Section 7: Flexibility Method - Trusses

**Plane Trusses**

A plane truss is defined as a two-dimensional framework of straight prismatic members connected at their ends by frictionless hinged joints, and subjected to loads and reactions that act only at the joints and lie in the plane of the structure. The members of a plane truss are subjected to axial compressive or tensile forces.

The objective is to develop the analysis of plane trusses using matrix methods. The analysis must be general in the sense that it can be applied to statically determinate, as well as indeterminate plane trusses. Methods of analysis will be presented based on both global and local coordinate systems.
Applied loads may consist of point loads applied at the joints as well as loads that act on individual truss members. Loads that are applied directly to individual truss members can be replaced by statically equivalent loads acting at the joints.

In addition to the axial loads typically computed for individual truss members bending moments and shear forces will be present in truss members where point loads are applied directly to the member.

Whereas numerous methods are available to compute elastic deformations for structures in general, i.e.,

1. Virtual work method (dummy load)
2. Castigliano’s theorem
3. Conjugate beam method
4. Moment area theorems
5. Double integration method

only the first two are applicable to trusses. So we begin with a review of computing joint displacements in a truss using virtual work.
Virtual Work, Virtual Forces and Trusses

The application of virtual work to any structure demands that the external work done by an applied load system is equal to the internal work stored. Consider the following truss with three loads applied along the top chord.

\[ N_m = N_m(P_1, P_2, P_3) \]

Force deflection curves are presented for each truss member in the figure.
We will focus on a single virtual (imagined) force acting on a truss with actual (real) displacements. When utilizing a virtual force we in essence apply a dummy load $P'$ at the point and in the direction of the desired displacement $\Delta$ in the truss. With the virtual force applied the real force system is then brought onto the structure. As the structure deforms under the real load system the virtual force does external virtual work, i.e.,

$$ U'_{\text{ext}} = P' \Delta $$

as it moves through a real displacement at the point of application of the dummy force in the truss. With loads applied in this sequence, dummy forces first and then the real load system then $D$ in the expression above is a displacement resulting from real loads. The dummy force also contributes to the internal energy of the structure. For a single truss member

$$ U'_{\text{int}} = \int_0^L N' \varepsilon(x) \, dx $$

This is a force displacement statement where $\varepsilon$ is the strain in the actual structure due to the system of applied real loads. Hence

$$ \varepsilon(x) = \frac{N}{AE} $$
and

\[ U_{\text{int}} = \int_{0}^{L} N' \left( \frac{N}{AE} \right) dx \]

Equating internal and external energy from the virtual force leads to the following relationship

\[ U'_{\text{ext}} = U'_{\text{int}} \]

\[ P' \Delta = \int_{0}^{L} N' \left( \frac{N}{AE} \right) dx \]

\[ = N' \left( \frac{N}{AE} \right) \int_{0}^{L} dx \]

\[ = \frac{N' N L}{AE} \]

For convenience we will take (and hence the name “unit load method”)

\[ P' = 1 \]
and this leads to

\[ \Delta = \frac{N' N L}{AE} \]

Applying this to the entire truss leads to

\[ \Delta = \sum_{m=1}^{n} N' N_m \left( \frac{L}{AE} \right)_m \]

where \( n \) is the number of truss members. Keep in mind that \( \Delta \) is the deflection associated with the dummy unit load. We will now carefully define \( N' \) and \( N_m \) in the following example problem in order to ascertain flexibility coefficients in a truss using the dummy unit load method.
Example 7.1
Virtual Work – Castigliano’s Theorem

The internal energy stored in the truss is given by the expression

\[ U_{\text{int}} = \sum_{m=1}^{n} \frac{1}{2} (N_m \Delta_m) \]

Assuming a linear response for each truss member, i.e.,

or

\[ f = k \Delta \]
For each truss member

\[ N_m = k_m \Delta_m = \left( \frac{AE}{L} \right)_m \Delta_m \]

Thus

\[
U_{\text{int}} = \sum_{m=1}^{n} \frac{N_m \Delta_m}{2}
\]
\[
= \sum_{m=1}^{n} \frac{k_m (\Delta_m)^2}{2}
\]
\[
= \sum_{m=1}^{n} \frac{(k_m \Delta_m)^2}{2k_m}
\]
\[
= \sum_{m=1}^{n} \frac{\left( \frac{L}{AE} \right)_m}{2}
\]

External work for this truss is given by the expression

\[
U_{\text{ext}} = \int_{0}^{\Delta_1} P_1 \, d\Delta_1 + \int_{0}^{\Delta_2} P_2 \, d\Delta_2 + \int_{0}^{\Delta_3} P_3 \, d\Delta_3
\]
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For a conservative system

\[ U_{\text{int}} = U_{\text{ext}} \]

\[ \sum_{m=1}^{n} \frac{(N_m)^2}{2} \left( \frac{L}{AE} \right)_m = \int_{0}^{\Delta_1} P_1 \, d\Delta_1 + \int_{0}^{\Delta_2} P_2 \, d\Delta_2 + \int_{0}^{\Delta_3} P_3 \, d\Delta_3 \]

Where \( n \) is the number of truss members. Focusing attention on the forces in each of the truss members it was pointed out on an earlier slide that

\[ N_m = N_m(P_1, P_2, P_3) \]

Some members will not be a function of all three forces, but that can easily be accounted for in a complete truss analysis. If we accept the functional relationship above then

\[ U_{\text{int}} = U_{\text{int}}(P_1, P_2, P_3) \]

Taking the partial derivative of the expression above with respect to \( P_1 \)

\[ \frac{\partial}{\partial P_1} \left[ \int_{0}^{\Delta_1} P_1 \, d\Delta_1 + \int_{0}^{\Delta_2} P_2 \, d\Delta_2 + \int_{0}^{\Delta_3} P_3 \, d\Delta_3 \right] = \frac{\partial}{\partial P_1} \left[ \sum_{m=1}^{n} \frac{(N_m)^2}{2} \left( \frac{L}{AE} \right)_m \right] \]
This is a statement of Castigliano’s first theorem, i.e., the displacement of a joint in a truss can be determined by taking the partial derivative of the internal energy stored in the truss with respect to the force applied to the truss at that joint.
Example 7.2
Example 7.3 – Finding Redundants

The plane truss shown below is statically indeterminate to the second degree. The horizontal reaction at support $B$ (positive to the right) and axial force in bar $AD$ (positive in tension) are selected as redundants. Find these redundants.

The cut bar remains part of the released structure since the deformation in the cut bar must be included in the calculations of displacements in the released structure.
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A displacement corresponding to $Q_2$ consists of the relative translation of the end of bar $AD$. When the ends of bar $AD$ displace toward one another the displacements are in the direction of $Q_2$ and thus are positive. When the joints move away the displacements are negative.

The first step in the analysis is determining the displacements that correspond to $Q_1$ and $Q_2$ in the released structure due to external loads. These displacements are denoted $D_{QL1}$ and $D_{QL2}$ and are depicted below.

Assuming that all the members have the same axial stiffness $EA$, then from application of Castigliano’s theorem

\[
D_{QL1} = -\frac{PL}{EA} \left(1 + 2\sqrt{2}\right)
\]

\[
= -3.828 \frac{PL}{EA}
\]

\[
D_{QL2} = -2 \frac{PL}{EA}
\]

Please verify these quantities for homework. The minus signs indicate that joints $A$ and $D$ move away from each other under the application of the external load in the released structure.
The next step will be the determination of the displacements associated with $Q_1$ and $Q_2$ in the released structure due to unit loads at $Q_1$ and $Q_2$, i.e., determine the flexibility coefficients. The flexibility coefficient $F_{11}$ is the displacement corresponding to $Q_1$ and caused by a unit value of $Q_1$. Thus

$$F_{11} = \frac{L}{EA} \left( 1 + 2\sqrt{2} \right)$$

$$= 3.828 \frac{L}{EA}$$

The flexibility coefficient $F_{21}$ is the displacement corresponding to $Q_2$ and caused by a unit value of $Q_1$. Thus

$$F_{21} = \frac{L}{2EA} \left( 4 + \sqrt{2} \right)$$

$$= 2.707 \frac{L}{EA}$$
The flexibility coefficient \( F_{22} \) is the displacement corresponding to \( Q_2 \) and caused by a unit value of \( Q_2 \). Thus

\[
F_{22} = \frac{2L}{EA} \left( 1 + \sqrt{2} \right)
\]

\[
= 4.828 \frac{L}{EA}
\]

The flexibility coefficient \( F_{12} \) is the displacement corresponding to \( Q_1 \) and caused by a unit value of \( Q_2 \). Thus

\[
F_{12} = \frac{L}{2EA} \left( 4 + \sqrt{2} \right)
\]

\[
= 2.707 \frac{L}{EA}
\]
The flexibility matrix is

\[
[F] = \frac{L}{EA} \begin{bmatrix} 3.828 & 2.707 \\ 2.707 & 4.828 \end{bmatrix}
\]

The inverse of this matrix is

\[
[F]^{-1} = \frac{EA}{L} \begin{bmatrix} 0.4328 & -0.2426 \\ -0.2426 & 0.3431 \end{bmatrix}
\]

There are no support displacements in the truss. Thus the displacement in the structure corresponding to \(Q_1\) is

\[
D_{Q_1} = 0
\]

In addition, the displacement in the structure corresponding to \(Q_2\) consists of a relative displacement of the joints A and D. In the original, or primary structure, the cut ends of bar \(AD\) occupy the same location in space before loads are applied. After loads are applied the cut ends still occupy the same point, however the point moved to another location. So relative to either cut end of the released structure, no translation takes place. Thus

\[
D_{Q_2} = 0
\]
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The compatibility equation is

\[ \{D_Q\} = \{D_{QL}\} + [F]\{Q\} \]

Manipulating this expression and substituting the inverse of the flexibility matrix leads to

\[ \{Q\} = [F]^{-1} \{D_Q\} - \{D_{QL}\} \]

or

\[ \{Q\} = \frac{EA}{L} \begin{bmatrix} 0.4328 & -0.2426 \\ -0.2426 & 0.3431 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{PL}{AE} \begin{bmatrix} -3.828 \\ -2 \end{bmatrix} \]

or

\[ \{Q\} = \frac{EA}{L} \begin{bmatrix} 0.4328 & -0.2426 \\ -0.2426 & 0.3431 \end{bmatrix} \frac{PL}{AE} \begin{bmatrix} 3.828 \\ 2 \end{bmatrix} \]

\[ = P \begin{bmatrix} 1.172 \\ -0.2432 \end{bmatrix} \]

The minus sign for \( Q_2 \) indicates that member \( AD \) is in compression.