EEC 686/785
Modeling & Performance Evaluation of Computer Systems

Lecture 12

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(based on Dr. Raj Jain's lecture notes)

Outline

- Review of lecture 11
- $2^k r$ Factorial Designs with Replication

Terminology

- **Response variables**: outcome
- **Factors**: variables that affect the response variable
- **Levels**: the values that a factor can assume
- **Primary factors** and **secondary factors**
- **Replication**: repetition of all or some experiments
- **Design**: the number of experiments, the factor level and number of replications for each experiment
- **Experimental Unit**
- **Interaction**: effect of one factor depends upon the level of the other

Types of Experimental Designs

- **Simple Designs**: vary one factor at a time
- **Full Factorial Design**: all combinations
- **Fractional Factorial Designs**: use only a fraction of the full factorial design

2^k Factorial Designs

- *k* factors, each at two levels
- Easy to analyze
- Helps in sorting out impact of factors
- Good at the beginning of a study
- Valid only if the effect of a factor is unidirectional, i.e., the performance either continuously decreases or continuously increases as the factor is increased from min to max
  - E.g., memory size, the number of disk drives

2^2 Factorial Designs: Model

- $y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$

  $15 = q_0 - q_A - q_B + q_{AB}$

  $45 = q_0 + q_A - q_B + q_{AB}$

  $25 = q_0 - q_A + q_B - q_{AB}$

  $75 = q_0 + q_A + q_B + q_{AB}$
- Unique solution for q_A and q_B:
  $y = 40 + 20x_A + 10x_B + 5x_A x_B$
- Interpretation:
  - Mean performance = 40 MIPS
  - Effect of memory = 20 MIPS
  - Effect of cache = 10 MIPS
  - Interaction between memory and cache = 5 MIPS

Performance in MIPS

Cache	Memory Size	
Size	4M Bytes	16M Bytes
1K	15	45
2K	25	75

$$x_A = \begin{cases} -1 & \text{if 4M bytes memory} \\ 1 & \text{if 16M bytes memory} \end{cases}$$

$$x_B = \begin{cases} -1 & \text{if 1K bytes cache} \\ 1 & \text{if 2K bytes cache} \end{cases}$$

Allocation of Variation

- Importance of a factor = proportion of the *variation* explained
- Sample Variance of $y = s_y^2 = \frac{\sum_{i=1}^{2^2}(y_i - \bar{y})^2}{2^2 - 1}$
- Variation of $y \triangleq Numerator$

$$= \sum_{i=1}^{2^2}(y_i - \bar{y})^2$$

$$= sum\ of\ \ squares\ total\ (SST)$$

- For a 2^2 design $SST = 2^2 q_A^2 + 2^2 q_B^2 + 2^2 q_{AB}^2$
- Variation due to A=SSA= $2^2 q_A^2$, etc.

Derivation

- Model:

$$y_i = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

- Notice
  - The sum of entries in each column is zero
  - The sum of the squares of entries in each column is 4
  - **The columns are orthogonal** (inner product of any two columns is zero):

$$\sum_{i=1}^{4} x_{Ai} x_{Bi} = 0; \sum_{i=1}^{4} x_{Ai}(x_{Ai} x_{Bi}) = 0; \sum_{i=1}^{4} x_{Bi}(x_{Ai} x_{Bi}) = 0$$

Derivation

- Variation of *y*

$$
\begin{aligned}
&= \sum_{i=1}^{4} (y_i - \bar{y})^2 \\
&= \sum_{i=1}^{4} (q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi})^2 \\
&= \sum_{i=1}^{4} (q_A x_{Ai})^2 + \sum_{i=1}^{4} (q_B x_{Bi})^2 + \sum_{i=1}^{4} (q_{AB} x_{Ai} x_{Bi})^2 + \textit{product terms} \\
&= q_A^2 \sum_{i=1}^{4} (x_{Ai})^2 + q_B^2 \sum_{i=1}^{4} (x_{Bi})^2 + q_{AB}^2 \sum_{i=1}^{4} (x_{Ai} x_{Bi})^2 + 0 \\
&= 4q_A^2 + 4q_B^2 + 4q_{AB}^2
\end{aligned}
$$

$2^k r$ Factorial Designs with Replication

- *r* replications of 2^k experiments
  - $2^k r$ observations
  - Allows estimation of experimental errors
- Model: $y = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$
  - e = experimental error

Computation of Effects

- Memory-Cache Study: Simply use means of r measurements

I	A	B	AB	y	Mean $\bar{y}$
1	−1	−1	1	(15, 18, 12)	15
1	1	−1	−1	(45, 48, 51)	48
1	−1	1	−1	(25, 28, 19)	24
1	1	1	1	(75, 75, 81)	77
164	86	38	20		total
41	21.5	9.5	5		total/4

  - Effects: $q_0 = 41$, $q_A = 21.5$, $q_B = 9.5$, $q_{AB} = 5$

Estimation of Experimental Errors

- Estimated Response:

$$\hat{y}_i = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

- Experimental error = estimated – measured

$$e_{ij} = y_{ij} - \hat{y}_i$$

$$= y_{ij} - q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi}$$

$$\sum_{i,j} e_{ij} = 0$$

- Sum of squared errors: $SSE = \sum_{i=1}^{2^2 r} e_i^2$

Experimental Errors: Example

- Estimated Response:

  $\hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15$

- Experimental errors:

$$e_{11} = y_{11} - \hat{y}_1 = 15 - 15 = 0$$

	Effect				Estimated	Measured					
i	I	A	B	AB	Response	Responses			Errors		
	41	21.5	9.5	5	$\hat{y}_i$	y_{i1}	y_{i2}	y_{i3}	e_{i1}	e_{i2}	e_{i3}
1	1	−1	−1	1	15	15	18	12	0	3	-3
2	1	1	−1	−1	48	45	48	51	-3	0	3
3	1	−1	1	−1	24	25	28	19	1	4	-5
4	1	1	1	1	77	75	75	81	-2	-2	4

$$\text{SSE} = 0^2 + 3^2 + (-3)^2 + (-3)^2 + \cdots + 4^2 = 102$$

Allocation of Variation

- $\bar{y}_{..}$ denotes the mean of responses from all replications of all experiments
  - The dots in the subscript indicate the dimension along which the averaging is done
  - $\bar{y}_{.i}$ denotes the means of responses in all replications of the ith experiment
- Total variation or **total sum of squares** (SST):

$$SST = \sum_{ij} (y_{ij} - \bar{y}_{..})^2$$

$$y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + e_{ij}$$

$$\sum_{ij} (y_{ij} - \bar{y}_{..})^2 = 2^2 r q_A^2 + 2^2 r q_B^2 + 2^2 r q_{AB}^2 + \sum_{i,j} e_{ij}^2$$

$$SST = SSA + SSB + SSAB + SSE$$

Derivation

- Model: $y_{ij} = q_0 + q_A x_{Ai} + q_B x_{Bi} + q_{AB} x_{Ai} x_{Bi} + e_{ij}$

$$\sum_{i,j} y_{ij} = \sum_{i,j} q_0 + \sum_{i,j} q_A x_{Ai} + \sum_{i,j} q_B x_{Bi} + \sum_{i,j} q_{AB} x_{Ai} x_{Bi} + \sum_{i,j} e_{ij}$$

- Since x's, their products, and all errors add to 0

$$\sum_{i,j} y_{ij} = \sum_{i,j} q_0 = 2^2 r q_0$$

- Mean response:

$$\bar{y}_{..} = \frac{1}{2^2 r} \sum_{i,j} y_{ij} = q_0$$

- Squaring both sides of the model and ignoring cross product terms:

$$\sum_{i,j} y_{ij}^2 = \sum_{i,j} q_0^2 + \sum_{i,j} q_A^2 x_{Ai}^2 + \sum_{i,j} q_B^2 x_{Bi}^2 + \sum_{i,j} q_{AB}^2 x_{Ai}^2 x_{Bi}^2 + \sum_{i,j} e_{ij}^2$$

$$SSY = SS0 + SSA + SSB + SSAB + SSE$$

Derivation

- Total variation:

$$\begin{aligned} SST &= \sum_{i,j} (y_{ij} - \bar{y}_{..})^2 \\ &= \sum_{i,j} y_{ij}^2 - \sum_{i,j} \bar{y}_{..}^2 \\ &= SSY - SS0 \\ &= SSA + SSB + SSAB + SSE \end{aligned}$$

- SSE $SSE = SSY - 2^2 r (q_0^2 + q_A^2 + q_B^2 + q_{AB}^2)$

Example: Memory-Cache Study

$$SSY = 15^2 + 18^2 + 12^2 + 45^2 + \cdots + 75^2 + 75^2 + 81^2 = 27204$$
$$SS0 = 2^2 r q_0^2 = 12 \times 41^2 = 20172$$
$$SSA = 2^2 r q_A^2 = 12 \times (21.5)^2 = 5547$$
$$SSB = 2^2 r q_B^2 = 12 \times (9.5)^2 = 1083$$
$$SSAB = 2^2 r q_{AB}^2 = 12 \times 5^2 = 300$$
$$SSE = 27204 - 2^2 \times 3(41^2 + 21.5^2 + 9.5^2 + 5^2) = 102$$
$$SST = SSY - SS0 = 27204 - 20172 = 7032$$

- SSA+SSB+SSAB+SSE = 5547+1083+300+102 = 7032 = SST

Example: Memory-Cache Study

- Factor A explains 5547/7032 or 78.88%
- Factor B explains 15.40%
- Interaction AB explains 4.27%
- 1.45% is unexplained and is attributed to errors

Confidence Intervals for Effects

- Effects are random variables
- Errors $\sim N(0, \sigma_e) \Rightarrow y \sim N(\bar{y}_{..}, \sigma_e)$

$$q_0 = \frac{1}{2^2 r} \sum_{i,j} y_{ij}$$

- q_0 = Linear combination of normal variates => q_0 is normal with variance $\sigma_e^2 / (2^2 r)$

Confidence Intervals for Effects

- Variance of errors:

$$s_e^2 = \frac{1}{2^2 (r-1)} \sum_{ij} e_e^2 = \frac{SSE}{2^2 (r-1)} \underline{\Delta} MSE$$

  - **Mean Square of Errors (MSE)**: quantity at the right hand side
  - Denominator = $2^2(r\text{-}1)$ = # of independent terms in SSE => degrees of freedom SSE has

Confidence Intervals for Effects

- Estimated variance of q_0: $s_{q_0}^2 = s_e^2 / (2^2 r)$
- Similarly, $s_{q_A} = s_{q_B} = s_{q_{AB}} = s_e \big/ \sqrt{2^2 r}$
- Confidence intervals (CI) for the effects:

$$q_i \mp t_{[1-\alpha/2; 2^2(r-1)]} s_{q_i}$$

- CI does not include a zero => significant

Example

- For memory-cache study:
- Standard deviation of errors:

$$s_e = \sqrt{\frac{SSE}{2^2(r-1)}} = \sqrt{\frac{102}{8}} = \sqrt{12.75} = 3.57$$

- Standard deviation of effects:

$$s_{q_i} = s_e \Big/ \sqrt{(2^2 r)} = 3.57 \Big/ \sqrt{12} = 1.03$$

- For 90% confidence: $t_{[0.95,8]} = 1.86$

Example

- Confidence intervals:
  - $q_i \pm (1.86)(1.03) = q_i \pm 1.92$
  - $q_0 = (39.08, 42.91)$
  - $q_A = (19.58, 23.41)$
  - $q_B = (7.58, 11.41)$
  - $q_{AB} = (3.08, 6.91)$
  - No zero crossing => all effects are significant

Confidence Intervals for Contrasts

- It is also possible to compute variance and confidence intervals for any contrast of effects.
- A **contrast** is any linear combination whose coefficients add up to zero
- Variance of $\sum h_i q_i$, where h_i=0, is:

$$s^2_{\sum h_i q_i} = \frac{s_e^2 \sum h_i^2}{2^2 r}$$

- For 100(1-α)% confidence interval, use $t_{[1-\alpha/2;2^2(r-1)]}$

Example 18.5

- Memory-cache study: the confidence interval for $u = q_A + q_B - 2q_{AB}$ is calculated as follows:
  - Coefficients = 0, 1, 1, and -2 => Contrast
  - Mean $\bar{u} = 21.5 + 9.5 - 2 \times 5 = 11$
  - Variance $s_u^2 = \frac{s_e^2 \times 6}{2^2 \times 3} = 6.375$
  - Standard deviation $s_u = \sqrt{6.375} = 2.52$
  - $t_{[0.95;8]}$=1.86
  - 90% confidence interval for u:

    $\bar{u} \mp t s_u = 11 \mp 1.86 \times 2.52 = (6.31, 15.69)$

CI for Predicted Responses

- Mean response $\hat{y} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B$
- The standard deviation of the mean of *m* responses:

$$s_{\hat{y}_m} = s_e \left(\frac{1}{n_{eff}} + \frac{1}{m} \right)^{1/2}$$

$$n_{eff} = \text{effective degree of freedom}$$

$$= \frac{\text{total number of runs}}{1 + \text{sum of DFs of params used in } \hat{y}}$$

$$= \frac{2^2 r}{5}$$

CI for Predicted Responses

- 100(1-α)% confidence interval:

$$\hat{y} \mp t_{[1-\alpha/2;2^2(r-1)]} s_{\hat{y}_m}$$

- A single run (m=1): $s_{\hat{y}_1} = s_e\left(\frac{5}{2^2 r}+1\right)^{1/2}$
- Population mean (m=∞): $s_{\hat{y}} = s_e\left(\frac{5}{2^2 r}\right)^{1/2}$

Example: Memory-Cache Study

- For x_A = -1 and x_B = -1:
  - **A single confirmation experiment**:

    $\hat{y}_1 = q_0 - q_A - q_B + q_{AB} = 41 - 21.5 - 9.5 + 5 = 15$

    Standard deviation of the predication:

    $s_{\hat{y}_1} = s_e\left(\frac{5}{2^2 r}+1\right)^{1/2} = 3.57\sqrt{\frac{5}{12}+1} = 4.25$

    Using $t_{[0.95;8]}$ = 1.86, the 90% confidence interval is:
    15±1.86×4.25=(8.09, 22.91)

Example: Memory-Cache Study

- For x_A = -1 and x_B = -1 (continued):
  - **Mean response for 5 experiments in future**:

$$s_{\hat{y}_1} = s_e \left(\frac{5}{2^2 r} + \frac{1}{m} \right)^{1/2} = 3.57 \sqrt{\frac{5}{12} + \frac{1}{5}} = 2.80$$

The 90% confidence interval is:
15±1.86×2.80=(9.79, 20.29)

Example: Memory-Cache Study

- For x_A = -1 and x_B = -1 (continued):
  - **Mean response for a large number of experiments** in future:

$$s_{\hat{y}} = s_e \left(\frac{5}{2^2 r} \right)^{1/2} = 3.57 \sqrt{\frac{5}{12}} = 2.30$$

  - The 90% confidence interval is:
15±1.86×2.30=(10.72, 19.28)

Example: Memory-Cache Study

- For x_A = -1 and x_B = -1 (continued):
  - **Current mean response**: not for future (use the formula for contrasts):

$$s_{\hat{y}} = \sqrt{\frac{s_e \sum h_i^2}{2^2 r}} = \sqrt{\frac{12.75 \times 4}{12}} = 2.06$$

  The 90% confidence interval is:
  15±1.86×2.06=(11.17, 18.83)
  - Notice: Confidence intervals become narrower

Visual Tests for Verifying the Assumptions

- In deriving the expressions for effects, we made essentially the same assumptions as in regression analysis:
  - Errors are statistically independent
  - Errors are additive
  - Errors are normally distributed
  - Errors have a constant standard deviation
  - Effects of factors are additive

  => Observations are independent and normally distributed with constant variance

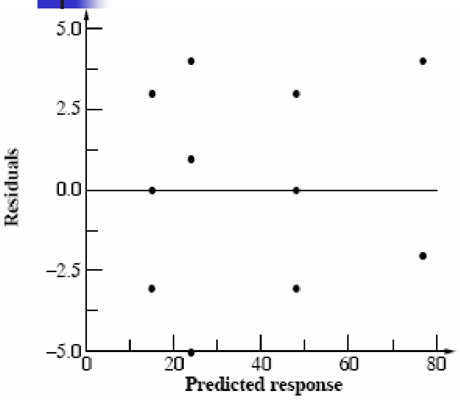
Visual Tests for Verifying the Assumptions

- Visual tests for independent errors:
  - Scatter plot of residuals versus the predicted response
  - Magnitude of residuals < magnitude of responses/10 => ignore trends
  - Plot the residuals as a function of the experiment number
  - Trend up or down => other factors or side effects
- Visual tests for Normally distributed errors:
  - Prepare normal quantile-quantile plot of errors
  - If the plot is approximately linear, the assumption is satisfied

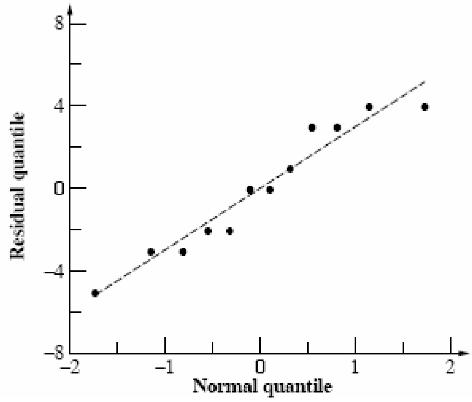
Visual Tests for Verifying the Assumptions

- Visual tests for constant standard deviation of errors:
  - Scatter plot of y for various levels of the factor
  - Spread at one level significantly different than that at other => Assumption of constant variance is not valid, need transformation

Visual Tests Example: Memory-Cache



Residues an order of magnitude smaller than the responses



Residuals appear to be approximately normally distributed

Multiplicative Models for 2^2r Experiments

- **Additive model**:

$$y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$$

  - Not valid if effects do not add
- E.g., **execution time of workloads**
  ith processor speed = v_i instructions/second
  jth workload size = w_j instructions
  Execution time $y_{ij} = v_i \times w_j$
  The two effects **multiply**. Logarithm => additive model:

$$\log(y_{ij}) = \log(v_i) + \log(w_j)$$

Multiplicative Models for 2^2r Experiments

- **Correct model**: $y'_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e_{ij}$

  where, $y'_{ij} = \log(y_{ij})$
- **Taking an antilog of additive effects *q*. to get the multiplicative effects *u*.**:

  $u_A = 10^{q_A}$, $u_B = 10^{q_B}$, and $u_{AB} = 10^{q_{AB}}$
  - u_A = ratio of MIPS rating of the two processors
  - u_B = ratio of the size of the two workload
  - Antilog of additive mean q_0 => geometric mean

$$\dot{y} = 10^{q_0} = (y_1 y_2 \cdots y_n)^{1/n} \quad n = 2^2 r$$

Example: Execution Times

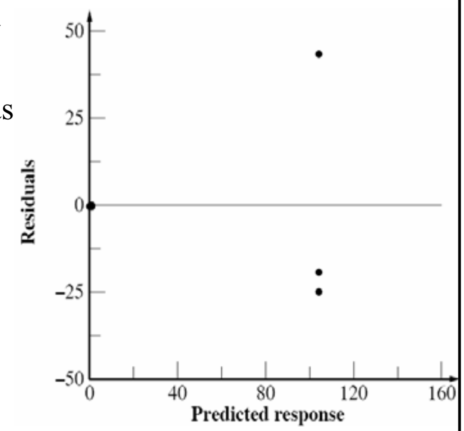
Analysis Using an Additive Model

I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	(85.10, 79.50, 147.90)	104.170
1	1	-1	-1	(0.891, 1.047, 1.072)	1.003
1	-1	1	-1	(0.955, 0.933, 1.122)	1.003
1	1	1	1	(0.0148, 0.0126, 0.0118)	0.013
106.19	-104.15	-104.15	102.17	total	
26.55	-26.04	-26.04	25.54	total/4	

- Additive model is not valid because of physical consideration
  - Effects of workload and processors do not add. They multiply
- Additive model is not valid because of large range for *y*
  - y_{max}/y_{min} =147.90/0.0118 or 12,534 => log transformation
  - Taking an arithmetic mean of 114.17 and 0.013 is inappropriate

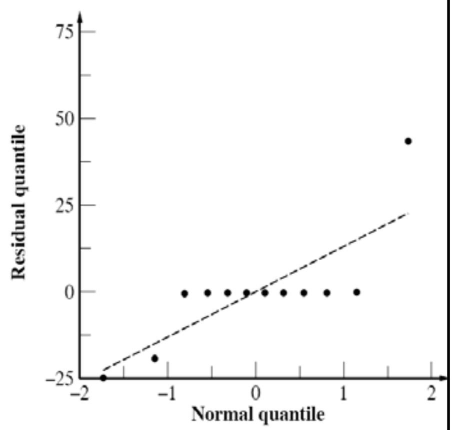
Example: Execution Times

- Additive model is not valid because:
  - The residuals are not small as compared to the response
  - The spread of residuals is large at larger value of the response => log transformation



Example: Execution Times

- Additive model is not valid because:
  - The residual distribution has a longer tail than normal



Analysis Using Multiplicative Model

- Transformed data for multiplicative model example

Data After Log Transformation

I	A	B	AB	y	Mean $\bar{y}$
1	-1	-1	1	(1.93, 1.90, 2.17)	2.00
1	1	-1	-1	(-0.05, 0.02, 0.03)	0.00
1	-1	1	-1	(-0.02, -0.03, 0.05)	0.00
1	1	1	1	(-1.83, -1.90, -1.93)	-1.89
0.11	-3.89	-3.89	0.11	total	
0.03	-0.97	-0.97	0.03	total/4	

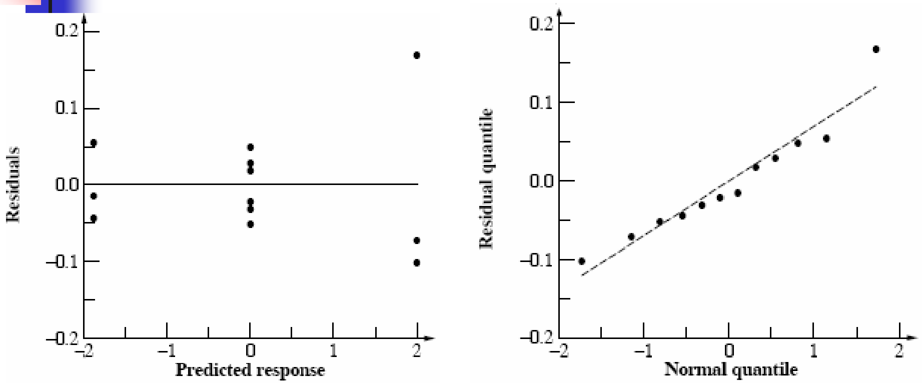
Analysis Using Multiplicative Model

	Additive Model			Multiplicative Model		
Factor	Effect	% Var.	Conf. Interval	Effect	% Var.	Conf. Interval
I	26.55		(16.35, 36.74)	0.03		(-0.02, 0.07)†
A	-26.04	30.1%	(-36.23, -15.84)	-0.97	49.9%	(-1.02, -0.93)
B	-26.04	30.1%	(-36.23, -15.84)	-0.97	49.9%	(-1.02, -0.93)
AB	25.54	29.0%	(15.35, 35.74)	0.03	0.0%	(-0.02, 0.07)†
e		10.8%			0.2%	

† ⇒ Not Significant

- Percentage of variation explained by the two models
- With multiplicative model
  - Interaction is almost zero
  - Unexplained variation is only 0.2%

Using Multiplicative: Visual Tests



- Conclusion: multiplicative model is better than the additive model

Using Multiplicative: Interpretation of Results

$$\log(y) = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + e$$

$$y = 10^{q_0} 10^{q_A x_A} 10^{q_B x_B} 10^{q_{AB} x_A x_B} 10^{e}$$

$$= 10^{0.03} 10^{-0.97 x_A} 10^{-0.97 x_B} 10^{0.03 x_A x_B} 10^{e}$$

$$= 1.07 \times 0.107^{x_A} \times 0.107^{x_B} \times 1.07^{x_A x_B} 10^{e}$$

- The time for an average processor on an average benchmark is 1.07
- The time on processor A_1 is 9 times (0.107^{-1}) that on an average processor. The time on A_2 is one ninth (0.107^{1}) of that on an average processor

Using Multiplicative: Interpretation of Results

- MIPS rate for A_2 is 81 times that of A_1
- Benchmark B_1 executes 81 times more instructions than B_2
- The interaction is negligible
  => Results apply to all benchmarks and processors

Transformation Considerations

- y_{max}/y_{min} small => multiplicative model results similar to additive model
- Many other transformation possible

Transformation Considerations

- **Box-Cox family of transformations**:

$$w = \begin{cases} \dfrac{y^a - 1}{a g^{a-1}}, & a \neq 0 \\ (\ln y) g, & a = 0 \end{cases}$$

Where g is the geometric mean of the responses:

$$g = (y_1 y_2 \cdots y_n)^{1/n}$$

  - w has the same units as y
  - a can have any real value, positive, negative, or zero
  - Plot SSE as a function of a => optimal a
  - Knowledge about the system behavior should always take precedence over statistical considerations

General $2^k r$ Factorial Design

- Model: $y_{ij} = q_0 + q_A x_A + q_B x_B + q_{AB} x_A x_B + \cdots + e_{ij}$
- Parameter estimation: $q_j = \dfrac{1}{2^k} \sum_{i=1}^{2^k} S_{ij} \bar{y}_i$

  S_{ij} = (i, j)th entry in the sign table
- Sum of squares: $SSY = \sum_{i=1}^{2^k} \sum_{j=1}^{r} y_{ij}^2$

$$SS0 = 2^k r q_0^2$$

$$SST = SSY - SS0$$

$$SSj = 2^k r q_j^2 - \sum_{j=1}^{2^k - 1} SSj$$

General $2^k r$ Factorial Design

- Percentage of y's variation explained by jth effect = (SSj/SST) × 100%
- Standard deviation of errors: $s_e = \sqrt{\dfrac{SSE}{2^k(r-1)}}$
- Standard deviation of effects:
  $$s_{q_0} = s_{q_A} = s_{q_B} = s_{q_{AB}} = s_e \big/ \sqrt{2^k r}$$
- Variance of contrast $\sum h_i q_i$, where h_i=0, is: $s^2_{\sum h_i q_i} = \dfrac{s_e^2 \sum h_i^2}{2^2 r}$
- Standard deviation of the mean of m future responses:
  $$s_{\hat{y}_m} = s_e \left(\frac{1+2^k}{2^k r} + \frac{1}{m} \right)^{1/2}$$

General $2^k r$ Factorial Design

- Confidence intervals are calculated using $t_{[1-\alpha/2;2^k(r-1)]}$
- Modeling assumptions
  - Errors are IID (independently and identically distributed) normal variates with zero mean
  - Errors have the same variance for all values of the predictors
  - Effects and errors are additive

General $2^k r$ Factorial Design

- Visual tests
  - The scatter plot of errors versus predicted responses should not have any trend
  - The normal quantile-quantile plot of errors should be linear
  - Spread of *y* values in all experiments should be comparable
  - If any of the above visual tests fail or if the ratio y_{max}/y_{min} is large, a multiplicative model should be investigated

Example: A $2^3 3$ Design

I	A	B	C	AB	AC	BC	ABC	y	Mean $\bar{y}$
1	−1	−1	−1	1	1	1	−1	(14, 16, 12)	14
1	1	−1	−1	−1	−1	1	1	(22, 18, 20)	20
1	−1	1	−1	−1	1	−1	1	(11, 15, 19)	15
1	1	1	−1	1	−1	−1	−1	(34, 30, 35)	33
1	−1	−1	1	1	−1	−1	1	(46, 42, 44)	44
1	1	−1	1	−1	1	−1	−1	(58, 62, 60)	60
1	−1	1	1	−1	−1	1	−1	(50, 55, 54)	53
1	1	1	1	1	1	1	1	(86, 80, 74)	80
319	67	43	155	23	19	15	−1	total	
39.87	8.375	5.375	19.37	2.875	2.375	1.875	-0.125	total/8	

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Example: A $2^3 3$ Design

- Sums of squares

Component	Sum of Squares	Percent Variation
y	4.9E4	
$\bar{y}$	3.8E4	
$y - \bar{y}$	1.1E4	100.00%
A	1683.0	14.06%
B	693.3	5.79%
C	9009.0	75.27%
AB	198.3	1.66%
AC	135.4	1.13%
BC	84.4	0.70%
ABC	0.4	0.00%
Errors	164.0	1.37%

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Example: A $2^3 3$ Design

- Confidence intervals of effects
  - The errors have $2^3(3-1)$ or 16 degrees of freedom. Standard deviation of errors:

$$s_e = \sqrt{\frac{SSE}{2^k(r-1)}} = \sqrt{\frac{164}{16}} = 3.20$$

  - Standard deviation of effects:

$$s_{q_i} = s_e \big/ \sqrt{2^3 3} = 3.20 \big/ \sqrt{24} = 0.654$$

  - $t_{[0.95,16]} = 1.337$

Example: A $2^3 3$ Design

- 90% confidence intervals for parameters:
  $q_i \pm (1.337)(0.654) = q_i \pm 0.874$
  - q_0 = (39.00, 40.74)
  - q_A = (7.50, 9.25)
  - q_B = (4.50, 6.25)
  - q_C = (18.50, 20.24)
  - q_{AB} = (2.00, 3.75)
  - q_{AC} = (1.50, 3.25)
  - q_{BC} = (1.00, 2.75)
  - q_{ABC} = (-1.00, 0.75)

  All effects except q_{ABC} are significant

Example: A $2^3 3$ Design

- **Predication**
  - For a single confirmation experiment (m=1)
  - With A = B = C = -1:

$$\hat{y} = 14$$

$$s_{\hat{y}} = s_e\left(\frac{5}{2^k r} + \frac{1}{m}\right)^{1/2} = 3.2\left(\frac{5}{24} + 1\right)^{1/2} = 3.52$$

  - 90% confidence interval:
    $14 \pm 1.337 \times 3.52 = 14 \pm 4.70 = (9.30, 18.70)$

Case Study: Garbage Collection

Factors and Levels

Variable	Factor	-1	1
A	Workload	Single Task	Several parallel tasks
B	Compiler	Simple	Deallocating
C	Limbo List	Enabled	Disabled
D	Chunk Size	4K bytes	16K bytes

Case Study: Garbage Collection

- Measured Data

I	A	B	C	D	y	Mean $\bar{y}$
1	-1	-1	-1	-1	(97, 97, 97)	97.00
1	1	-1	-1	-1	(31, 31, 32)	31.33
1	-1	1	-1	-1	(97, 97, 97)	97.00
1	1	1	-1	-1	(31, 32, 31)	31.33
1	-1	-1	1	-1	(97, 97, 97)	97.00
1	1	-1	1	-1	(32, 32, 31)	31.67
1	-1	1	1	-1	(97, 97, 97)	97.00
1	1	1	1	-1	(32, 32, 32)	32.00
1	-1	-1	-1	1	(407, 407, 407)	407.00
1	1	-1	-1	1	(135, 136, 135)	135.33
1	-1	1	-1	1	(409, 409, 409)	409.00
1	1	1	-1	1	(135, 135, 136)	135.33
1	-1	-1	1	1	(407, 407, 407)	407.00
1	1	-1	1	1	(139, 140, 139)	139.33
1	-1	1	1	1	(409, 409, 409)	409.00
1	1	1	1	1	(139, 139, 140)	139.33
2695.67	-1344.33	4.33	9.00	1667.00		total
168.48	-84.02	0.27	0.56	104.19		total/8

Case Study: Garbage Collection

- Effects and variation explained

Factor	Effect	% Variation	Conf. Interval
I	168.48	138.1%	(168.386, 168.573)
A	-84.02	34.4%	(-84.114, -83.927)
B	0.27	0.0%	(0.177, 0.364)
C	0.56	0.0%	(0.469, 0.656)
D	104.19	52.8%	(104.094, 104.281)
AB	-0.23	0.0%	(-0.323, -0.136)
AC	0.56	0.0%	(0.469, 0.656)
AD	-51.31	12.8%	(-51.406, -51.219)
BC	0.02	0.0%	(-0.073, 0.114)†
BD	0.23	0.0%	(0.136, 0.323)
CD	0.44	0.0%	(0.344, 0.531)
ABC	0.02	0.0%	(-0.073, 0.114)†
ABD	-0.27	0.0%	(-0.364, -0.177)
ACD	0.44	0.0%	(0.344, 0.531)
BCD	-0.02	0.0%	(-0.114, 0.073)†
$ABCD$	-0.02	0.0%	(-0.114, 0.073)†

† ⇒ Not Significant

Case Study: Garbage Collection

- Conclusions
  - Most of the variation is explained by factors A (workload), D (chunk size), and the interaction AD between the two
  - The variation due to experimental error is small => Several effects that explain less than 0.05% of variation (listed as '0.0%') are statistically significantly
  - Only effects A, D, and AD are **both practically significant and statistically significant**