The Algebra (al-jabr) of Matrices

Algebra as a branch of mathematics is much broader than elementary algebra all of us studied in our high school days. In a sense “an algebra” is a set of rules. As we learned from our secondary education the algebra of scalars studies the operations and relationships of scalar numbers. When the rules of addition and multiplication are generalized, their precise definitions lead to the notions of algebraic structures that can lead to the more esoteric concepts in scalar algebra such as groups, rings and fields – fields of study in the realm of mathematics known as abstract algebra, In that same sense we can also talk in terms of the algebra of complex numbers, algebra of vectors, and tensor algebra.

Algebra is a set of definitions, rules, and operations that govern mathematical quantities.
Section 2: Matrix Algebra and Calculus

Matrix Notation

Given the following system of equations

\[ b_1 = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \]
\[ b_2 = a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \]
\[ \vdots \]
\[ b_m = a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \]

in short hand the above system can be expressed as

\[ \{B\} = [A]\{X\} \]

where

\[ \{B\} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \{A\} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad \{X\} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]
Thus a matrix is an ordered arrangement of values (scalar, vector, higher order tensor) in a row-column format

\[
\{ A \} = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\]

The matrix above consists of \( m \) rows and \( n \) columns. We can identify the elements in a matrix using the notation \( a_{ij} \). The first subscript designates row in which the element is in and the second subscript identifies the column. Repeated subscripts (indices) indicates the element is on the diagonal.

In later courses the subscripted notation will be used to represent the matrix itself when some rules are incorporated on how to employ indicial notation (see Elasticity or Continuum Mechanics).
Although the number of rows and columns of a matrix may vary from problem to problem, two cases deserve attention. When $m = 1$ the matrix consists of one row of elements. This is called a row matrix and is denoted

$$\{ B \} = \{ b_1 \ b_2 \ \ldots \ b_m \}$$

When $n = 1$ the matrix consists of one column of elements and is referred to as a column matrix. It is denoted

$$\{ X \} = \begin{Bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{Bmatrix}$$
Fundamental Types of Matrices

A **square matrix** has the same number of rows as columns.

A **symmetric matrix** is one in which the off diagonal elements are reflected about the diagonal. Using subscript notation

\[ a_{ij} = a_{ji} \quad i \neq j \]

or in a full matrix format

\[
[A] = \begin{bmatrix}
1 & 4 & 5 \\
4 & 2 & 6 \\
5 & 6 & 3
\end{bmatrix}
\]

Symmetric square matrices play a special role in engineering mathematics. Can a matrix be symmetric if \( m \) does not equal \( n \)?
When all elements of the main diagonal are equal to one, and all the off diagonal entries are equal to zero, i.e.,

\[
[I] = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

the square matrix is referred to as the \textit{identity matrix}.

The \textit{transpose of a matrix} is defined as the reordering of the elements of the matrix such that the columns of the original matrix become the rows of the new matrix. The following notation is utilized

\[
\{ A \} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

\[
\{ A \} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]
An example of a specific matrix and its transpose is

\[
\{A\} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \{A\}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}
\]

The product of a transpose is defined as

\[
\{\{A\}{B}\}^T = \{A\}^T \{B\}^T
\]

For a square symmetric matrix

\[
[A]^T = [A] \\
a_{ji} = a_{ij} \quad i \neq j
\]
Example 2.1

Show that

\[
\{\{A\} \{B\}\}^T = \{B\}^T \{A\}^T
\]

Proof by example: If

\[
\{A\} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ 0 & -2 & 1 \\ 2 & -3 & 2 \end{pmatrix} \quad \{B\} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}
\]

So that

\[
\{A\}^T = \begin{pmatrix} 1 & 4 & 0 & 2 \\ 2 & -1 & -2 & -3 \\ 3 & -2 & 1 & 2 \end{pmatrix} \quad \{B\}^T = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}
\]
Section 2: Matrix Algebra and Calculus

Then

\[
\{A\} \{B\} = \begin{pmatrix}
1 & 2 & 3 \\
4 & -1 & -2 \\
0 & -2 & 1 \\
2 & -3 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
2 & -1 \\
0 & 3
\end{pmatrix}
\]

\[
= \begin{cases}
(1)(1) + (2)(2) + (3)(0) = 5 \\
(4)(1) + (-1)(0) + (-2)(0) = 2 \\
(0)(1) - (2)(2) + (1)(0) = -4 \\
(2)(1) - (3)(2) + (2)(0) = -4
\end{cases}
\]

\[
= \begin{pmatrix}
5 & 8 \\
2 & -1 \\
-4 & 5 \\
-4 & 11
\end{pmatrix}
\]

\[
\{\{A\}\{B\}\}^T = \begin{pmatrix}
5 & 2 & -4 & -4 \\
8 & -1 & 5 & 11
\end{pmatrix}
\]
Similarly

\[
\begin{bmatrix}
B^T \{A^T\}
\end{bmatrix} =
\begin{bmatrix}
1 & 2 & 0 \\
1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & 4 & 0 & 2 \\
2 & -1 & -2 & -3 \\
3 & -2 & 1 & 2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(1)(1) + (2)(2) + (0)(3) = 5 \\
(1)(1) + (-1)(2) + (3)(3) = 8
\end{bmatrix}
\begin{bmatrix}
(1)(4) + (2)(-1) + (0)(-2) = 2 \\
(1)(4) + (-1)(-1) + (3)(-2) = -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(1)(0) + (2)(-2) + (0)(1) = -4 \\
(1)(0) + (-1)(-2) + (3)(1) = 5
\end{bmatrix}
\begin{bmatrix}
(1)(2) + (2)(-3) + (0)(2) = -4 \\
(1)(2) + (-1)(-3) + (3)(2) = 11
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5 & 2 & -4 & -4 \\
8 & -1 & 5 & 11
\end{bmatrix}
= \{\{A\}\{B\}\}^T = \begin{bmatrix}
5 & 2 & -4 & -4 \\
8 & -1 & 5 & 11
\end{bmatrix}
\]

hence

\[
\{\{A\}\{B\}\}^T = \{B\}^T \{A\}^T
\]
It is easy to conceptualize the transpose of the transpose is the matrix itself

\[
\left\{ B^T \right\}^T = \{B\}
\]

With rules of addition and multiplication

\[
\left\{ A + B + C + \cdots \right\}^T = \{A\}^T + \{B\}^T + \{C\}^T + \cdots
\]

\[
\{\alpha \ A\}^T = \alpha \{A\}^T
\]

and

\[
\left\{ A^n \right\}^T = \left(\{A\}^T\right)^n
\]

where

\[
\{A\}^n = \{A\}\{A\}\{A\} \cdots \{A\}
\]
A square matrix is a **diagonal matrix** when the entries along the diagonal are non-zero and all off diagonal are zero, i.e.,

\[
[D] = \begin{bmatrix}
d_{11} & 0 & \cdots & 0 \\
0 & d_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & d_{mm}
\end{bmatrix}
\]

A **scalar matrix** is a diagonal matrix whose diagonal elements all contain the same scalar \( \lambda \)

\[
[S] = \begin{bmatrix}
\lambda & 0 & \cdots & 0 \\
0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda
\end{bmatrix} = \lambda [I]
\]
The *null matrix* is a matrix whose entries are all zero, i.e.,

\[ \{0\} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \]

The null matrix does not have to be a square matrix.

The number of rows and columns that a matrix has is called its *order* or its dimension. By convention, rows are listed first and then columns. Thus, we would say that the order (or dimension) of the matrix below is 3 x 4, meaning that the matrix has 3 rows and 4 columns.

\[
\begin{pmatrix}
57 & 23 & 103 & 42 \\
0 & 47 & 17 & 6 \\
51 & 89 & 1 & 9
\end{pmatrix}
\]
A **lower triangular matrix** is a square matrix with entries equal to zero above the diagonal, i.e.,

\[
\{ L \} = \begin{bmatrix}
    l_{11} & 0 & \cdots & 0 \\
    l_{21} & l_{22} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    l_{m1} & l_{m2} & \cdots & l_{mm}
\end{bmatrix}
\]

An **upper triangular matrix** is a square matrix with entries equal to zero below the diagonal, i.e.,

\[
\{ u \} = \begin{bmatrix}
    u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
    0 & u_{22} & u_{23} & \cdots & u_{2n} \\
    0 & 0 & u_{33} & \cdots & \vdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & u_{nn}
\end{bmatrix}
\]
Matrix Addition and Subtraction

The matrix \( A \)

\[
\{A\} = \{a_{ij}\}_{m \times n}
\]

can be added to the matrix \( B \)

\[
\{B\} = \{b_{ij}\}_{m \times n}
\]

to produce the matrix \( C \)

\[
\{C\} = \{c_{ij}\}_{m \times n} = \{a_{ij} + b_{ij}\}_{m \times n}
\]

This points out that to form a sum of two matrices the matrices must be of the same order (the matrices are said to be conformable for addition) and that the elements of the sum are determined by adding the corresponding elements of the matrices forming the sum.
Matrix addition is both commutative and associative

\[
\{A\} + \{B\} = \{B\} + \{A\}
\]

\[
\{A\} + (\{B\} + \{C\}) = (\{A\} + \{B\}) + \{C\}
\]

The matrix \(\{B\}\)

\[
\{B\} = \{b_{ij}\}_{m \times n}
\]

can be subtracted from the matrix \(\{A\}\)

\[
\{A\} = \{a_{ij}\}_{m \times n}
\]

to produce the matrix \(\{C\}\)

\[
\{C\} = \{c_{ij}\}_{m \times n} = \{a_{ij} - b_{ij}\}_{m \times n}
\]

To carry this one step further

\[
\{A\} + \{B\} = \{C\} \quad \rightarrow \quad \{A\} = \{C\} - \{B\}
\]
Matrix Multiplication

We will see that the matrix methods in structural analysis requires solving large systems of linear equations using matrix algebra tools. In an earlier section, the large systems of linear equations was represented simply as

\[ \{ B \} = [A]\{X\} \]

where \( A \) was an \( m \times n \) coefficient matrix, \( B \) was an \( m \times 1 \) vector, and \( X \) was an \( n \times 1 \) vector. Now let \( m = n = 3 \). The matrices take on the following forms:

\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
\end{bmatrix} = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33} \\
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
\end{bmatrix}
\]

A row by column element product and summation is clearly evident:
A row by column element product and summation is clearly evident:

\[
\begin{align*}
b_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
b_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
b_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3
\end{align*}
\]

that is, each element of \( B \) is obtained by multiplying the corresponding element of \( A \) by the appropriate element in \( X \) and adding the result. Notice that the forgoing procedure does not work if the number of columns in \( A \) does not equal the number of rows of \( B \). This suggests a general definition for the multiplication of two matrices. If \( A \) is an \( m \times n \) matrix, and \( B \) is a \( p \times q \) matrix, then

\[
\{A\}\{B\} = \{C\}
\]

exists if \( n \) is equal to \( p \).
If this is the case the elements of $C$ are given by

$$i = 1, 2, 3, \ldots, m$$

$$\{c_{ij}\}_{m \times q} = \sum_{k=1}^{n=p} a_{ik} b_{kj}$$

$$j = 1, 2, 3, \ldots, q$$

Under these conditions the matrices $A$ and $B$ are said to be conformable for multiplication. In general

$$\{A\}{B} \neq \{B\}{A}$$

However, it can be proven with some effort that

$$\{(A\{B\})\{C\} = \{A\}(\{B\}\{C\})$$

$$\{A\}(\{B\} + \{C\}) = \{A\}\{B\} + \{B\}\{C\}$$

$$\{(A) + \{B\}\{C\} = \{A\}\{C\} + \{B\}\{C\}$$
Example 2.2
If the product of two matrices $A$ and $B$ yields the null matrix, that is,

$$\{A\} \{B\} = \{0\}$$

it cannot be assumed that $A$ or $B$ is the null matrix. Furthermore, if

$$\{A\} \{B\} = \{A\} \{C\}$$

or

$$\{C\} \{A\} = \{B\} \{A\}$$

it cannot be assumed that

$$\{B\} = \{C\}$$

This infers that in general cancellation of matrices in a manner similar to multiplication of scalar algebra is not permissible.
Example 2.3
Matrix Multiplication Applied to Structural Analysis

For displacement based structural analysis

\[
\begin{pmatrix}
    f_1 \\
    f_2 \\
    \vdots \\
    f_n
\end{pmatrix}
= 
\begin{pmatrix}
    k_{11} & k_{12} & \cdots & k_{1n} \\
    k_{12} & k_{22} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    k_{1n} & \cdots & \cdots & k_{nn}
\end{pmatrix}
\begin{pmatrix}
    d_1 \\
    d_2 \\
    \vdots \\
    d_n
\end{pmatrix}
\]

Stiffness matrix - \textit{Symmetric} since \( k_{ij} = k_{ji} \)

We will need matrix multiplication concepts extensively in order to formulate the expression above. Later we will solve this expression for \( \{d\} \).
Powers and Roots of Square Matrices

Because the matrix $A$ is conformable with itself for matrix multiplication we can form powers of the matrix as follows:

$$\{ A \}^n = \{ A \} \{ A \} \{ A \} \cdots \{ A \}$$

In addition it is easy to see that the law of exponents holds

$$\{ A \}^m \{ A \}^n = \{ A \}^{(m+n)}$$

and the zero power of a matrix is the identity matrix. Negative powers of a matrix can be defined as:

$$\{ A \}^{-m} = \left( \{ A \}^{-1} \right)^m$$

and are defined as:

$$\{ A \}^{\frac{1}{m}} = \sqrt[m]{A}$$
Matrix Differentiation

One can differentiate a matrix by differentiating every element of the matrix in the conventional manner. Consider

\[
[A] = \begin{bmatrix}
x^3 & 2x^2 & 3x \\
2x^2 & x^4 & x \\
3x & x & x^5
\end{bmatrix}
\]

The derivative \( \frac{d[a]}{dx} \) of this matrix is

\[
\frac{d[A]}{dx} = \begin{bmatrix}
3x^2 & 4x & 3 \\
4x & 4x^3 & 1 \\
3 & 1 & 5x^4
\end{bmatrix}
\]
Similarly, one can take the partial derivative of a matrix as follows

\[
\frac{\partial [A]}{\partial x} = \frac{\partial}{\partial x} \begin{bmatrix}
  x^3 & y & 2x^2z & 3xy^2 \\
  2x^2z & xz^4 & y \\
  3xy^2 & y & z^5 \\
  3x^2y & 4xz & 3y^2
\end{bmatrix} = \begin{bmatrix}
  3x^2y & 4xz & 3y^2 \\
  4xz & z^4 & 0 \\
  3y^2 & 0 & 0
\end{bmatrix}
\]

In structural analysis we differentiate strain energy potential functions that have the form

\[
U = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix} x \\
  y
\end{bmatrix}
\]

via matrix multiplication

\[
U = \frac{1}{2} (a_{11}x^2 + 2a_{12}xy + a_{22}y^2)
\]
Partial differentiation leads to
\[
\frac{\partial U}{\partial x} = a_{11} x + a_{12} y \\
\frac{\partial U}{\partial y} = a_{12} x + a_{22} y
\]

Or in matrix format
\[
\begin{bmatrix}
\frac{\partial U}{\partial x} \\
\frac{\partial U}{\partial y}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

If
\[
U = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\
a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \{X\}^T [A] \{X\}
Then

\[ \frac{\partial U}{\partial x_i} = [A] \{X\} \]

Here \( x_i \) represents \( x \) and \( y \) using index notation. The above holds only if \([A]\) is a symmetric matrix.
**Matrix Integration**

One can differentiate a matrix by differentiating every element of the matrix in the conventional manner. Consider

\[
[A] = \begin{bmatrix}
3x^2 & 4x & 3 \\
4x & 4x^3 & 1 \\
3 & 1 & 5x^4
\end{bmatrix}
\]

The integration of this matrix is

\[
\int [A] \, dx = \begin{bmatrix}
x^3 & 2x^2 & 3x \\
2x^2 & x^4 & x \\
3x & x & x^5
\end{bmatrix}
\]

We often integrate the expression

\[
\iiint \{X\}^T [A] \{X\} \, dx \, dy \, dz
\]

This triple product will be symmetric if [A] is symmetric
Special Categories of Matrices

A square matrix is said to be a **skew matrix** if all diagonal elements are not zero and

\[ a_{ij} = -a_{ji} \quad i \neq j \]

This matrix becomes skew-symmetric if all diagonal elements are zero. Here

\[ A = -A^T \]

Any matrix can be composed of complex elements

\[
C = \begin{bmatrix} c_{ij} \end{bmatrix} = \begin{bmatrix} a_{ij} + j b_{ij} \end{bmatrix} \quad j = \sqrt{-1}
\]
A complex matrix has a conjugate

\[
\bar{C} = \begin{bmatrix}
\bar{c}_{ij}
\end{bmatrix} = \begin{bmatrix}
a_{ij} & -jb_{ij}
\end{bmatrix}
\]
Other Matrix Terminology

• **Banded matrix**
  – If all non-zero terms are contained within a band along the diagonal, the matrix is said to be banded

\[
\begin{bmatrix}
  a_{11} & a_{12} & 0 & 0 & 0 & 0 & \ldots & 0 \\
  a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \ldots & 0 \\
  0 & a_{32} & a_{33} & a_{34} & 0 & 0 & \ldots & 0 \\
  0 & 0 & a_{43} & a_{44} & a_{34} & 0 & \ldots & 0 \\
  0 & 0 & 0 & a_{54} & a_{55} & a_{34} & \ldots & 0 \\
  0 & 0 & 0 & 0 & a_{65} & a_{66} & \ldots & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & 0 & \ldots & a_{m,m-1} & a_{mm}
\end{bmatrix}
\]

Later you will find that banded matrices are quite common in structural analysis. Special band storage techniques are used to avoid finding space for all the zero entries.
• **Sparse matrix**
  – If a matrix has relatively few non-zero terms (as is common in FEA), the matrix is said to be sparse

• **Singular matrix**
  – If the determinant of the matrix equals zero, the matrix is said to be singular. As we saw earlier, if \([A]\) is singular, then the system of equations \([A]\{x\} = \{b\}\) has no unique solution.