PROOF

Given a finite number of events \( x_1, x_2, x_3, \ldots, x_m \) that each occur a nonnegative, countable number of times, using the algorithm I developed in problem 2.17 (a clarifying example is given as an appendix to this proof) the following expression gives the total number of possible combinations:

(i.) \( \frac{(x_1)(x_1+x_2) \cdots (x_1+x_2+\cdots+x_m)}{x_1!(x_1-x_1)!x_2!(x_1+x_2-x_2)!x_3!(x_1+x_2+x_3-x_3)!\cdots x_m!(x_1+x_2+x_3+\cdots+x_m-x_m)!} \)

Using the definition of a combination, this expands to

\[
\frac{x_1!(x_1+x_2)!(x_1+x_2+x_3)!\cdots(x_1+x_2+x_3+\cdots+x_m)!}{x_1!(x_1-x_1)!x_2!(x_1+x_2-x_2)!x_3!(x_1+x_2+x_3-x_3)!\cdots x_m!(x_1+x_2+x_3+\cdots+x_m-x_m)!}
\]

Next I simplify the denominator, obtaining

\[
\frac{x_1!(x_1+x_2)!(x_1+x_2+x_3)!\cdots(x_1+x_2+x_3+\cdots+x_m)!}{x_1!0!x_2!x_1!x_3!(x_1+x_2)!\cdots x_m!(x_1+x_2+x_3+\cdots+x_{m-1})!}
\]

Now I remove the 0! since it equals 1, and replace \( x_1+x_2+x_3+\cdots+x_m \) with \( n \), which is defined in the problem definition as the sum of all the events.

\[
\frac{x_1!(x_1+x_2)!(x_1+x_2+x_3)!\cdots n!}{x_1!x_2!x_1!x_3!(x_1+x_2)!\cdots x_m!(x_1+x_2+x_3+\cdots+x_{m-1})!}
\]

In the numerator, I write out the term that is in sequence immediately before the \( n! \) term I also line up equivalent terms in the numerator and the denominator, yielding:

\[
\frac{x_1!(x_1+x_2)!(x_1+x_2+x_3)!\cdots (x_1+x_2+\cdots+x_{m-1})!n!}{x_1!x_1!x_2!x_3!(x_1+x_2)!\cdots x_m!(x_1+x_2+x_3+\cdots+x_{m-1})!}
\]

From this setup it is obvious that part of each denominator term cancels the factorial in the numerator term that precedes it (that is, the second denominator cancels the first numerator and so on). The first denominator yields a 0! in this place, which has no effect. The result is that all factorials in the numerator are erased except the last one (since there is no \( n+1 \) term to eliminate it). Likewise all the denominator terms that correspond with the \( (n-r)! \) part of the combination formula are cancelled. This leaves

\[
\frac{n!}{x_1!x_2!x_3!\cdots x_m!}
\]

which of course is the combination part of the multinomial formula in its general form.

Appendix: The algorithm that produced the initial formula

The algorithm seeks to deal with each group of events separately. It finds the total number of combinations of all occurrences of event \( A \), and then combines this with all occurrences of event \( B \), and so on until all events are accounted for.
Begin by taking all events $A$ (say there are 3). Since only three events can happen, and by definition they must all be event $A$, there is only one way in which they could occur:

$$AAA \quad \text{(which is } \binom{3}{3}\text{)}$$

Next consider all events $B$ (say there is only 1). Since now four events will occur, it follows that $B$ can occur in any one of the four positions (e.g. there are $\binom{4}{1}$ places for $B$)

$$B \_ \_ \_ \_ \_ \text{ or } B \_ \_ \_ \_ \_ \text{ or } B \_ \_ \_ \_ \_ \text{ or } B \_ \_ \_ \_ \_$$

This placement does not introduce any more possible combinations among $A$ only. However, the combination of all $A$s and $B$s is given by $\binom{3}{3}\binom{4}{1}$. Likewise, all events $C$ (say there are 2) have 6 total positions that they can be placed in. For example, these are some of the possibilities:

$$C \_ \_ \_ \_ \_ \_ \text{ or } C \_ \_ \_ \_ \_ \_ \text{ or } C \_ \_ \_ \_ \_ \_ \_ \_ \_ \text{ etc.}$$

It is easily seen that the total number of such placements is $\binom{6}{2}$. More important to note, however, is the fact that, if you ignore the locations of the $C$s in the illustrations above, you have four blank spaces remaining in which to place the $A$s and $B$s. However, it is already known that the total number of combinations of $A$s and $B$s in four spots equals $\binom{3}{3}\binom{4}{1}$. Therefore, for each placement of the two $C$s the remaining $A$s and $B$s can be placed in $\binom{3}{3}\binom{4}{1}$ ways. It follows that the total number of combinations for $A$s, $B$s, and $C$s is $\binom{3}{3}\binom{4}{1}\binom{6}{2}$.

Now, for all subsequent events, one need only consider how many placements of $D$, $E$, etc. are possible, and multiply that number by the total number of combinations of all letters (i.e. events) that have already been accounted for. In general, each new letter requires

$$\binom{\text{Total number of events so far}}{\text{Number of times the next event has occurred}}$$

or

$$\binom{A + B + C + \ldots \text{up to the last event group Xlast}}{\text{Xlast}}$$

where it is understood that $A, B, C, \ldots, \text{Xlast}$ denote the actual number of times that event $A$, $B$, $C$, $\ldots$, $	ext{Xlast}$ have actually occurred. Since each such term is multiplied by the last to generate the total number of combinations for all events up to the current event, formula (i.) in the proof follows logically.
Notation note:

\[
\binom{N}{R} = \binom{N}{R} = \frac{N!}{(N-R)! R!},
\]

read “\(N\) choose \(R\)”, is the number of ways of dividing \(N\) distinct objects into two groups: one of size \(R\) and one of size \(N - R\). Both of the first two notations are in common use. The former notation is used in our text, and the latter in the probability handout.