# Chapter 28 Random-Variate Generation

- Two steps
  - Random number generation
  - Random variate generation

- How to generate non-uniform random variables
  - Inverse Transformation : use $F^{-1}$
  - Rejection : when $F^{-1}$ is difficult to find/use
  - Composition
  - Convolution
  - Characterization

# Inverse Transformation

- Example: packet size and the corresponding probability

| size | 64 | 128 | 512 |
|---|---|---|---|
| prob. | 0.7 | 0.1 | 0.2 |

  - Generate u in [0,1]
  - 64 if $0<u \leq 0.7$, 128 if $0.7<u \leq 0.8$, 512 if $0.8<u \leq 1.0$

- Given any random variable x with a CDF F(x), the variable u=F(x) is uniformly distributed in [0,1]
  - Generate uniform random number u in [0,1]
  - Compute $x = F^{-1}(u)$

# Inverse Transformation(cont'd)

- Example : exponential random variate
  - pdf: $f(x) = \lambda e^{-\lambda x}$
  - CDF: $F(x) = 1- e^{-\lambda x} = u$

  => $x = -\ln(1-u)/ \lambda$ or $x = -\ln(u)/ \lambda$

- Example : Service time distribution by a repairman

| time (hour) | frequency | relative freq. | Cumul.freq. |
|---|---|---|---|
| $0 \le x \le 0.5$ | 31 | 0.31 | 0.31 |
| $0.5 \le x \le 1.0$ | 10 | 0.10 | 0.41 |
| $1.0 \le x \le 1.5$ | 25 | 0.25 | 0.66 |
| $1.5 \le x \le 2.0$ | 34 | 0.34 | 1.00 |

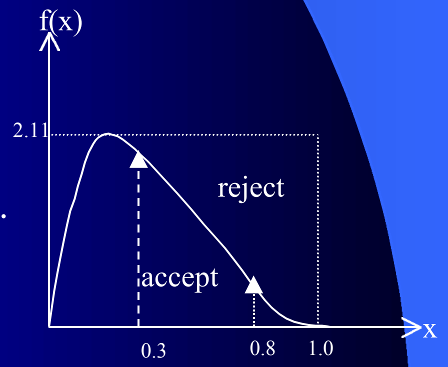
  - Generate u in [0,1], for example 0.54
  - Then what ? - Linear interpolation

# Rejection

- Generate random# x with pdf f(x)
  - Find pdf g(x) where $f(x) \le c \bullet g(x)$ for all values of x
  - Generate x with pdf g(x)
  - Generate y with uniform distribution on $[0, c \bullet g(x)]$
  - If $y \le f(x)$, then output x. Otherwise repeat.

- Example: $f(x)=20x(1-x)^3$, $0 \le x \le 1$
  - Select g(x)=1 on [0,1] and c=2.11
  - Generate x, uniform on [0,1]
  - Generate y, uniform on [0,2.11]
  - If $y \le 20x(1-x)^3$, then output x. Otherwise repeat.

* At x=0.3, high probability of acceptance
at x=0.8, high probability of rejection

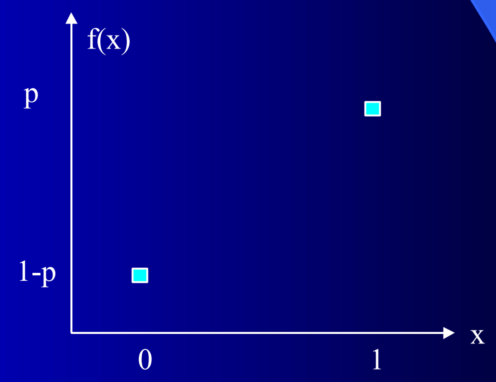


# Other Techniques

- Composition or Decomposition
  - $F(x) = \sum p_i F_i(x)$ or $f(x) = \sum p_i f_i(x)$
  - Steps : generate a random integer i (using Inverse Transformation)
    generate x with the ith CDF ($F_i(x)$) or pdf ($f_i(x)$)
- Convolution
  - $x = \sum y_i$ : random variable x is a sum of other random variables
  - pdf of x is a convolution of pdf of $y_i$'s
  - Steps: generate random variables $y_i$'s and sum them
  - Example : Erlang-k variate is a sum of k exponential variates
- Characterization
  - Interarrival time is exponential dist. => Poisson arrivals

# Chapter 29 Distributions
# (1) Bernoulli Distribution

- Simplest discrete distribution
  - failure(0 with 1-p) or success(1 with p) : Bernoulli trials
  - Models : system up or down
    packet delivery or not
    packet with noise or not
  - pdf : f(x) = 1-p  if x=0
    p  if x=1
    0  otherwise
  - mean = p,  variance = p(1-p)



# (2) Geometric Distribution

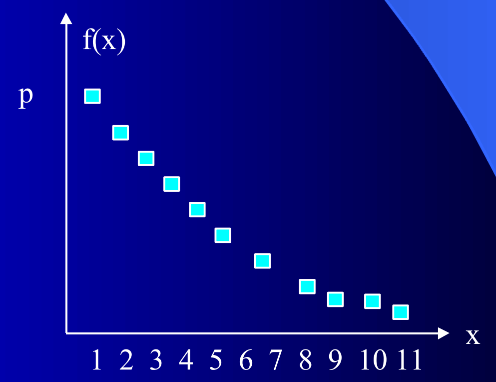
- Example : Number of coin tosses until head is tossed (p)
  - If p is the probability of success, # throws for a success is

    1: p

    2:(1-p)p

    3: $(1-p)^2p$

    …
  - $f(x)= (1-p)^{x-1}p$
  - mean = 1/p, variance = $(1-p)/p^2$




- "Memoryless" property
  - Distribution of # future throws is unaffected by the history

# (3) Binomial Distribution

- Example : Number of heads in n flips
  - Probability of success (1 with p), failure (0 with (1-p)) for each trial
  - Given n (Bernoulli) trials, probability of x successes is

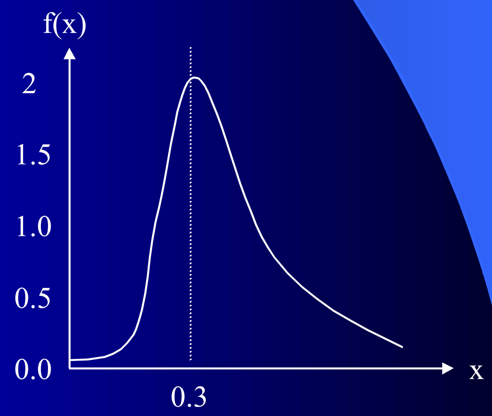
    $f_{n,p}(x) = nCx\ p^x(1-p)^{n-x}$
  - Models : # processors up in a parallel system

    # packets delivered to the destination without loss

    # packets without
  - mean = np, variance = np(1-p)
  - The variance is always smaller than the mean

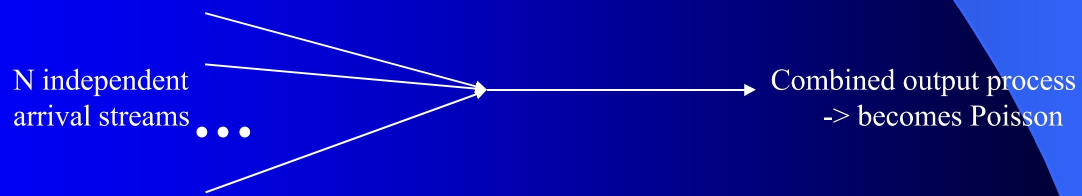
# * Continuous Probability Distribution

- f(x) is "not" the probability that x occurs at all !!!
  - f(x) "is" the probability with the three proceeding discrete distributions
  - f(0.3) > 1
  - To get a probability, you must integrate in a interval
  - The only requirement is ∫ f(x)dx = 1



# (4) Poisson Distribution

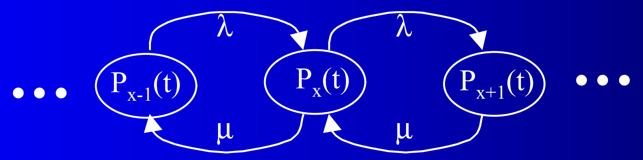
- Arrival processes
  - If each of the N input streams has independent, identically distributed interarrival times, and the N streams are independent with each other, the output process approaches Poisson as N->∝




- Probability of that x events occurs during time interval t
  - $f(x,t) = (\lambda t)^x e^{-\lambda t} / x!$ , where $\lambda$ is the mean arrival rate
  - mean = $\lambda t$, variance = $\lambda t$

# (4) Poisson Distribution (cont'd)

- Pure-birth process when $\mu=0$




  - $dP_x(t)/dt = -\lambda\, P_x(t) + \lambda\, P_{x-1}(t)$ ,   $dP_0(t)/dt = -\lambda\, P_0(t)$
    $\Rightarrow P_x(t) = (\lambda t)^x\, e^{-\lambda t} / x!$

- Properties
  - Sum of Poisson arrival processes is also a Poisson process
  - Poisson arrival process has "exponential" interarrival times

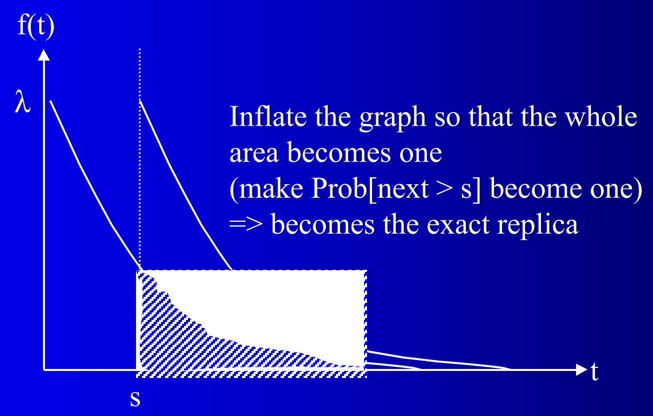
# (5) Exponential Distribution

- Interarrival time distribution of a Poisson arrival process
  - Prob[next arrival time $\leq$ t] : CDF
    $= 1$ - Prob[no arrivals during time t]
    $= 1$ - Poisson(0,t)
    $= 1 - (\lambda t)^0 e^{-\lambda t} / 0!$
    $= 1 - e^{-\lambda t}$
  - $f(t) = d(1- e^{-\lambda t})/dt = \lambda e^{-\lambda t}$ : "exponential distribution"
  - mean $= 1/\lambda$, variance $= 1/\lambda^2$, COV=1
- "Memoryless" property
  - Prob[next $\leq$ t+s | next > s] = Prob[next $\leq$ t]
  - Left = Prob[s<next $\leq$ t+s] / Prob[next > s]
    $= [(1- e^{-\lambda(t+s)})-(1-e^{-\lambda s})] / [1-(1- e^{-\lambda s})]$
    $= 1- e^{-\lambda t}$ = right

# (5) Exponential Distribution (cont'd)

- Memoryless property
  - At a bus stop, I have waited 15 minutes already.  But …..
  - The only continuous distribution with this property

    (The only discrete distribution with this property : Geometric distribution)



# (5) Exponential Distribution (cont'd)

- Maximum entropy distribution
  - Shannon defined entropy as a measure of information/uncertainty
  - You are waiting for a message made up of binary string. How much information does each bit you receive give you about the string ?
  - Generally, Entropy $S = -\sum p_i \ln p_i$
  - If P[0]=0 and P[1]=1, then each bit gives no new information

    $S = -(0 \ln 0 + 1 \ln 1) = 0$
  - If P[0]=P[1]=0.5, then each new bit is most valuable

    $S = -(0.5 \ln 0.5 + 0.5 \ln 0.5) = 0.69315$
  - Maximum entropy

    = new information has high value or old information has no influence

    = memoryless property => Exponential distribution (if mean is known)
  - If both mean and variance are known, it is Normal distribution

# * Discrete/Continuous

| | Discrete | Continuous |
|---|---|---|
| Interarrival time distribution | Geometric | Exponential |
| Distribution of # arrivals in [0,t] | Binomial | Poisson |

# (6) Normal (Gaussian) Distribution

- Discovered in 1733 (Moivre) and in 1809 (Gauss), 1812 (Laplace)
- $f(x) = 1/(\sigma\sqrt{(2\pi)}) \exp(-(x-\mu)^2/2\sigma^2)$ is called "$N(\mu,\sigma)$"
- Standard normal distribution $N(0,1)$
- Sum of n independent normal variates is a normal variate :

  $N(\Sigma\mu, \sqrt{(\Sigma\sigma^2)})$
- Used whenever randomness is caused by several independent sources acting additeively
  - Errors in measurement
  - Modeling error
  - Sample means of a large number of independent observations

# (7) Other Distributions

- Beta : Bounded random variable,   beta(a,b)
  - Models: fraction of packets requiring retransmission
    fraction of RPCs taking more than a specified time
  - pfd: $f(x) = [x^{a-1}(1-x)^{b-1}]/\beta(a,b)$ , $0 \le x \le 1$ (can be generalized)
    where $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ : beta function
  - mean = $a/(a+b)$,   variance = $ab/[(a+b)^2(a+b+1)]$
- Erlang : extension to the exponential (COV<1)
- Gamma : generalization of Erlang
- Student t
- Weibull