EEC 686/785
Modeling & Performance Evaluation of Computer Systems

Lecture 18

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(based on Dr. Raj Jain's lecture notes)

Outline

- Review of lecture 17
- Random-Variate Generation
- Commonly Used Distributions

Chi-Square Test

- Most commonly used test. Can be used for any distribution
- Prepare a histogram of the observed data
- Compare observed frequencies with theoretical D=…
- D=0 => exact fit
- D has a chi-square distribution with $k - 1$ degrees of freedom => compare D with $\chi^2_{[1-\alpha;k-1]}$
  Pass with confidence α if D is less

Kolmogorov-Smirnov Test

- Designed for continuous distributions
- Difference between the observed CDF (cumulative distribution function) $F_o(x)$ and the expected cdf $F_e(x)$ should be small
- K^+ = maximum observed deviation above the expected cdf
  K^- = maximum observed deviation below the expected cdf

  $$K^+ = \sqrt{n}\max_x(F_o(x) - F_e(x)) \quad K^- = \sqrt{n}\max_x(F_e(x) - F_o(x))$$
- $K^+ < K_{[1-\alpha;n]}$ and $K^- < K_{[1-\alpha;n]}$ => pass at α level of significance
- For $U(0, 1)$: $F_e(x) = x$, $F_o(x) = j/n$,
  where $x > x_1, x_2, \ldots, x_{j-1}$ $\quad K^+ = \sqrt{n}\max_j(\frac{j}{n} - x_j) \quad K^- = \sqrt{n}\max_j(x_j - \frac{j-1}{n})$

Chi-Squire vs. K-S Test

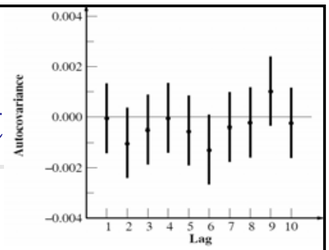
K-S Test	Chi-Square Test
Small samples	Large sample
Continuous distributions	Discrete distributions
Differences between observed and expected CDFs	Differences between observed and hypothesized probabilities (pdfs or pmfs)
Use each observation in the sample without any grouping => makes a better use of the data	Groups observations into a small number of cells
Cell size is not a problem	Cell sizes affect the conclusion but no firm guidelines
Exact	Approximate

k-Dimensional Uniformity or *k*-Distribution

- ***k*-distributed** if:

  $$P(a_1 \le u_{n-1} < b_1, ..., a_k \le u_{n+k-1} < b_k) = (b_1 - a_1) \cdots (b_k - a_k)$$

  For all choices of a_i, b_i in [0, 1), with $b_i > a_i$, i=1,2,…, k.
- k-distributed sequence is always (k – 1)-distributed. The inverse is not true
- Two tests:
  - Serial test
  - Spectral test

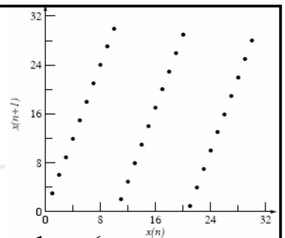


Serial-Correlation Test

- Nonzero covariance => dependence
  - The inverse is not true
- R_k = autocovariance at lag k = Cov[x_n, x_{n+k}]
- For large n, R_k is normal distributed with a mean of zero and a variance of 1/[144(n-k)]
- 100(1-α)% confidence interval for the autocovariance is

$$R_k \mp z_{1-\alpha/2}/(12\sqrt{n-k})$$

- Check if CI includes zero



Spectral Test

- **Goal**: to determine how densely the k-tuples $\{x_1, x_2, \ldots, x_n\}$ can fill up the k-dimensional hyperspace
- The k-tuples from an LCG all on a finite number of parallel hyperplanes
- Successive pairs would lie on a finite number of lines
- In three dimensions, successive triplets lie on a finite number of planes
- **Spectral test**: determine the max distance between adjacent hyperplanes
  - Larger distance => worse generator

Random-Variate Generation

- General techniques
  - Inverse Transformation
  - Rejection
  - Composition
  - Convolution
  - Characterization

Inverse Transformation

- $u = F(x) \sim U(0,1)$
  $x = F^{-1}(u)$
- Given any random variable x with a CDF F(x), the variable u=F(x) is uniformly distributed between 0 and 1 => can get x from u: $x = F^{-1}(u)$
  - $F_Y(y)=P(Y\leq y)=P(X\leq g^{-1}(\text{y}))=F_X(g^{-1}(y))$
  - Let $g(x)=F(x)$, or $y=F(x)$ => $F(y)=F(F^{-1}(y))=y$ => $f(y)=\text{d}F/\text{d}y=1$
- Used when F^{-1} can be determined either analytically or empirically

Example 28.1

- For exponential variates
- pdf $f(x) = \lambda e^{-\lambda x}$
- CDF $F(x) = 1 - e^{-\lambda x} = u \Rightarrow x = -1/\lambda \ln(1 - u)$
  $\Rightarrow x_i = -1/\lambda \ln(1 - u_i)$
  $\Rightarrow$ generate u_i and compute x_i
- U ~ U(0,1) => $1 - u \sim$ U(0,1)
  $x = -1/\lambda \ln(1 - u)$

Example 28.2

Size	Probability
64 Bytes	0.7
128 Bytes	0.1
512 Bytes	0.2

- The packet sizes (trimodal) probabilities
- The CDF for this distribution is:

$$F(x) = \begin{cases} 0.0 & 0 \le x < 64 \\ 0.7 & 64 \le x < 128 \\ 0.8 & 128 \le x < 512 \\ 1.0 & 512 \le x \end{cases}$$

- The inverse function is: $F^{-1}(u) = \begin{cases} 64 & 0 < u \le 0.7 \\ 128 & 0.7 < u \le 0.8 \\ 512 & 0.8 < u \le 1 \end{cases}$

Example 28.2

- Generate *u*~U(0,1)
  - $u \le 0.7$ => size = 64
  - $0.7 < u \le 0.8$ => size = 128
  - $0.7 < u$ => size = 512
- Note: CDF is continuous from the right
  - The value on the right of the discontinuity is used
  - The inverse function is continuous from the left
  - $u = 0.7$ => $x = 64$

Applications of the Inverse-Transformation Technique

Distribution	CDF $F(x)$	Inverse
Exponential	$1-e^{-x/a}$	$-a\ln(u)$
Extreme value	$1-e^{-e^{\frac{x-a}{b}}}$	$a+b\ln\ln u$
Geometric	$1-(1-p)^x$	$\left\lceil\frac{\ln(u)}{\ln(1-p)}\right\rceil$
Logistic	$1-\frac{1}{1+e^{\frac{x-\mu}{b}}}$	$\mu-b\ln(\frac{1}{u}-1)$
Pareto	$1-x^{-a}$	$1/u^{1/a}$
Weibull	$1-e^{(x/a)^b}$	$a(\ln u)^{1/b}$

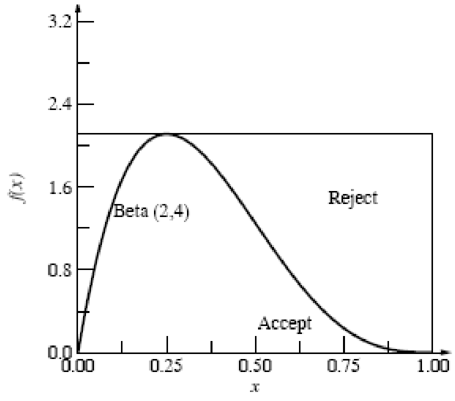
Rejection

- Can be used if a pdf $g(x)$ exists such that $cg(x)$ majorizes the pdf $f(x)$ => $cg(x) \geq f(x) \quad \forall x$
- Steps:
  - 1. Generate x with pdf $g(x)$
  - 2. Generate y uniform on $[0, cg(x)]$
  - 3. If $y \leq f(x)$, then output x and return. Otherwise, repeat from step 1

  Continue rejecting the random variates x and y until $y \leq$ f(x)

Example 28.3

- Beta(2,4) density function: $f(x) = 20x(1-x)^3 \quad 0 \leq x \leq 1$
- Bounded inside a rectangle of height 2.11 => c=2.11 and $g(x) = 0 \leq x \leq 1$
- Steps:
  - 1. Generate x unform on [0, 1]
  - 2. Generate y uniform on [0, 2.11]
  - 3. If $y \leq 20x(1-x)^3$, then output x and return. Otherwise, repeat from step 1



Example 28.3

- Steps 1 and 2 generate a point (x, y) uniformly distributed over the rectangle. If the point falls above the beta pdf, then step 3 rejects x
- Efficiency = how closely $cg(x)$ envelopes $f(x)$ Large area between $cg(x)$ and $f(x)$ => large percentage of (x, y) generated in steps 1 and 2 are rejected
- If generation of $g(x)$ is complex, this method may not be efficient

Composition

- Can be used if CDF F(x) = Weighted sum of *n* other CDFs

  $$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

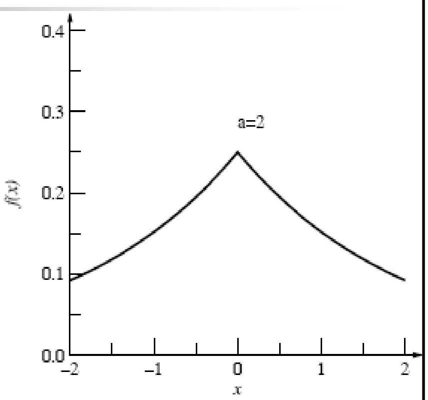
  Here, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, and F_i's are distribution functions
- *n* CDFs are composed together to form the desired CDF. Hence, the name of the technique
- The desired CDF is decomposed into several other CDFs => Also called decomposition
- Can also be used if the pdf $f(x)$ is a weighted sum of *n* other pdfs

Composition Steps

- Generate a random integer I such that $P(I=i) = p_i$
  This can easily be done using the inverse-transformation method
- Generate x with ith pdf $f_i(x)$ and return

Example 28.4

- pdf: $f(x) = 1/2a\ e^{-|x|/a}$
- With $a = 2$ => plot
- Composition of two exponential pdf's
  - 1. Generate $u_1 \sim U(0, 1)$ and $u_2 \sim U(0, 1)$
  - If $u_1 < 0.5$, return $x = -a \ln u_2$; otherwise return $x = a \ln u_2$



Convolution

- Sum of n variables: $x = y_1 + y_2 + \cdots + y_n$
- Generate n random variate y_i's and sum
- For sums of two variables, pdf of x = convolution of pdfs of y_1 and y_2. Hence the name. Although no convolution in generation
- If pdf or CDF = Sum => Composition
  Variable x = Sum => Convolution
- Example: the sum of m geometric variates is a Pascal variate

Characterization

- Use special characteristics of distributions => characterization
- Exponential interarrival times: mean $1/\lambda$
  - The number of arrivals n over a given period T has a Poisson distribution with parameter λT
  - Continuously generate exponential variates until their sum exceeds T and return the number of variates generated as the Poisson variate
- The ath small number in a sequence of $a + b + 1$ U(0,1) uniform variates has a beta(a,b) distribution

Characterization

- The ratio of two unit normal variates is a Cauchy(0,1) variate
- A chi-square variate with even degrees of freedom $\chi^2(\nu)$ is the same as a gamma variate $\gamma(2,\nu/2)$
- If x_1 and x_2 are two gamma variates $\gamma(a,b)$ and $\gamma(a,c)$, respectively, the ratio $x_1/(x_1+x_2)$ is a beta variates $\beta(b,c)$
- If x is a unit normal variate, $e^{\mu+\sigma x}$ is a lognormal (μ,σ) variate

How to Select a Random-Variate Generation Technique

Commonly Used Distributions

- Random number generation algorithms for distributions commonly used by computer systems performance analysts
- Organized alphabetically for reference
- For each distribution
  - Key characteristics
  - Algorithm for random number generation
  - Examples of applications

Bernoulli Distribution

- Takes only two values: failure and success or *x*=0 and *x*=1, respectively
- Key characteristics
  - Parameters: p = probability of success (x=1) $0 \le p \le 1$
  - Range: x=0,1
  - pmf: $$f(x) = \begin{cases} 1-p, & \text{if } x = 0 \\ p, & \text{if } x = 1 \\ 0, & \text{Otherwise} \end{cases}$$
  - Mean: p
  - Variance: $p(1-p)$

Bernoulli Distribution

- Applications: to model the probability of an outcome having a desired class or characteristics
  - A computer system is up or down
  - A packet in a computer network reaches or does not reach the destination
  - A bit in the packet is affected by noise and arrives in error
- Can be used only if trials are independent and identical
- Generation: inverse transformation
  - Generate $u \sim U(0,1)$
  - If $u \leq p$ return 0. Otherwise, return 1

Beta Distribution

- Used to represent random variates that are bounded
- Key characteristics
  - Parameters: a, b = shape parameters, $a > 0$, $b > 0$
  - Range: $0 \leq x \leq 1$
  - pdf: $f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}$

    β(.) is the beta function and is related to the gamma function as follows: $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)}$
  - Mean: $a/(a+b)$
  - Variance: $ab/\{(a+b)^2(a+b+1)\}$

Beta Distribution

- Substitute $(x - x_{min})/(x_{max} - x_{min})$ in place of x for other ranges
- Applications: to model random proportions
  - Fraction of packets requiring retransmissions
  - Fraction of remote procedure calls (RPC) taking more than a specified time

Beta Distribution

- Generation:
  - Generate two gamma variates $\gamma(1,a)$ and $\gamma(1,b)$, and take the ratio:

$$BT(a,b) = \frac{\gamma(1,a)}{\gamma(1,a) + \gamma(1,b)}$$

  - If a and b are integers
    - Generate $a + b + 1$ uniform U(0,1) random numbers
    - Return the ath smallest number as BT(a,b)

Beta Distribution

- Generation:
  - If *a* and *b* are less than one:
    - Generate two uniform U(0,1) random numbers u_1 and u_2
    - Let $x = u_1^{1/a}$ and $x = u_2^{1/b}$. If (x+y) > 1, go back to the previous step. Otherwise, return $x/(x+y)$ as BT(a,b)
  - If *a* and *b* are greater than 1: use an algorithm based on rejection techniques to generate beta variates

Binomial Distribution

- The number of successes *x* in a sequence of *n* Bernoulli trials has a binomial distribution
- Characteristics:
  - Parameters
    - p = probability of success in a trial, $0 < p < 1$
    - n = number of trials, n must be a positive integer
  - Range: $x = 0,1,\dots,n$
  - pdf: $$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$
  - Mean: np
  - Variance: $np(1-p)$

Binomial Distribution

- Applications: to model the number of successes
  - The number of processors that are up in a multiprocessor system
  - The number of packets that reach the destination without loss
  - The number of bits in a packet that are not affected by noise
  - The number of items in a batch that have certain characteristics
- Variance < mean => Binomial
  Variance > mean => Negative Binomial
  Variance = mean => Poisson

Binomial Distribution

- Generation:
  - Composition: generate n U(0,1). The number of RNs that are less than p is BN(p,n)
  - For small p:
    - Generate geometric random numbers: $G_i(p) = \left\lceil \frac{\ln(u_i)}{\ln(1-p)} \right\rceil$
    - If the sum of geometric RNs so far is less than or equal to n, go back to the previous step. Otherwise, return the number of RNs generated minus one.

    If $\sum_{i=1}^{m} G_i(p) > n,$ return $m-1$

Binomial Distribution

- Generation
  - Inverse transformation method:
    - Compute the CDF F(x) for x=0,1,2,…, n and store in an array
    - For each binomial variate, generate a U(0,1) variate u and search the array to find x so that $F(x) \le u < F(x+1)$
    - Return x

Chi-Square Distribution

- Sum of squares of several unit normal variates
- Key characteristics
  - Parameters: ν=degrees of freedom, ν must be a positive integer
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \dfrac{x^{(\nu-2)/2} e^{-x/2}}{2^{\nu/2}\Gamma(\nu/2)}$

    Here, Γ(.) is the gamma function defined as follows:

    $$\Gamma(b) = \int_0^\infty e^{-x} x^{b-1} dx$$
  - Mean: ν
  - Variance: 2ν

Chi-Square Distribution

- Application: to model sample variances
- Generation:
  - $\chi^2(\nu) = \gamma(2, \nu/2)$

    For ν even: $\chi^2(\nu) = -\frac{1}{2}\ln(\prod_{i=1}^{\nu/2} u_i)$

    For ν odd: $\chi^2(\nu) = \chi^2(\nu-1) + [N(0,1)]^2$
  - Generate ν N(0,1) variates and return the sum of their squares

Erlang Distribution

- Used in queueing models
- Key characteristics
  - Parameters:
    - a = scale parameters, $a > 0$
    - m = shape parameter, m is a positive integer
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \frac{x^{m-1}e^{-x/a}}{(m-1)!a^m}$
  - CDF: $F(x) = 1 - e^{-x/a}\left[\sum_{i=0}^{m-1} \frac{(x/a)^i}{i!}\right]$
  - Mean: am
  - Variance: a^2m

Erlang Distribution

- Applications: extension to the exponential distribution if the coefficient of variation is less than one
  - To model service times in a queueing network model
  - A server with Erlang(*a*,*m*) service times can be represented as a series of *m* servers with exponentially distributed service times
  - To model time-to-repair and time-between-failures
- Generation: convolution
  - Generate *m* U(0,1) random numbers u_i and then:

$$Erlang(a,m) \sim -a \ln\left(\prod_{i=1}^{m} u_i\right)$$

Exponential Distribution

- Used extensively in queueing models
- Key characteristics
  - Parameters: a = scale parameter = mean, $a > 0$
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \frac{1}{a} e^{-x/a}$
  - CDF: $F(x) = 1 - e^{-x/a}$
  - Mean: a
  - Variance: a^2

Exponential Distribution

- **Memoryless property**: past history is not helpful in predicting the future
- Applications: to model time between successive events
  - Time between successive request arrivals to a device
  - Time between failures of a device

  The service times at devices are also modeled as exponentially distributed
- Generation: inverse transformation
  - Generate a U(0,1) random number u and return $-a\ln(u)$ as Exp(a)

Memoryless Property

- Remembering the past does not help in predicting the time till the next event

$$F(\tau) = P(\tau < t) = 1 - e^{-\lambda x} \quad t \geq 0$$

- If we see an arrival and start out clock at $t = 0$, the mean time to the next arrival is $1/\lambda$
- If we do not see an arrival until $t = x$, the distribution of the time remaining till the next arrival is:

$$P(\tau - x < t \mid \tau > x) = \frac{P(x < \tau < x+t)}{P(\tau > x)} = \frac{P(\tau < x+t) - P(\tau < x)}{P(\tau > x)}$$

$$= \frac{(1-e^{-\lambda(x+t)}) - (1-e^{-\lambda x})}{e^{-\lambda x}} = 1 - e^{-\lambda x}$$

This is identical to the situation at t=0

F Distribution

- The ratio of two chi-square variates has an F distribution
- Key characteristics
  - Parameters:
    - n = numerator degrees of freedom, n should be a positive integer
    - m = denominator degrees of freedom, m should be a positive integer
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \dfrac{(n/m)^{n/2}}{\beta(n/2, m/2)} x^{(n-2)/2} (1 + \dfrac{n}{m} x)^{-(n+m)/2}$
  - Mean: $m/(m-2)$, provided $m > 2$
  - Variance: $\dfrac{2m^2(n+m-2)}{n(m-2)^2(m-4)}$, provided $m > 4$

F Distribution

- Tables A.6 to A.8 list the high quantiles of the F distribution. The low quantiles can be calculated:

$$F_{[1-\alpha;n,m]} = \frac{1}{F_{[\alpha;m,n]}}$$

- Applications: to model ratio of sample variances
  - E.g., in the F-test for regression and analysis of variance
- Generation:
  - Generate two chi-square variates $\chi^2(n)$ and $\chi^2(m)$ and compute: $F(n,m) = \dfrac{\chi^2(n)/n}{\chi^2(m)/m}$

Gamma Distribution

- Generalization of Erlang distribution. Allows noninteger shape parameters
- Key characteristics
  - Parameters:
    - a = scale parameter, $a > 0$
    - b = scale parameter, $b > 0$
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \dfrac{(x/a)^{b-1} e^{-x/a}}{a\Gamma(b)}$, $\Gamma(.)$ is the gamma function
  - Mean: ab
  - Variance: $a^2 b$

Gamma Distribution

- Applications: to model service times and repair times
- Generation:
  - If b is an integer, the sum of b exponential variates has a gamma distribution $\gamma(a,b) \sim -a \ln[\prod_{i=1}^{b} u_i]$
  - For $b < 1$, generate a beta variate x~BT(b,1-b) and an exponential variate y~Exp(1). The product axy has a gamma(a,b) distribution
  - For non-integer values of b:

$$\gamma(a,b) \sim \gamma(a, \lfloor b \rfloor) + \gamma(a, b - \lfloor b \rfloor)$$

Geometric Distribution

- Discrete equivalent of the exponential distribution
- Key characteristics
  - Parameters: p = probability of success, $0 < p < 1$
  - Range: $x = 1,2,\ldots, \infty$
  - pmf: $f(x) = (1 - p)^{x-1}p$
  - CDF: $F(x) = 1 - (1 - p)x$
  - Mean: $1/p$
  - Variance: $(1 - p)/p^2$

Geometric Distribution

- Memoryless
- Applications:
  - Number of trials up to and including the first success in a sequence of Bernoulli trials
  - Number of attempts between successive failures (or successes)
    - Number of local queries to a database between successive accesses to the remote database
    - Number of packets successfully transmitted between those requiring a retransmission
    - Number of successive error-free bits between in-error bits in a packet received on a noisy link

Geometric Distribution

- Generation: inversion transformation
  - Generate u~U(0,1) and compute:

$$G(p) = \left\lceil \frac{\ln(u)}{\ln(1-p)} \right\rceil, \quad \lceil . \rceil \text{=> rounding up}$$

Lognormal Distribution

- Log of a normal variate
- Key characteristics
  - Parameters
    - μ = mean of ln(x), μ > 0
    - σ = standard deviation of ln(x), σ > 0
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{\frac{-(\ln x - \mu)^2}{2\sigma^2}}$
  - Mean: $e^{\mu + \sigma^2/2}$
  - Variance: $e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$

Lognormal Distribution

- Note: μ and σ are the mean and standard deviation of ln(x), not x
- Applications:
  - The product of a large number of positive random variables tends to have an approximate lognormal distribution
  - To model multiplicative errors that are a product of effects of a large number of factors
- Generation: log of a normal variate
  - Generate x~N(0,1) and return exponent of x: $e^{\mu+\sigma x}$

Negative Binomial Distribution

- Number of failures x before the mth success
- Key characteristics
  - Parameters:
    - p = probability of success, $0 < p < 1$
    - m = number of success, m must be a positive integer
  - Range: $x = 1,2,\ldots,\infty$
  - pmf: $f(x)=\binom{m+x-1}{m-1}p^m(1-p)^x=\frac{\Gamma(m+x)}{(\Gamma m)(\Gamma x)}p^m(1-p)^x$
    The second expression allows a negative binomial to be defined for noninteger values of x
  - Mean: $m(1-p)/p$
  - Variance: $m(1-p)/p^2$

Negative Binomial Distribution

- Applications: used if variance > mean, otherwise use Binomial or Poisson
  - Number of local queries to a database system before *m*th remote query
  - Number of retransmissions for a message consisting of *m* packets
  - Number of error-free bits received on a noisy link before the *m*th in-error bit

Negative Binomial Distribution

- Generation:
  - Generate u_i~U(0,1) until m of the u_i's are greater than p. Return the count of u_i's less than or equal to p as NB(p,m)
  - The sum of m geometric variates G(p) gives the total number of trials for m successes

$$NB(p,m) \sim (\sum_{i=1}^{m} G(p)) - m$$

  - Composition:
    - Generate a gamma variate y~G(p/(1 – p), m)
    - Generate a Poisson variate x~Poisson(y)
    - Return x as NB(p,m)

Normal Distribution

- Also known as Gaussian distribution
  - Discovered by Abraham De Moivre in 1733
  - Rediscovered by Gauss in 1809 and by Laplace 1812
- N(0,1) = unit normal distribution or standard normal distribution

Normal Distribution

- Key characteristics
  - Parameters
    - μ = mean
    - σ = standard deviation, $\sigma > 0$
  - Range: $-\infty \leq x \leq \infty$
  - pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
  - Mean: μ
  - Variance: σ^2

Normal Distribution

- Applications:
  - Errors in measurement
  - Error in modeling to account for a number of factors that are not included in the model
  - Sample means of a large number of independent observations from a given distribution
- Generation:
  - Using the sum of a large number of uniform u_i~U(0,1) variates
  - Box-Muller method
  - Polar method
  - Rejection method

Pareto Distribution

- Pareto CDF is a power curve => fit to observed data
- Key characteristics
  - Parameters: a = shape parameter, a > 0
  - Range: $1 \leq x \leq \infty$
  - Pdf: $f(x) = ax^{-(a+1)}$
  - CDF: $F(x) = 1 - x^{-a}$
  - Mean: $a/(a-1)$, provided $a > 1$
  - Variance: $\frac{a}{(a-1)^2(a-2)}$, provided $a > 2$

Pareto Distribution

- Application: to fit a distribution
  - The maximum likelihood estimate

$$a = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \ln x_i}$$

- Generation: inverse transformation
  - Generate u~U(0,1) and return $1/u^{1/a}$

Pascal Distribution

- Extension of the geometric distribution
- Number of trials up to and including the *m*th success
- Key characteristics
  - Parameters:
    - p = probability of success, $0 < p < 1$
    - m = number of success, m must be a positive integer
  - Range: $x = m, m+1, \ldots, \infty$
  - pmf: $f(x) = \binom{x-1}{m-1} p^m (1-p)^{x-m}$
  - Mean: m/p
  - Variance: $m(1-p)/p^2$

Pascal Distribution

- Applications:
  - Number of attempts to transmit an *m* packet message
- Generation:
  - Generate *m* geometric variates $G(p)$ and return their sum as Pascal(p,m)

Poisson Distribution

- Limiting form of the binomial distribution
- Key characteristics
  - Parameters: λ=mean, $\lambda > 0$
  - Range: x=0,1,…, ∞
  - pmf: $f(x) = P(X = x) = \lambda^x \frac{e^{-\lambda}}{x!}$
  - Mean: λ
  - Variance: λ

Poisson Distribution

- Applications: to model the number of arrivals over a given interval
  - Number of requests to a server in a given time interval *t*
  - Number of component failures per unit time
  - Number of queries to a database system over *t* seconds
  - Number of typing errors per form
  - Particularly appropriate if the arrivals are from a large number of independent sources

Poisson Distribution

- Generation:
  - Inverse transformation method: Compute the CDF $F(x)$ for $x = 0,1,2,\ldots$ up to a suitable cutoff and store in an array. For each Poisson random variate, generate a U(0,1) variate u and search the array to find x such that $F(x) \leq u < F(x+1)$, return x

Student's *t*-Distribution

- Derived by W. S. Gosset, published under a pseudonym of 'Student' used symbol *t*
- Key characteristics
  - Parameters: ν = degree of freedom, ν must be positive integer
  - Range: $-\infty \le x \le \infty$
  - pmf: $f(x) = \dfrac{\{\Gamma[(\nu+1)/2]\}[1+(x^2/\nu)]^{-(\nu+1)/2}}{(\pi\nu)^{1/2}\Gamma(\nu/2)}$
  - Variance: $\nu/(\nu-2)$, for $\nu > 2$

Student's t-Distribution

- Very similar to normal distribution.
  For large *n* > 30, it can be approximated by normal distribution: *t*~N(0,1)
- Applications: in setting confidence intervals and in *t*-tests
- Generation: characterization. Generate *x*~N(0,1) and y~$\chi^2(\nu)$ and return $x/\sqrt{y/\nu}$ as t(ν)

Uniform Distribution (Continuous)

- Key characteristics
  - Parameters:
    - a = lower limit
    - b = upper limit, $b > a$
  - Range: $a \le x \le b$
  - Pdf: $f(x) = 1/(b - a)$
  - CDF: $$F(x) = \begin{cases} 0, & \text{If } x < a \\ \frac{x-a}{b-a}, & \text{If } a \le x < b \\ 0, & \text{If } b \le x \end{cases}$$
  - Mean: $(a+b)/2$
  - Variance: $(b - a)^2/12$

Uniform Distribution (Continuous)

- Applications: bounded random variables with no further information
  - Distance between source and destinations of messages on a network
  - Seek time on a disk
- Generation: To generate U(a,b), generate u~U(0,1) and return a+(b-a)u

Uniform Distribution (Discrete)

- Discrete version of the uniform distribution
- Takes a finite number of values, each with the same probability

Uniform Distribution (Discrete)

- Key characteristics
  - Parameters:
    - m = lower limit, m must be an integer
    - n = upper limit, n must be an integer and $n > m$
  - Range: $x = m, m+1, m+2, \ldots, n$
  - Pdf: $f(x) = 1/(n - m+1)$
  - CDF: $$F(x) = \begin{cases} 0, & \text{If } x < m \\ \dfrac{x-m+1}{n-m+1}, & \text{If } m \le x < n \\ 1, & \text{If } n \le x \end{cases}$$
  - Mean: $(n+m)/2$
  - Variance: $[(n - m+1)^2 - 1]/12$

Uniform Distribution (Discrete)

- Applications:
  - Track numbers for seeks on a disk
  - I/O device number selected for the next I/O
  - The source and destination node for the next packet on a network
- Generation:
  - Generate u~U(0,1), return $\lfloor m+(n-m+1)u \rfloor$

Weibull Distribution

- Weibull distribution is commonly used in reliability analysis
- Key characteristics
  - Parameters:
    - a = scale parameter, $a > 0$
    - b = scale parameter, $b > 0$
  - Range: $0 \le x \le \infty$
  - pdf: $f(x) = \frac{bx^{b-1}}{a^b} e^{-(x/a)^b}$ CDF: $F(x) = 1 - e^{-(x/a)^b}$
  - Mean: $\frac{a}{b}\Gamma(1/b)$
  - Variance: $\frac{a^2}{b^2}\left[2b\Gamma(2/b) - \{\Gamma(1/b)\}^2\right]$

Weibull Distribution

- If b = 3.602, the Weibull distribution is close to a normal
- For b > 3.602, it has a long left tail
- For b > 3.602, it has a long right tail
- For b ≤ 1, the Weibull pdf is L-shaped
- For b > 1, it is bell-shaped
- For large b, the Weibull pdf has a sharp peak at the mode

Weibull Distribution

- Applications: to model lifetimes of components
  - b < 1 => failure rate increasing with time
  - b > 1 => failure rate decreases with time
  - b = 1 => failure rate is constant
  - ⇒ life times are exponentially distributed
- Generation: inverse transformation
  - Generate u~U(0,1) and return $a(\ln u)^{1/b}$ as Weibull(a,b)

Relationships among Distributions

Relationships among Distributions