Review

- 큰 백화점에서 평균 이틀에 하나씩 전구가 고장이 난다. 한달 후 고장난 전등의 숫자는 평균 몇 개인가? 그 숫자가 20개가 넘을 확률은?
  - 한달 후에 1/2x30 = 15개 (?)
  - 20개 넘을 확률 ???

Review - Poisson

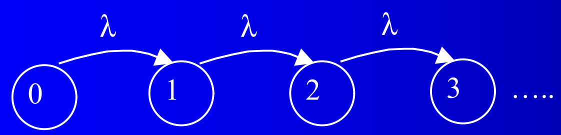
- 큰 백화점에서 평균 이틀에 하나씩 전구가 고장이 난다. 한달 후 고장난 전등의 숫자는 평균 몇 개인가 ? 그 숫자가 20개가 넘을 확률은 ?
  - Poisson process with λ=0.5/day, t=30 days, $P_k(t) = (\lambda t)^k e^{-\lambda t} /k!$
  - 한달 후 평균 = $\sum_{k=0}^{\infty} kP_k(t) = \lambda t = 0.5 x 30 = 15$ 개
  - $Pr\{k>=20\} = \sum_{k=20}^{\infty} P_k(t=30) = \sum_{k=20}^{\infty} (0.5x30)^k e^{-0.5x30} /k!$

- Poisson process는 매우 일반적으로 관측 가능한 현상임
  - 어떠한 사건이 일어나는 확률 (또는 비율)이 일정하고 그 사건들이 독립적으로 일어날 때 (“constant birth rate”)

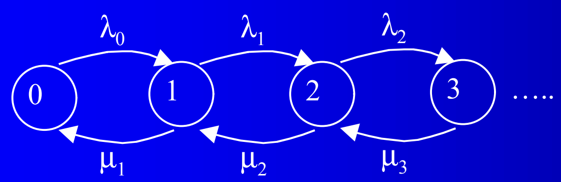
Review - Constant Birth Rate

- Constant birth rate
  - 전등의 숫자가 20개라면 고장나서 줄어들수록 λ가 작아질 것이다. 또한 이 경우, Pr{k>20}=0 (for all t)
  - 전체 군인 수가 10만명이고, 말에 채여 죽는 군인의 수는 평균 1/day 이다. 만일 전체가 5만명으로 줄어든다면 말에 채여 죽는 군인의 수는 평균 0.5/day로 줄어들 것이다.
- Limit theorem
  - "Sum of a large number of independent stationary renewal processes (each with a arbitrary distribution of renewal time) will tend to a Poisson process" [Palm & Khinchin, 1943 & 1960]
  - Aggregation of a "large number" of individuals or particles will become a Poisson process

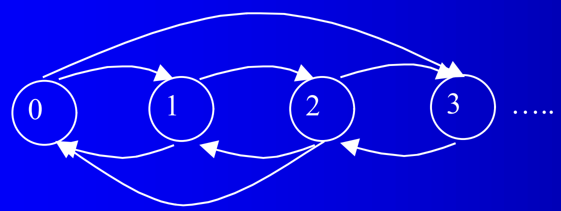
Review - Stocastic Processes



Poisson process



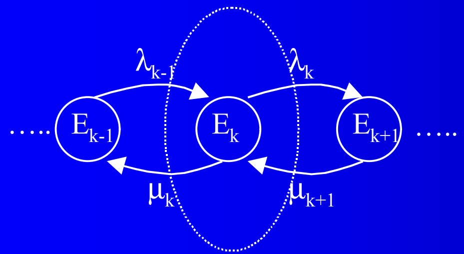
Birth-death process
(M/M/1 when $\lambda_k = \lambda$, $\mu_k = \mu$)



Markov process
- 머무르는 시간의 확률분포가 memoryless
- 상태천이가 과거와 무관함

Semi-Markov process
- 상태천이가 과거와 무관함

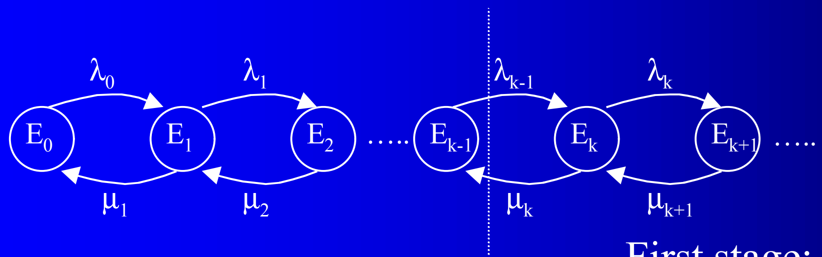
Review - Local Balance Equation



Conservation of flow:

- Flow into $E_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}$
- Flow from $E_k = \lambda_k p_k + \mu_k p_k$

$\Rightarrow \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} = (\lambda_k + \mu_k)p_k$



First stage: $\mu_1 p_1 = \lambda_0 p_0$

Second stage: $\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1)p_1$

$\Rightarrow \mu_2 p_2 = \lambda_1 p_1$

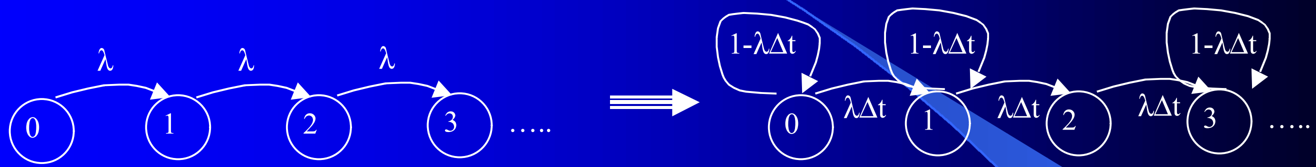
Therefore,

$$\mu_k p_k = \lambda_{k-1}p_{k-1} \text{ (steady-state)}$$

Review - Solution from p_k

- M/M/1
  - Constant birth rate : exponential interarrival time distribution $A(t)=1-e^{-\lambda t}$
  - Constant death rate : exponential service time distribution $B(x)=1- e^{-\mu x}$
  - $p_k = (1-\rho)\rho^k$ where $\rho = \lambda/\mu$
  - $N = \sum_{k=1}^{\infty} kp_k = \rho/(1-\rho)$
  - $T = N/\lambda$ (Little's Law) $= \rho/(1-\rho)\lambda = 1/(1-\rho)\mu$
  - $N_q = \sum_{k=1}^{\infty} (k-1)p_k = \rho^2/(1-\rho)$
  - $T_q = N_q/\lambda$ (Little's Law) $= \rho^2/(1-\rho)\lambda$
  - $Pr\{k>=n\} = \sum_{k=n}^{\infty} p_k$

Review - Rate vs Probability



- 평균 3초에 1대씩 들어온다면, $\lambda = 1/3$ customers/sec
  - Discrete-time (step = 1 sec)인 경우,
    다음 step (1 sec)에 state transition이 있을 확률
    = 다음 step (1 sec)에 새로운 customer가 들어올 확률
    $= 1/3 = \lambda\Delta t = (1-p_{ii})$
  - 현재 state에 머무를 확률은 $2/3 = 1- \lambda\Delta t = p_{ii}$
  - 현재 state에 정확히 m steps (seconds)동안 머무를 확률은 $(1-p_{ii})\ p_{ii}^{m}$
  - 이미 m steps동안 있었는데 정확히 m+n steps 있을 확률은
    = Pr{m+n steps | m steps} = Pr{m+n steps & m steps}/Pr{m steps}
    = Pr{m+n steps}/Pr{m steps} $= (1-p_{ii})p_{ii}^{m+n}/p_{ii}^{m} = (1-p_{ii})p_{ii}^{n}$
    = 현재 state에 앞으로 m steps (seconds)동안 머무를 확률

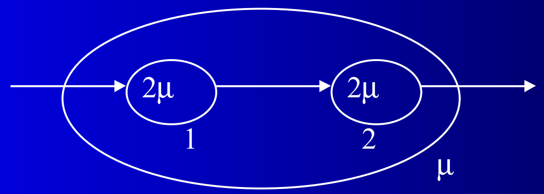
Ch.31 1/2 Markovian Queues

- Exponential service time distribution
  - $B(x) = 1 - e^{-\mu x}$, $b(x) = \mu e^{-\mu x}$ ==> COV = 1

    (steady-state solution's average = standard deviation = $1/\mu$)
  - This is not always the case with regard to service times
  - If more general distribution is used, we need more complicated solutions
- Erlangian Distribution E_r (by Erlang)
  - Decomposing the service time distribution into a collection of structured exponential distributions

Method of Stages - Erlang

- Service facility with stages
  - only one customer can exist in the service facility
  - a customer stays in each stage with exponential distribution
  - there is no queueing between stages



- How long will it stay in the current state ?
  - $X = X_1 + X_2$, where X_i's distribution is exponential (mean = $1/\mu$)
  - $f_X(x) = f_{X1}(x1) \otimes f_{X2}(x2)$: convolution

    => $b(x) = 2\mu(2\mu x)e^{-2\mu x}$

    => steady-state solution's average = $1/\mu$, standard deviation = $1/\sqrt{2}\mu$

r-stage Erlangian Distribution

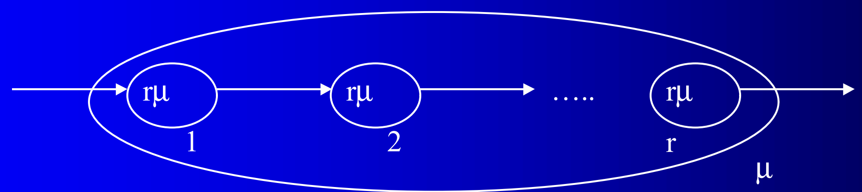
- r-stage exponential server facility
  - $b(x) = r\mu(r\mu x)^{r-1}e^{-r\mu x}/(r-1)!$

    => steady-state : average = $1/\mu$

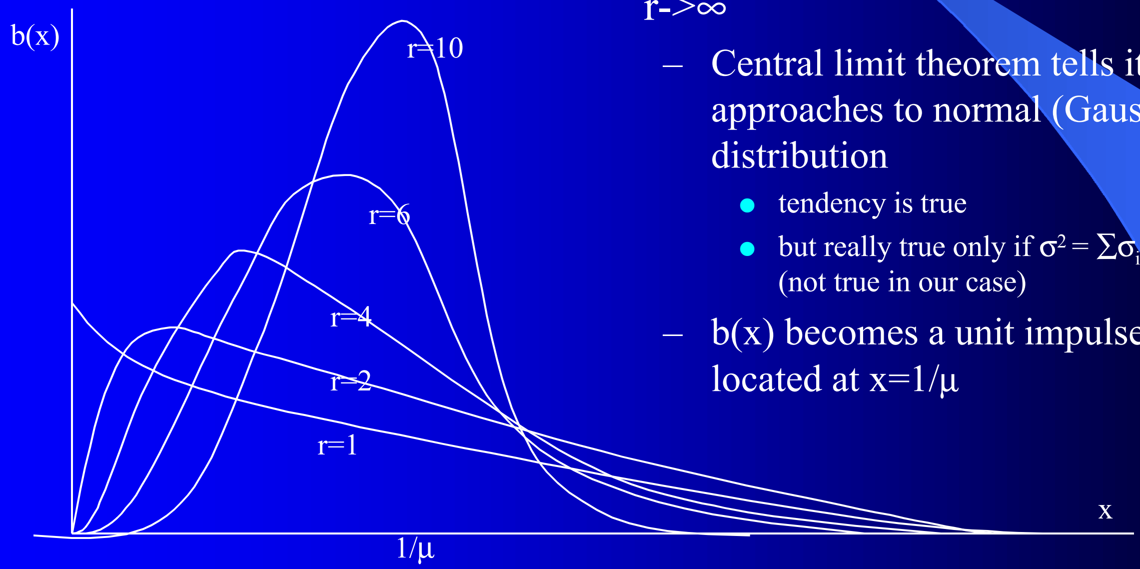
    standard deviation = $1/\sqrt{r}\mu$

    COV = $1/\sqrt{r}$

  - Less variance than exponential (single-stage Erlang) distribution (more centered around the mean value $1/\mu$)



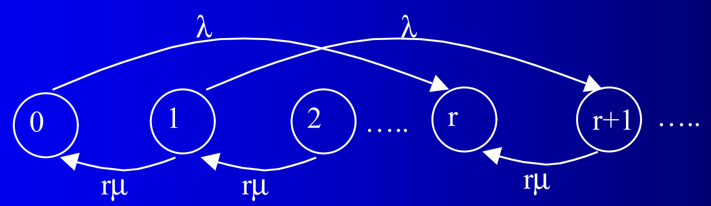
Limit Case



- r-stage Erlangian distribution when r->∞
  - Central limit theorem tells it approaches to normal (Gaussian) distribution
    - tendency is true
    - but really true only if $\sigma^2 = \sum \sigma_i^2$ -> ∞ (not true in our case)
  - b(x) becomes a unit impulse located at x=1/μ

Queue $M/E_r/1$

- Distributions:
  - Exponential interarrival time distribution: $a(t)= \lambda e^{-\lambda t}$
  - Erlangian service time distribution: $b(x) = r\mu(r\mu x)^{r-1}e^{-r\mu x}/(r-1)!$
- State-transition-rate diagram
  - the system has k customers and i-th stage contains the customer in service
  - define j as "number of stages left in total system": $j = rk-i+1$
  - $p_k = \Pr\{k \text{ customers in the system}\} = \Pr\{rk-r+1 \le j \le rk\} = \sum P_j$



Bulk Arrival Systems

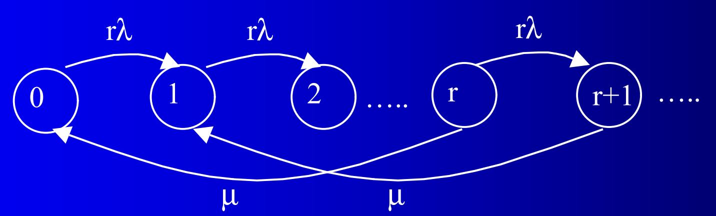
- $M/E_r/1$ = M/M/1 with bulk arrivals of size r
  - customers arrive in size r
  - State in the state-transition-rate diagram becomes # customers
  - transition 0->r with rate λ

- Generalization
  - not a fixed bulk size
  - g_i = Pr{bulk size is i}
  - $\sum g_i = 1$
  - State-transition-rate diagram ?

Queue $E_r/M/1$

- Distributions:
  - Erlangian interarrival time distribution: $a(t) = r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}/(r-1)!$
  - Exponential service time distribution: $b(x) = \mu e^{-\mu x}$

- State-transition-rate diagram
  - the system has k customers and the customer is in the i-th arrival stage
  - define j as "# completed stages of arrival in the system": $j = rk+i-1$
  - p_k = Pr{k customers in the system} = $\Pr\{rk+r-1 \le j \le rk\} = \sum P_j$



Bulk Service Systems

- E_r/M/1 = M/M/1 which provides service to group of size r
  - server accepts bulk of exactly r customers and serve at once
  - State in the state-transition-rate diagram becomes # customers
  - transition r->0 with rate μ
  - transition (r-1)->0 is not possible : if <r, server does not start service
- Generalization
  - not a fixed bulk size
  - server starts service whenever there are customers
  - State-transition-rate diagram ?

Parallel Stages

- When to apply the "Erlangian" distribution ?
  - Service time distribution's COV=1 : exponential distribution (M)
  - Service time distribution's COV<1 : Erlangian distribution (E_r)
  - Service time distribution's COV>1 : hyperexponential distribution (H_R)

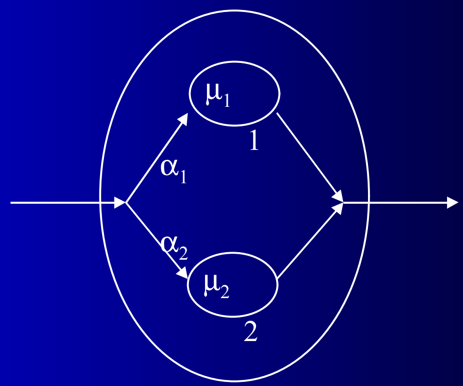
- 2-stage parallel server facility
  - $b(x) = \alpha_1\mu_1 e^{-\mu_1 x} + \alpha_2\mu_2 e^{-\mu_2 x}$,
    where $\alpha_1 + \alpha_2 = 1$

  => steady-state :

  average = $\alpha_1/\mu_1 + \alpha_2/\mu_2$

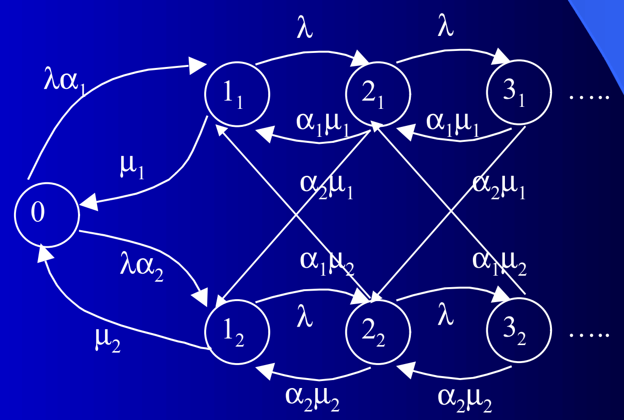
  variance = $2(\alpha_1/\mu_1^2 + \alpha_2/\mu_2^2)$

  COV > 1



$M/H_R/1$, $H_R/M/1$, $H_R/H_R/1$

- State-transition-rate diagram for $M/H_2/1$
  - 2-stage parallel server (hyperexponential distribution)
  - k_i : the system contains k customers and
    that the customer in service is located in stage I



- Generalization:
  - Series-Parallel Stages