Chapter 6: Analog Filter Design

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Ideal Filtering in the Frequency Domain

- Unaliased samples
- Broadband signal to be sampled
- 1/2 of available sampling freq.

Diagram:
- Filter freq. response
- X(jw)
- Unaliased samples
- X(k)
Basic Ideal Filters

<table>
<thead>
<tr>
<th>Type</th>
<th>Frequency Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>Constant gain</td>
</tr>
<tr>
<td>Highpass</td>
<td>Constant gain</td>
</tr>
<tr>
<td>Bandpass</td>
<td>Constant gain</td>
</tr>
<tr>
<td>Bandstop (notch)</td>
<td>Constant gain</td>
</tr>
</tbody>
</table>

Passive Filter Realizations

- Ideal filters cannot be exactly realized due to discontinuities.
- RLC networks can be designed to provide approximate LP, HP, BP and notch characteristics.

For example, the network $SE \rightarrow 1 \rightarrow 0$ has the transfer function

$$\frac{1}{RCs + 1}$$

The Bode plot of this TF rolls off at -20 dB/dec with a corner frequency (-3dB point) of $\frac{1}{\tau} = \frac{1}{RC}$ and has a low-frequency gain of one.

Similarly, the network $SE \rightarrow C \rightarrow 0$ is the mirror image of the previous one. It has a high-pass characteristic.
Pros/Cons of Passive Realizations

- Second order passive stages require inductors for their realization.
- Cascading several stages to increase the order or to create BP or notch characteristics results in output loading and deviation from desired behavior.
- Passive filters, however, do not require power supplies.
- High $R$ is desirable to minimize loading effects in passive filters. For a given corner frequency, increasing $R$ requires a corresponding increase in $C$, usually resulting in large, bulky capacitors.

Real Filter Characteristics

- $|H(j\omega)|$ dB
- $1/\sqrt{2}$ (−3dB)
- Passband
- Transition Band
- Stopband
- Stopband ripple
- Corner (cutoff) frequency
- w
Butterworth Filter

- The Butterworth transfer function has no ripple in the passband or stopband (maximally flat).
- An $N$-th order Butterworth stage has $N$ complex-conjugate pole pairs. $2N$ poles are equally spaced on a circle with radius equal to the corner frequency $w_c$ and centered at the origin. The $N$ stable poles belong to the filter transfer function.
- The normalized ($w_c = 1$) 2nd-order LP Butterworth filter has the TF
  \[ H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \]
- The numerator will always be one, while the denominator for any order can be found in tables or by using Matlab. HP stages are obtained by adding $N$ zeros at zero.

Chebyshev Type I and II Filters

- The Chebyshev filters have a sharper transition but have ripple in either passband (type I) or stopband (type II).
- The ripple is uniform (equiripple) and becomes an extra design specification.
- An $N$-th order Chebyshev stage has $N$ complex-conjugate pole pairs spaced on a half-ellipse centered at the origin. The half-ellipse lies on the left half of the complex plane.
- The normalized ($w_c = 1$) 2nd-order LP Chebyshev Type I filter with 3dB of ripple in the passband has the TF
  \[ H(s) = \frac{1}{1.4142s^2 + 0.911s + 1} \]
- This TF has been normalized to have the same gain as the Butterworth filter for comparison. The numerator will always be one. HP stages are again obtained by adding $N$ zeros at zero.
### Normalized Denominator Polynomials

<table>
<thead>
<tr>
<th>Order</th>
<th>Chebyshev I (3dB ripple)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.00s + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$1.41s^2 + 0.911s + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$3.98s^3 + 2.38s^2 + 3.70s + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$5.65s^4 + 3.29s^3 + 6.60s^2 + 2.29s + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$15.9s^5 + 9.11s^4 + 22.5s^3 + 8.71s^2 + 6.48s + 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Order</th>
<th>Butterworth</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$s^2 + 1.41s + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$s^3 + 2.00s^2 + 2.00s + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$s^4 + 2.61s^3 + 3.41s^2 + 2.61s + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$s^5 + 3.24s^4 + 5.24s^3 + 5.24s^2 + 3.24s + 1$</td>
</tr>
</tbody>
</table>

### Gain and Corner Frequency Adjustment

- To adjust the gain of a filter TF just include a constant in the numerator.

- To adjust the corner frequency in Butterworth or Chebyshev filters use the frequency transformations:
  - **LP to LP**: $s \mapsto s/\omega_c$
  - **LP to HP**: $s \mapsto \omega_c/s$ where $\omega_c$ is the desired corner frequency.

- Additionally, transformations can be defined to produce BP and notch filters from LP.
Op-Amp realization of a LP 2nd-order stage

\[ H(s) = \frac{A}{R_2 C^2 s^2 + RC(3-A)s + 1} \]

\[ A = \frac{R_2 + R_1}{R_2} \]

Op-Amp realization of a HP 2nd-order stage

\[ H(s) = \frac{A R_2^2 C^2 s^2}{R_2 C^2 s^2 + (3-A)RCs + 1} \]

\[ A = \frac{R_1 + R_2}{R_2} \]
Design Procedure - Butterworth

- Select corner frequency based on design requirements (antialiasing, etc.)

- Choose order of filter considering performance/simplicity tradeoff. Matlab’s `butterord` command can find the required order according to specifications.

- For even-order filters, cascade a number of 2nd order stages.

- The corner frequency of a 2nd-order Butterworth stage can be obtained from the standard denominator form \( s^2 + 2\zeta w_c s + w_c^2 \). Based on the previous Op Amp realizations we have

\[
 w_c = \frac{1}{RC}
\]

- Select suitable values of \( R \) and \( C \) to obtain the desired corner frequency. Use the same \( R \) and \( C \) in all stages.

Design Procedure - Butterworth

- Select \( R_1 \) and \( R_2 \) in every stage to obtain the filter gain \( A \) that places the poles evenly spaced on the half circle of radius \( 1/RC \).

- Remember basic controls courses: A pair of complex conjugate poles from \( s^2 + 2\zeta w_c s + w_c^2 \) have magnitude \( w_c \) and form an angle \( \beta = \cos(\zeta) \) with the negative real axis. Therefore, the damping ratio is given by

\[
 \zeta = \frac{3 - A}{2}
\]

- For odd order, cascade an appropriate RC stage. The real pole at \( s = -1/RC \) is one of the poles that need to be evenly spaced along the circle of radius \( 1/RC \).

- The ordering of 2nd order sections is unimportant. However, it is preferable to have the real pole sections “first”. (Why?)

- The total filter gain is the product of individual gains. It can be modified by cascading a noninverting stage.
Examples

1. Design a 4-pole Butterworth filter with $f_c = 10kHz$. Choose values for $R, C, R_1$ and $R_2$. What is the filter gain?
2. Same as above but for $N = 5$. Check the denominator polynomial and plot the filter TF in Matlab.
3. Simulate the above filter by filtering the signal $x(t) = 10 \sin(2\pi \times 10^2 t) + \sin(\pi \times 10^4 t)$