+x_3+x_4+x_5+x_6)/5 \dots$$

$$m_{n-2} = (x_{n-4}+x_{n-3}+x_{n-2}+x_{n-1}+x_n)/5$$

....  
some point becomes a knee, where the transient interval ends

# Batch Means

- Run a long simulation and divide into equal duration part, Study variance of batch means as a function of the batch size (Fig.25.10/11)
- Plot the variance as a function of batch size and find the decreasing point where the steady-state begins



## (4) Stopping Criteria-Variance Estimation

- Run until 95% confidence interval is narrow enough
  - $(\mu \pm 1.96\sigma/\sqrt{n})$  if the observations are independent
  - Independence not applicable to most simulations
  - (Ex) Large waiting time for  $i$ th job makes large waiting time for  $(i + 1)$ th job
- What if the observations are correlated ?
  - Independent Replications
    - dependent over time but not over replications
    - check confidence interval over replications
  - Batch Means
    - check confidence interval over batches
  - Method of Regeneration
    - some system is regenerative (start over again from scratch - memoryless)
    - check confidence interval over regeneration cycles

# Statistics of Samples - review

- $x_1, x_2, \dots$  are instances of random variable  $x$

- $E(x) = \mu$
- $E\{(x - E(x))^2\} = \sigma^2$

$$\Rightarrow E\{(x - \mu)^2\} = E(x^2) - 2\mu^2 + \mu^2 \quad \text{or} \quad E(x^2) = \sigma^2 + \mu^2$$

- If we have  $n$  samples  $x_1, x_2, \dots, x_n$ ,

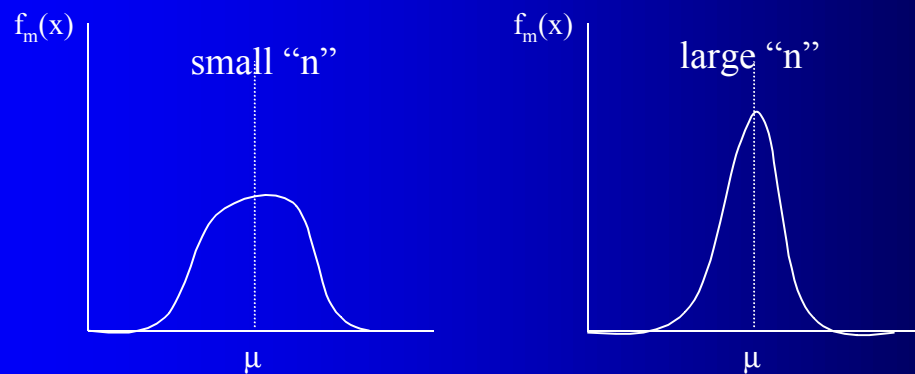
we'll choose  $m$  as the natural estimator for the mean of  $x$  and  $s^2$  as that of  $\sigma^2$   
(sample mean, sample variance)

- $m = \Sigma x_i / n$
- $s^2 = \Sigma (x_i - m)^2 / n$

- Our goal is to infer true distribution ( $\mu$  and  $\sigma^2$ )  
from our observations ( $m$  and  $s^2$ )

## Statistics of Samples (cont'd)

- $m, s$  are random variable
  - $E(m) = \mu$
  - $E\{(m-E(m))^2\} = \sigma^2/n$  : error of the estimate of the mean
  - $E(s^2) = (n-1)/n * \sigma^2$
  - confidence interval  $(\mu \pm 1.96\sigma\sqrt{n})$ ,  
use  $m$  for  $\mu$ ,  $s$  for  $\sigma$  since same expected value
- The larger “ $n$ ”, the better the estimate
  - As  $n \rightarrow \infty$ ,  $E\{(m-E(m))^2\} \rightarrow 0$  &  $E(s^2) \rightarrow \sigma^2$



# Statistics of Samples (cont'd)

- Derivations

mean of the s.mean

$$E(m) = E(\sum x_i / n) = \sum E(x_i) / n = n * E(x) / n = E(x) = \mu$$

variance (error) of the s.mean

$$\begin{aligned} & E\{(m-E(m))^2\} \\ &= E\{(m-\mu)^2\} \\ &= E\{m^2 - 2m\mu + \mu^2\} = E(m^2) - \mu^2 \\ &= E\{(\sum x_i / n)^2\} - 2\mu^2 + \mu^2 \\ &= \{E(\sum x_i^2) + 2 * \sum_{i=1}^{n-1} \sum_{j=i+1}^n E(x_i x_j)\} / n^2 - \mu^2 \\ &= n * E(x^2) / n^2 + 2 * \sum_{i=1}^{n-1} \sum_{j=i+1}^n E(x_i) E(x_j) / n^2 - \mu^2 \quad (\text{if } x_i \text{ and } x_j \text{ are uncorrelated}) \\ &= (\sigma^2 + \mu^2) / n + 2 * n(n-1) * \mu^2 / 2 / n^2 - \mu^2 \\ &= \sigma^2 / n \quad \Rightarrow E(m^2) = \sigma^2 / n + \mu^2 \end{aligned}$$

mean of the s.variance

$$E(s^2) = E\{\sum (x_i - m)^2 / n\} = E\{\sum (x_i - \sum x_i / n)^2 / n\} = \dots = (n-1) / n * \sigma^2$$

## Statistics of Samples (cont'd)

- Uncorrelated
  - if uncorrelated,  $P(x_i | x_j) = P(x_i)$
  - left =  $P(x_i \& x_j) / P(x_j)$
- Therefore, if uncorrelated,
  - $P(x_i \& x_j) = P(x_i) P(x_j)$
  - $E(x_i x_j) = \iint x_i x_j P(x_i \& x_j) dx_i dx_j$   
 $= \iint x_i x_j P(x_i) P(x_j) dx_i dx_j$   
 $= \int x_i P(x_i) dx_i \int x_j P(x_j) dx_j$   
 $= E(x_i) E(x_j)$
- Since observations from most simulation are correlated,
  - we cannot say that  $E\{(m-E(m))^2\} = \sigma^2/n$
  - neither to apply confidence interval ( $\mu \pm 1.96\sigma/\sqrt{n}$ )

# Comments

- Example- Job scheduling on MP
  - There are several job scheduling policies to compare (FCFS, Shortest job first, ...)
  - For each alternative, make 100 runs with 100 jobs, or what ?
  - How to stop the simulation ?
- How to model the job arrivals ?
  - Same generator provide the arrivals for the alternatives ?
  - How to model the bursts ?
- Modeling the Arrival Process
  - Arrival process is assumed stationary
    - = its statistical property does not change with time
  - Self-similarity
    - = the statistical properties of the arrival process are similar at all time scales