Homework 1, Selected Problems

1.1 6. Suppose two students memorize lists according to the same model
\[ \frac{dL}{dt} = 2(1 - L). \]
a) If one of the students knows one-half of the list at time \( t = 0 \) and the other knows none of the list, which student is learning most rapidly at this instant?

**Solution:** Let \( L_1(t) \) represent the portion of the list the first student has memorized by time \( t \), and let \( L_2(t) \) represent the portion of the list the second student has memorized by time \( t \). Then \( L_1(0) = 1/2 \), and \( L_2(0) = 0 \). Substituting in to the differential equation for the model, we have
\[ \frac{dL_1}{dt} \bigg|_{t=0} = 2(1 - L_1(0)) = 2(1 - 1/2) = 1, \]
and
\[ \frac{dL_2}{dt} \bigg|_{t=0} = 2(1 - L_2(0)) = 2(1 - 0) = 2, \]
so the second student is learning faster initially.

b) Will the student who starts out knowing none of the list ever catch up to the student who starts out knowing half of the list?

**Solution:** No, not according to the model. Although both students will tend toward knowing the entire list, neither will actually reach that goal, and the second student will always trail the first in how much (s)he knows.

1.1 18. For the following predator-prey systems, identify which dependent variable, \( x \) or \( y \), is the prey population and which is the predator population. Is the growth of the prey limited by any factors other than the number of predators? Do the predators have sources of food other than the prey? (Assume that the parameters \( \alpha, \beta, \gamma, \delta, \) and \( N \) are all positive.)

a) \[
\frac{dx}{dt} = -\alpha x + \beta xy \\
\frac{dy}{dt} = \gamma y - \delta xy
\]

**Solution:** The interaction term, \( \beta xy \) is positive for \( \frac{dx}{dt} \), meaning that chance meetings between the two species causes the population of \( x \) to increase, and the interaction term, \( -\delta xy \), is negative for \( \frac{dy}{dt} \), so interaction is bad for \( y \). Therefore, \( x \) is predator and \( y \) is prey. The growth of prey is only limited by predators (to see this take \( x(t) = 0 \) and note that the resulting differential equation for \( y \) is the exponential growth equation. The predators do not have a source of food other than the prey (take \( y(t) = 0 \) and note the resulting exponential decay equation for \( x \)).
b)

\[
\frac{dx}{dt} = \alpha x - \frac{\alpha x^2}{N} - \beta xy \\
\frac{dy}{dt} = \gamma y + \delta xy
\]

**Solution:** The interaction term, \(-\beta xy\) is negative for \(\frac{dx}{dt}\), meaning that chance meetings between the two species causes the population of \(x\) to decrease, and the interaction term, \(\delta xy\), is positive for \(\frac{dy}{dt}\) so interaction is good for \(y\). Therefore, \(x\) is prey and \(y\) is predator. The growth of prey is not only limited by predators (to see this take \(y(t) = 0\) and note that in the resulting differential equation for \(x\) is the logistic equation and the population declines if it exceeds the carrying capacity \(N\)). The predators have a source of food other than the prey (take \(x(t) = 0\) and note the resulting exponential growth equation for \(y\)).

1.2 6.

**Solution:**

\[y(t) = ke^{t^5/5}\]

1.2 18.

\[
\frac{dy}{dt} = \frac{1}{ty + t + y + 1}
\]

**Solution:**

\[
\frac{dy}{dt} = \frac{1}{ty + t + y + 1} = \frac{1}{t(y + 1) + y + 1} = \frac{1}{y + 1 + t + 1} \\
\int (y + 1)dy = \int (t + 1)dt \\
\frac{y^2}{2} + y = \ln |t + 1| + k \\
\frac{y^2}{2} + y - \ln |t + 1| - k = 0
\]

Applying the quadratic formula, we have

\[y(t) = \frac{-1 \pm \sqrt{1 + 2(\ln |t + 1| + k)}}{2(1/2)} = -1 \pm \sqrt{c + 2 \ln |t + 1|}\]

where \(c = 2k + 1\), and the sign in front of the radical is determined by the initial condition.
1.2 36. Consider the following very simple model of blood cholesterol levels based on the fact that cholesterol is manufactured by the body for use in the construction of cell walls and is absorbed from foods containing cholesterol: Let \( C(t) \) be the amount in milligrams per deciliter) of cholesterol in the blood of a particular person at time \( t \) (in days). Then
\[
\frac{dC}{dt} = k_1(C_0 - C) + k_2E,
\]
where

\[
\begin{align*}
C_0 &= \text{the person’s natural cholesterol level} \\
k_1 &= \text{production parameter} \\
E &= \text{daily rate at which cholesterol is eaten} \\
k_2 &= \text{absorption parameter}.
\end{align*}
\]

(a) Suppose \( C_0 = 200 \), \( k_1 = 0.1 \), \( k_2 = 0.1 \), \( E = 400 \), and \( C(0) = 150 \). What will the person’s cholesterol level be after two days on this diet?

**Solution:**
\[
\begin{align*}
\frac{dC}{dt} &= k_1(C_0 - C) + k_2E \\
\int \frac{1}{k_1(C_0 - C) + k_2E} \, dC &= \int dt \\
-\frac{1}{k_1} \ln |k_1(C_0 - C) + k_2E| &= t + B_1 \\
\ln |k_1(C_0 - C) + k_2E| &= -k_1(t + B_1) \\
|k_1(C_0 - C) + k_2E| &= e^{-k_1(t + B_1)} \\
k_1(C_0 - C) + k_2E &= B_2e^{-k_1t} \\
k_1C &= B_2e^{-k_1t} - k_2E - k_1C_0 \\
C(t) &= \frac{-1}{k_1} \left( B_2e^{-k_1t} - k_2E - k_1C_0 \right),
\end{align*}
\]

where \( B_1 \) is a constant of integration, and \( B_2 = e^{-k_1B_1} \) if the quantity in the absolute value is positive, and \( B_2 = -e^{-k_1B_1} \) otherwise. One could solve this differential equation for the particular values of the parameters given, but then we would have to solve it again when the parameters are changed later in the problem.

\[
\begin{align*}
C(0) &= 150 = \frac{-1}{k_1} \left( B_2 - k_2E - k_1C_0 \right) \\
150 &= \frac{-1}{0.1} \left( B_2 - 0.1(400) - 0.1(200) \right) \\
&= -10(B_2 - 40 - 20) \\
B_2 &= 45.
\end{align*}
\]
\[ C(t) = -10 \left( 45e^{-0.1t} - 60 \right) \]
\[ C(2) = -10 \left( 45e^{-0.2} - 60 \right) \approx 231.57. \]

After 2 days on this diet, the person’s cholesterol level will be about 232.

(b) With the initial conditions as above, what will the person’s cholesterol level be after 5 days on this diet?

**Solution:**

\[ C(5) = -10 \left( 45e^{-0.5} - 60 \right) \approx 327.06. \]

After 5 days on this diet, the person’s cholesterol level will be about 327.

(c) What will the person’s cholesterol level be after a long time on this diet?

**Solution:**

\[ \lim_{t \to \infty} C(t) = -10 \left( 45e^{-0.1t} - 60 \right) = 600. \]

After a long time on this diet, the person’s cholesterol level will be about 600.

(d) High levels of cholesterol in the blood are known to be a risk factor for heart disease. Suppose that, after a long time on the high cholesterol diet described above, the person goes on a very low cholesterol diet, so \( E \) changes to \( E = 100 \). (The initial cholesterol level at the starting time of this diet is the result of part (c)). What will the person’s cholesterol level be after 1 day on the new diet, after five days on the new diet, and after a very long time on the new diet?

**Solution:**

\[ C(0) = 600 = \frac{-1}{0.1} \left( B_2 - 0.1(100) - 0.1(200) \right) \]
\[ = -10(B_2 - 30) \]
\[ B_2 = -30. \]

\[ C(t) = 300 \left( e^{-0.1t} + 1 \right) \]
\[ C(1) = 300 \left( e^{-0.1} + 1 \right) \approx 571.45. \]
\[ C(5) = 300 \left( e^{-0.5} + 1 \right) \approx 481.96. \]

\[ \lim_{t \to \infty} C(t) = 300. \]

After 1 day on this diet, the person’s cholesterol level will be about 571. After 5 days on this diet, the person’s cholesterol level will be about 482. After a long time on this diet, the person’s cholesterol level will be about 300.
(e) Suppose the person stays on the high cholesterol diet but takes drugs that block some of the
uptake of cholesterol from food, so \( k_2 \) changes to \( k_2 = 0.075 \). With the cholesterol level from part (c),
what will the person’s cholesterol level be after 1 day, after five days, and after a very long time?

**Solution:**

\[
C(0) = 600 = -\frac{1}{0.1} \left( B_2 - 0.075(400) - 0.1(200) \right) \\
= -10(B_2 - 50) \\
B_2 = -10.
\]

\[
C(t) = \left( 100e^{-0.1t} + 500 \right) \\
C(2) = \left( 100e^{-0.2} + 500 \right) \approx 590.48. \\
C(5) = \left( 100e^{-0.5} + 500 \right) \approx 560.65. \\
\lim_{t \to \infty} C(t) = 500.
\]

After 1 day on this diet, the person’s cholesterol level will be about 590. After 5 days on this diet, the
person’s cholesterol level will be about 561. After a long time on this diet, the person’s cholesterol
level will be about 500.

1.2 37. A cup of hot chocolate is initially 170°F and is left in a room with an ambient temperature
of 70°F. Suppose that at time \( t = 0 \) it is cooling at a rate of 20 degrees per minute.

a) Assume that Newton’s law of cooling applies: The rate of cooling is proportional to the difference
between the current temperature and the ambient temperature. Write an initial value
problem that models the temperature of the hot chocolate.

Let \( T(t) \) represent the temperature of the hot chocolate at time \( t \), and \( A \) represent the ambient
temperature of the room, then

\[
\frac{dT}{dt} = -k(T - A),
\]

where \( k \) is the proportionality constant.

b) How long does it take the hot chocolate to cool to a temperature of 110°F?

First we compute \( k \).

\[
\left. \frac{dT}{dt} \right|_{t=0} = -20 = -k(T(0) - A) = -k(170 - 70) \\
k = 0.2,
\]

so

\[
\frac{dT}{dt} = -0.2(T - 70)
\]
\[
\int \frac{1}{T-70} dT = -\int 0.2 dt \\
\ln |T-70| = -0.2t + c \\
|T-70| = e^{-0.2t+c} \\
T - 70 = c_2 e^{-0.2t} \\
T = c_2 e^{-0.2t} + 70,
\]

and from the initial condition,

\[
T(0) = 170 = c_2 + 70 \\
c_2 = 100.
\]

The solution for \(T(t)\) is

\[
T(t) = 100e^{-0.2t} + 70.
\]

We must solve

\[
110 = 100e^{-0.2t} + 70
\]

for \(t\).

\[
40 = 100e^{-0.2t} \\
0.4 = e^{-0.2t} \\
\ln(0.4) = -0.2t \\
t = -5 \ln(0.4)
\]

\(t \approx 4.58\).

It takes the hot chocolate about 4.58 minutes (4 minutes, 35 seconds) to cool to a temperature of 110 degrees Fahrenheit.

Note: that your solution is not complete without some sentence to this effect!