CELL ATTACHMENTS AND THE HOMOLOGY OF LOOP SPACES AND
DIFFERENTIAL GRADED ALGEBRAS

by

Peter G. Bubenik

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Abstract

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Peter G. Bubenik
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
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Let $R$ be a subring of $\mathbb{Q}$ containing $\frac{1}{6}$ or $R = \mathbb{F}_p$ with $p > 3$. Let $(A,d)$ be a differential graded algebra (dga) such that its homology, $H(A,d)$, is $R$-free and is the universal enveloping algebra of some Lie algebra $L_0$. Given a free $R$-module $V$ and a map $d : V \to A$ such that there is an induced map $d' : V \to L_0$, let $B$ denote the canonical dga extension $(A \oplus TV, d)$ where $TV$ is the tensor algebra generated by $V$. $HB$ is studied in the case that the Lie ideal $[d'V] \subset L_0$ is a free Lie algebra. This is a broad generalization of the previously studied inert condition. An intermediate semi-inert condition is introduced under which the algebraic structure of $HB$ is determined.

This problem is suggested by the following cell attachment problem, first studied by J.H.C. Whitehead around 1940. For a simply-connected finite-type CW-complex $X$, denote by $Y$ the space obtained by attaching a finite-type wedge of cells to $X$. Then how is the loop space homology $H_*(\Omega Y; R)$ related to $H_*(\Omega X; R)$? In some cases this topological problem is described by the previous algebraic situation. Under a further hypothesis it is shown that if $H_*(\Omega X; R)$ is generated by Hurewicz images then so is $H_*(\Omega Y; R)$ and if $R \subset \mathbb{Q}$ then the localization of $\Omega Y$ at $R$ is homotopy equivalent to a product of spheres and loop spaces on spheres.

It is well-known that if the coefficient ring is a field, Lie subalgebras of free Lie algebras are also free. To help implement the above results this fact is generalized, giving a simple condition guaranteeing that Lie subalgebras of more general Lie algebras are free.
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Part I

Introduction and Mathematical Review
Chapter 1

Introduction and Summary of Results

In the chapter we introduce and summarize our results and give an outline of the thesis.

1.1 Introduction

In algebraic topology one seeks to understand topological spaces by studying associated algebraic objects. For example each space comes with a set of homology groups and homotopy groups. These groups are invariants of the collection of spaces homotopy equivalent to the given space.

It is often the case that the homology groups are easy to calculate. Unfortunately they do not carry as much information as one might like about the associated space. The homotopy groups on the other hand typically carry much more information but are usually impossible to calculate.

An intermediate approach is to study the loop space homology groups of a given space. Under the Pontrjagin product these invariants have an algebra structure, giving us much more information about the space under consideration [Fel92]. For example, knowing the loop space homology is generated by Hurewicz images may allow one to
write the loop space as a product of spheres and loops on spheres (see Section 4.3). If this happens it gives the homotopy groups of the space in terms of the homotopy groups of spheres, which have been extensively studied.

This thesis presents a method for calculating the loop space homology in some instances of the cell attachment problem which we now describe.

Since topological spaces are weak homotopy equivalent to spaces which are built out of cells (see Section 3.1), there is an obvious need to understand how attaching one or more cells affects the algebraic invariants associated to the space.

The effect on homology is straight-forward. In the torsion-free case attaching a cell will either add or remove one homology generator. However the effect on homotopy and loop space homology is much more mysterious.

**The Cell Attachment Problem:** *Given a simply-connected topological space $X$ what is the effect on the loop space homology and the homotopy type if one attaches one or more cells to $X$?*

In some cases this problem is equivalent to the following purely algebraic problem.

**The Differential Graded Algebra Extension Problem:** *Given a differential graded algebra $(A,d)$ what is the effect on the homology if one adds one or more generators to $(A,d)$?*

This thesis analyzes these problems under some assumptions.

The cell attachment problem was perhaps first considered by J.H.C. Whitehead [Whi41], [Whi39, Section 6], around 1940 and more recently these two problems have been studied by Anick, Félix, Halperin, Hess, Lemaire and Thomas among others.

There are parallel theories analyzing the two problems. For the topological problem, let $X$ be a finite-type simply-connected topological space and let $Y$ be some space obtained by attaching cells to $X$. For the algebraic problem let $(A,d)$ be a finite-type connected differential graded algebra (dga) and let $B$ be some dga obtained by adding generators to $(A,d)$. In the first case let $I$ denote the two-sided ideal of $H_*(\Omega X; R)$
generated by the image of the attaching maps (see Section 4.1). In the second case let $I$ denote the two-sided ideal of $H(A,d)$ generated by the boundaries of the added generators. The conditions under which $H_*(\Omega Y; R) \cong H_*(\Omega X; R)/I$ and $HB \cong H(A,d)/I$ have been extensively studied [Lem78, Ani82b, HL87, FT89, HL95]. This is called the inert condition. The slightly weaker lazy condition and the related nice condition have also been studied [FL91, HL95, HL96]. (We will not use or define lazy, but we will define nice in Remark 1.4 and in Section 4.1.) For more details see Sections 4.1 and 2.4. Most of the results in the above papers were proved for the case where the coefficient ring is a field. We will prove our results for both fields and subrings of the rational numbers.

In this thesis we study a broad generalization of the inert condition which we call the free condition, and which we now define. We assume that $H(A,d)$ is $R$-free and as algebras $H(A,d) \cong UL_0$ for some Lie algebra $L_0$. As explained in the next section this assumption is often satisfied when the algebra extension is derived from a topological cell attachment. Let $J \subset L_0$ be the Lie ideal generated by the boundaries of the added generators. The dga extension $B$ satisfies the free condition if $J$ is a free Lie algebra. If $B$ is a dga extension then it has an obvious filtration, which we define in the next section. If $B$ is a dga extension which satisfies the free condition then we calculate $HB$ as an $R$-module and give the multiplicative structure of the graded object associated to the filtration (see Section 2.1).

For dga extensions which satisfy the free condition we define a semi-inert condition which is a weaker generalization of the inert condition. Under this condition we are able to determine $HB$ including its multiplicative structure.

In case the topological problem for a space $Y$ is equivalent to the algebraic problem for $B$ we obtain analogous results for $H_*(\Omega Y; R)$. Conditions under which this happens are described in the next section. Previously studied special cases include the case where $R = \mathbb{Q}$ and the case where $X$ is a wedge of spheres.

Under further topological conditions we show that if $\Omega X$ is homotopy-equivalent to
a product of spheres and loop spaces of spheres then so is \( \Omega Y \).

It follows from our results that in certain cases the Lie algebra of Hurewicz images automatically contains a free Lie algebra on two generators.

Most of our results depend on showing that certain Lie subalgebras \( J \) of a Lie algebra \( L \) are free Lie algebras. When \( L \) is a free Lie algebra it is a well know fact that \( J \) is a free Lie algebra (when the coefficient ring is a field). However in our cases \( L \) is generally not free. Nevertheless we were able to prove a simple condition under which Lie subalgebras of the given Lie algebras are free.

Our approach to the cell attachment problem generalizes work that has been done in the special case that \( X \) is a wedge of spheres, in which case \( Y \) is called a \((spherical) \ two \ cone\). This situation has been closely studied [Lem74, Ani82a, FHT84, Bøg85, FT86, Ani89, FT89, FHT90, BFT92, FL92, FT93, Pop00]. Our approach is particularly indebted to [Ani89].

We apply our results to various topological examples, construct an infinite family of finite CW-complexes using only semi-inert cell attachments, and give an algebraic version of Ganea’s fiber-cofiber construction.

## 1.2 Summary of results

In this summary we briefly introduce notation and definitions that will be used throughout this thesis. A more detailed exposition will be given in Chapters 2, 3 and 4.

Let \( R \) be a principal ideal domain containing \( \frac{1}{6} \). If \( R \) is a subring of \( \mathbb{Q} \) let \( P \) be the set of invertible primes in \( R \) and let \( \hat{P} = \{ p \in \mathbb{Z} \mid p \text{ is prime and } p \notin P \} \cup \{0\} \). All of our \( R \)-modules \( M \) are graded, connected and have finite type. Even when we do not explicitly mention it, our notation and definitions depend on the choice of \( R \).

Let \((\hat{A}, \hat{d})\) be a differential graded algebra (dga) over \( R \). Let \( Z\hat{A} \) denote the subalgebra of cycles.
Let $V_1$ be a free $R$-module together with a map $d : V_1 \to Z \tilde{A}$. Let $\tilde{T}V_1$ denote the tensor algebra on $V_1$ and let $\tilde{A} \boxtimes \tilde{T}V_1$ denote the coproduct or free product of the algebras $\tilde{A}$ and $\tilde{T}V_1$. $\tilde{d}$ and $d$ can be canonically extended to a differential on the coproduct which we also denote $d$. Let $A = (\tilde{A} \boxtimes \tilde{T}V_1, d)$ be the resulting dga.

There is an increasing filtration $\{F_k A\}$ on $A$ defined by taking $F_{-1} A = 0$, $F_0 A = \tilde{A}$ and for $k \geq 0$, $F_{k+1} = \sum_{i=0}^{k} F_i A \cdot V_1 \cdot F_{k-i} A$. This induces a filtration on $H A$, the homology of $A$. Each of these filtrations will be denoted by $\{F_k(\_\_\_\_)\}$. If $M$ has an increasing filtration $\{F_k M\}$ then there exists an associated graded object

$$\text{gr}(M) = \bigoplus_k (F_k M / F_{k-1} M).$$

Let $\text{gr}_k(M) = F_k M / F_{k-1} M$.

Assume that $H(\tilde{A}, \tilde{d})$ is $R$-free and that as algebras $H(\tilde{A}, \tilde{d}) \cong U L_0$ for some Lie algebra $L_0$. By [Hal92, Theorem 8.3], if $(\tilde{A}, \tilde{d}) \cong U(\tilde{L}, \tilde{d})$ and $H(\tilde{A}, \tilde{d})$ is $R$-free then $H(\tilde{A}, \tilde{d}) \cong U L_0$ as algebras (indeed as Hopf algebras) for some free $R$-module $L_0$. There is an induced map

$$d' : V_1 \to Z \tilde{A} \to H(\tilde{A}, \tilde{d}) \xrightarrow{\cong} U L_0.$$

Assume one can choose $L_0$ so that $d'(V_1) \subset L_0$. In this case we say that $A$ is a \textit{dga extension of} $((\tilde{A}, \tilde{d}), L_0)$, which we will sometimes abbreviate to a \textit{dga extension of} $\tilde{A}$.

Letting $d' L_0 = 0$ one can canonically extend $d'$ to a differential on $L_0 \boxtimes L V_1$. Let $L = (L_0 \boxtimes L V_1, d')$. $L$ is bigraded as a dgL, where the usual grading is called dimension and the second grading, called degree, is given by letting $L_0$ and $V_1$ be in degrees 0 and 1, respectively. The sign conventions in a bigraded dgL use dimension. $d'$ is a differential of bidegree $(-1, -1)$. There is an induced bigrading on $U L$, $H L$ and $H U L$. We record the following observation.

\textbf{Lemma 1.1.} If $\tilde{d} = 0$ and $\tilde{A} \cong U \tilde{L}$ then one can choose $L_0 = \tilde{L}$. Then $A \cong U L$ and hence $A$ and $HA$ are bigraded. As a result $HA \cong \text{gr}(HA)$ as algebras in this case.
We will use the following notation throughout the paper: If $M \subset X$ where $X$ is a Lie algebra or algebra then $[M]$ and $(M)$ are respectively, the Lie ideal and two-sided ideal in $X$ generated by $M$. If $M \subset L$ it is a standard lemma that $U(L/[M]) \cong UL/(M)$ (see [Jac79, p.153; Theorem V.1(4)] for example). If $M$ is bigraded then $M_i$ will denote the component of $M$ in degree $i$. Let $\mathbb{F}_p$ denote the finite field with $p$ elements and we use the convention that $\mathbb{F}_0 = \mathbb{Q}$. For the $R$-module $M$, given any $p \in \tilde{P}$ denote $M \otimes \mathbb{F}_p$ by $\tilde{M}$ and denote $d' \otimes \mathbb{F}_p$ by $\tilde{d}$, with $p$ omitted from the notation.

Given Lie algebras $A$ and $B$ over $R$ such that $B$ is an $A$-module, one can define the semi-direct product $L = A \ltimes B$ as follows. As an $R$-module $L \cong A \times B$. The bracket between $A$ and $B$ is given by the action of $A$ on $B$. Equivalently (see Lemma 2.2) there is a short exact sequence of Lie algebras

$$0 \to B \to L \xrightarrow{g} A \to 0$$

with a Lie algebra section for $g$.

If $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$, define a dga extension of $((\tilde{A}, \tilde{d}), L_0)$, $A = (\tilde{A} \amalg TV_1, d)$, to be free if $[d'V_1] \subset L_0$ is a free Lie algebra. If $R$ is a subring of $\mathbb{Q}$ define a dga extension of $((\tilde{A}, \tilde{d}), L_0)$, $A = (\tilde{A} \amalg TV_1, d)$, to be free if for each $p \in \tilde{P}$, $[d'V_1] \subset L_0$ is a free Lie algebra. In either case we introduce the following new condition. Call $A$ semi-inert if furthermore $\text{gr}_1(\text{HA})$ is a free $\text{gr}_0(\text{HA})$-bimodule. We will show that this is equivalent to both the condition that $(H\mathcal{L})_1$ is a free $(H\mathcal{L})_0$-module and the condition that

$$(H\mathcal{L})_0 \ltimes \mathcal{L}(H\mathcal{L})_1 \cong (H\mathcal{L})_0 \amalg \mathcal{L}K$$

as Lie algebras for some free $R$-module $K \subset (H\mathcal{L})_1$.

This is a generalization of the inert condition (defined in Section 2.4) of Anick [Ani82b] and Halperin and Lemaire [HL87] (see Lemma 2.12). It is a strict generalization. For example we will show (see Example 2.13) that when $L = (\mathcal{L}(x, y, a, b), d)$ where $|x| = |y| = 2$, $dx = dy = 0$, $da = [[x, y], x]$ and $db = [[x, y], y]$, then $UL$ is not an inert
extension of $T(x, y)$ but is a semi-inert extension of $T(x, y)$.

Our first main theorem is the following.

**Theorem A.** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $(\tilde{A}, \tilde{d})$ be a connected finite-type dga and let $V_1$ be a connected finite-type $\mathbb{F}$-module with a map $d : V_1 \rightarrow \tilde{A}$. Let $A = (\tilde{A} \oplus TV_1, d)$. Assume that there exists a Lie algebra $L_0$ such that $H(\tilde{A}, \tilde{d}) \cong U L_0$ as algebras and $d V_1 \subseteq L_0$ where $d'$ is the induced map. Also assume that $[d' V_1] \subseteq L_0$ is a free Lie algebra. That is, $A$ is a dga extension of $((\tilde{A}, \tilde{d}), L_0)$ which is free. Let $L = (L_0 \oplus \mathbb{L} V_1, d')$.

(i) Then as algebras

$$\text{gr}(H A) \cong U((H L)_0 \ltimes \mathbb{L}(H L)_1)$$

with $(H L)_0 \cong L_0/[d' V_1]$ as Lie algebras.

(ii) Furthermore if $A$ is semi-inert then as algebras

$$HA \cong U((H L)_0 \oplus \mathbb{L} K')$$

for some $K' \subseteq F_1 H A$.

**Remark 1.2.** Baues, Félix and Thomas [BFT92, FT93] showed that if $B$ is a two-level dga then $HB \cong \mathbb{T}_{H_0 B} H_1 B$ as algebras. Using Theorem A(i) one can show that $\text{gr}(HA) \cong \mathbb{T}_{\text{gr}_0(HA)} \text{gr}_1 HA$ as algebras.

If $R$ is a subring of $\mathbb{Q}$, then we prove Theorem B, a slightly stronger analogue of Theorem A.

Let $\psi : HL \rightarrow HUL$ be the map induced by the inclusion $L \hookrightarrow UL$. This is a map of Lie algebras where $HUL$ is a Lie algebra under the commutator bracket. Theorem B is stronger than Theorem A in that in addition to the conclusions of Theorem A, we will identify $\underline{\psi} HL$.

**Theorem B.** Let $R \subseteq \mathbb{Q}$ be a subring containing $\frac{1}{6}$. Let $(\tilde{A}, \tilde{d})$ be a connected finite-type dga and let $V_1$ be a connected finite-type free $R$-module with a map $d : V_1 \rightarrow \tilde{A}$. Let
\[ \mathbf{A} = (\hat{A} \amalg TV_1, d). \] Assume that there exists a Lie algebra \( L_0 \) such that \( H(\hat{A}, d) \cong U L_0 \) as algebras and \( d'V_1 \subset L_0 \) where \( d' \) is the induced map. Also assume that \( L_0 / [d'V_1] \) is a free \( R \)-module and that for any \( p \in \hat{P} \), \([d'V_1] \subset L_0 \) is a free Lie algebra. That is, \( \mathbf{A} \) is a dga extension of \(((\hat{A}, d), L_0)\) which is free. Let \( \mathbf{L} = (L_0 \amalg LV_1, d') \).

(i) Then \( H \mathbf{A} \) and \( \text{gr}(H \mathbf{A}) \) are \( R \)-free and as algebras

\[
\text{gr}(H \mathbf{A}) \cong U((H \mathbf{L})_0 \ltimes \text{Lie}(H \mathbf{L})_1)
\]

with \((H \mathbf{L})_0 \cong L_0 / [d'V_1] \) as Lie algebras. Additionally \((H \mathbf{L})_0 \ltimes \text{Lie}(H \mathbf{L})_1 \cong \text{Lie} H \mathbf{L} \) as Lie algebras.

(ii) Furthermore if \( \mathbf{A} \) is semi-inert then as algebras

\[
H \mathbf{A} \cong U((H \mathbf{L})_0 \amalg \mathcal{K}')
\]

for some \( \mathcal{K}' \subset F_1 H \mathbf{A} \).

Let \( M(z) \) denote the Hilbert series (see Section 2.1) for the \( R \)-module \( M \). If \( M(0) = 1 \) then let \( M(z)^{-1} \) denote the series \( 1 / M(z) \). Using Anick’s formula [Ani82a, Theorem 3.7], the following corollary follows from both Theorems A and B.

**Corollary 1.3.** Let \( \mathbf{A} \) be a semi-inert dga extension satisfying the hypotheses of either Theorems A or B. Let \( \mathcal{K}' \) be the \( R \)-module in (ii) of the Theorem. Then

\[
\mathcal{K}'(z) = V_1(z) + z[UL_0(z)^{-1} - U(H \mathbf{L})_0(z)^{-1}] .
\]

(1.1)

The above equation provides a necessary condition for semi-inertness. If a dga extension is semi-inert then the right hand side of (1.1) should give a Hilbert series all of whose terms have non-negative coefficients. While this condition is not sufficient, for a ‘generic’ dga extension which is not semi-inert there is no reason to expect that the terms in the right hand side of (1.1) should have non-negative coefficients.

We now introduce a topological situation for which we prove Theorems C and D analogous to Theorems A and B and, with an additional hypothesis, the stronger Theorems E and F.
Let $X$ and $W$ be simply-connected topological spaces with the homotopy type of a finite-type CW-complex. Using the Samelson product $\pi_*(\Omega X) \otimes R$ is a Lie algebra, and under the commutator bracket $H_*(\Omega X; R)$ is a Lie algebra. Using these products the Hurewicz map $h_X : \pi_*(\Omega X) \otimes R \to H_*(\Omega X; R)$ is a Lie algebra map [Sam53]. Let $L_X$ denote its image. A map $f : W \to X$ induces a map $L_W \to L_X$. Denote the image of this map by $L^W_X$, and let $[L^W_X]$ be the Lie ideal in $L_X$ generated by $L^W_X$. Note that the map $f$ is omitted from the notation.

Consider a continuous map $W \xrightarrow{f} X$, where $W = \bigvee_{j \in J} S^{n_j}$ is a finite-type wedge of spheres and $f = \bigvee_{j \in J} \alpha_j$. The attaching map construction (see Section 3.1) gives the adjunction space

$$Y = X \cup_f \left( \bigvee_{j \in J} e^{n_j+1} \right).$$

Assume that $H_*(\Omega X; R)$ is torsion-free and that $H_*(\Omega X; R) \cong U L_X$ as algebras. If $R$ is a field then there is no torsion and if $R = \mathbb{Q}$ then the latter condition is trivial by the Milnor-Moore Theorem (Theorem 3.5) [MM65]. If $R \subset \mathbb{Q}$ then in many examples one can reduce to the torsion-free case by localizing (see Section 4.3) away from a finite set of primes. McGibbon and Wilkerson [MW86] have shown that any finite CW-complex $X$ such that $\pi_*(X) \otimes \mathbb{Q}$ is finite satisfies both of these conditions after localizing away from finitely many primes (see Section 4.3). Even if the loop space homology has torsion for infinitely many primes [Ani86, Avr86], one might be able to study the space by including it in a larger torsion-free space [Ani89]. If $H_*(\Omega X; R)$ is torsion-free then it is has been conjectured that $H_*(\Omega X; R') \cong U L_X$ [Ani92] where $R'$ is obtained from $R$ by inverting finitely many primes. This thesis will provide methods for verifying this conjecture in certain cases (see Theorem F and Corollary 1.11).

Let $\hat{\alpha}_j : S^{n_j-1} \to \Omega X$ denote the adjoint of $\alpha_j$. Let $\mathbb{L} = (L_X \amalg \mathbb{L}(y_j)_{j \in J}, d')$, where $d'L_X = 0$ and $d' y_j = h_X(\hat{\alpha}_j)$. We will justify reusing the notation $\mathbb{L}$ in Section 4.2. $\mathbb{L}$ is a bigraded differential graded Lie algebra.

If $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$, define the attaching map $f$ to be free if the
Lie ideal $[L^W_X] \subset L_X$ is a free Lie algebra. If $R$ is a subring of $\mathbb{Q}$ define the attaching map $f$ to be free if for each $p \in P$, $[L^W_X]$ is a free Lie algebra. In either case we introduce the following new condition. Call $f$ semi-inert if in addition $(H\mathcal{L})_1$ is a free $(H\mathcal{L})_0$-module. (We will see in Section 4.2 why this corresponds to same terminology defined earlier.) We will show that this is equivalent to the condition that

$$(H\mathcal{L})_0 \ltimes (H\mathcal{L})_1 \cong (H\mathcal{L})_0 \ltimes K$$

as Lie algebras for some free $R$-module $K \subset (H\mathcal{L})_1$. We will also show that a certain Adams-Hilton model (see Section 3.2) provides a filtration for $H_*(\Omega Y; R)$ under which a free attaching map is semi-inert if and only if $\text{gr}_1(H_*(\Omega Y; R))$ is a free $\text{gr}_0(H_*(\Omega Y; R))$-bimodule.

Again this is a generalization of the inert condition [Ani82b, HL87] (see Section 4.1). For example we will show that the attaching map of the top cells in the 6-skeleton of $S^3 \times S^3 \times S^3 (= (S^3_s \vee S^3_b \vee S^3_c) \cup_f (\bigvee_{i=1}^3 e^b))$, where $f = [\ell_b, \ell_c] \vee [\ell_c, \ell_a] \vee [\ell_a, \ell_b]$) is not inert but is semi-inert (see Example 4.8).

In the statements of Theorems C and D below $\text{gr}(H_*(\Omega Y; R))$ refers to the graded object associated to the filtration mentioned above.

The following are two of our four main topological theorems. Recall that $L^X_Y$ is the image of the induced map $L_X \to L_Y$ between Hurewicz images. Corresponding to Theorem A we have

**Theorem C.** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $X$ be a finite-type simply-connected CW-complex such that $H_*(\Omega X; R)$ is torsion-free and as algebras $H_*(\Omega X; R) \cong UL_X$ where $L_X$ is the Lie algebra of Hurewicz images. Let $W = \bigvee_{j \in J} S^{e_j}$ be a finite-type wedge of spheres and let $f : W \to X$. Let $Y = X \cup_f \left( \bigvee_{j \in J} e^{e_j+1} \right)$. Assume that $[L^W_X] \subset L_X$ is a free Lie algebra. That is, $f$ is free.

(i) Then as algebras

$$\text{gr}(H_*(\Omega Y; \mathbb{F})) \cong U(L^X_Y \ltimes (H\mathcal{L})_1)$$
with \( L^X_Y \cong L_X/[I^W_X] \) as Lie algebras.

(ii) Furthermore if \( f \) is semi-inert then as algebras

\[
H_*(\Omega Y; \mathbb{F}) \cong U(L^X_Y \oplus K')
\]

for some \( K' \subset F_1H_*(\Omega Y; \mathbb{F}) \).

The following theorem corresponds to Theorem B.

**Theorem D.** Let \( R \subset \mathbb{Q} \) be a subring containing \( \frac{1}{6} \). Let \( X \) be a finite-type simply-connected CW-complex such that \( H_*(\Omega X; R) \) is torsion-free and as algebras \( H_*(\Omega X; R) \cong UL_X \) where \( L_X \) is the Lie algebra of Hurewicz images. Let \( W = \bigvee_{j \in J} S^{n_j} \) be a finite-type wedge of spheres and let \( f : W \to X \). Let \( Y = X \cup_f \left( \bigvee_{j \in J} e^{n_j+1} \right) \). Assume that \( L_X/[L^W_X] \) is \( R \)-free and that for each \( p \in \bar{P} \), \([L^W_X] \subset \bar{L}_X \) is a free Lie algebra. That is, \( f \) is free.

(i) Then \( H_*(\Omega Y; R) \) and \( gr(H_*(\Omega Y; R)) \) are torsion-free and as algebras

\[
gr(H_*(\Omega Y; R)) \cong U(L^X_Y \ltimes \mathbb{L}(HL)L)_1)
\]

with \( L^X_Y \cong L_X/[L^W_X] \) as Lie algebras.

(ii) If in addition \( f \) is semi-inert then as algebras

\[
H_*(\Omega Y; R) \cong U(L^X_Y \oplus K')
\]

for some \( K' \subset F_1H_*(\Omega Y; R) \).

**Remark 1.4.** Since \( UL_X/(L^W_X) \cong U(L_X/[L^W_X]) \) it follows from (i) in both Theorems C and D that \( UL_X/(L^W_X) \) injects in \( H_*(\Omega Y; R) \). This is the definition of Hess and Lemaire’s [HL96] nice condition for \( f \).

**Remark 1.5.** In Theorems A and B (respectively Theorems C and D) if one assumes that \( H(\bar{A}, \bar{d}) \cong UL_0 \) (respectively \( H_*(\Omega X; R) \cong UL_X \)) as Hopf algebras then it is straightforward to check that the isomorphisms in parts (i) the Theorems are indeed Hopf algebra isomorphisms. We will not need to use this fact.
The following corollary follows from Theorems C and D.

**Corollary 1.6.** If \( f \) is a non-inert free cell attachment satisfying the conditions of either Theorem C or Theorem D and \( \dim(HL)_1 \neq 1 \) then \( H_*(\Omega Y; R) \) contains a tensor algebra on two generators.

**Remark 1.7.** Félix and Lemaire [FL91, Corollary 1.4] showed\(^1\) that for a non-inert free attachment either there exists an \( N \) such that \( \dim H_i(\Omega Y; R) \leq N \) for all \( i \), or \( H_*(\Omega Y; R) \) contains a tensor algebra on two generators.

Let \( \tilde{H}_*(W)(z) \) be the Hilbert series (see Section 2.1) for the reduced homology of \( W \). The following corollary follows from both Theorems C and D.

**Corollary 1.8.** Let \( f \) be a semi-inert attaching map satisfying the hypotheses of either Theorems C or D. Let \( K' \) be the \( R \)-module in (ii) of the Theorem. Then

\[
K'(z) = \tilde{H}_*(W)(z) + z[U L_X(z)^{-1} - U(L_Y^X)(z)^{-1}].
\]  

(1.2)

Like Corollary 1.3, this corollary provides a necessary condition for an attaching map to be semi-inert. Note that \( W \) and \( L_X \) are already known and \( L_Y^X \cong L_X/[L_X^W] \), which in some cases is easy to calculate.

We now proceed to our next two main topological theorems.

The following is an important collection of spaces. Let \( \mathcal{S} = \{S^{2m-1}, \Omega S^{2m+1} | m \geq 1 \} \). A space \( Y \) is called **atomic** if for some \( r \) it is \( r \)-connected, \( \pi_{r+1}(Y) \) is a cyclic abelian group, and any self-map \( f : Y \to Y \) inducing an isomorphism on \( \pi_{r+1}(Y) \) is a homotopy equivalence. The spaces in \( \mathcal{S} \) are atomic. Let \( \prod \mathcal{S} \) be the collection of spaces homotopy equivalent to a **weak product** (see Section 3.1 for the definition) of spaces in \( \mathcal{S} \).

Let \( W, X, \) and \( Y \) be the spaces in Theorems C and D. If in addition to the hypotheses of Theorems C and D there exists a Lie algebra map \( \sigma_X : L_X \to \pi_*(\Omega X) \otimes R \) right inverse to \( h_X \) then we also have the stronger topological results below. In the case

\(^1\)The trivial case, which is illustrated by Example 2.13, is omitted in [FL91].
where $X$ is a wedge of spheres, studied by Anick [Ani89], such a map always exists (see Example 10.3). In Theorem F we will give a sufficient conditions for extending such a map from a subspace, but for other cases its existence is an open problem. When $R \subseteq \mathbb{Q}$, in Section 4.4 we will use such a map to define the set of implicit primes labeled $P_Y$ which contains the invertible primes in $R$. Intuitively the implicit primes are those primes $p$ for which $p$-torsion is used in the cell-attachments. If $W$ is finite and the set of invertible primes in $R$ is finite then $P_Y$ is finite as well (see Lemma 4.13). The next two theorems are analogues of the fundamental Milnor-Moore Theorem (Theorem 3.5) which holds for $R = \mathbb{Q}$.

**Theorem E.** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $\bigvee_{j \in J} S^{n_j} \xrightarrow{\alpha_j} X$ be a cell attachment satisfying the hypotheses of Theorem C. Let $Y = X \cup \bigvee_{\alpha_j} (\vee e^{n_j+1})$ and let $\hat{\alpha}_j$ denote the adjoint of $\alpha_j$. In addition assume that there exists a map $\sigma_X$ right inverse to the Hurewicz map $h_X$ and that $\forall j \in J, \sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$. Then the canonical algebra map

$$UL_Y \to H_*(\Omega Y; \mathbb{F})$$

is a surjection. That is, $H_*(\Omega Y; \mathbb{F})$ is generated as an algebra by Hurewicz images.

Let $R \subseteq \mathbb{Q}$ be a subring with invertible primes $P$ and let $X$ be a simply-connected topological space. Then there is a topological analogue to the localization of $\mathbb{Z}$-modules at $R$ (see Section 4.3). We will call this the localization at $R$ or the localization away from $P$. We denote the localization of $X$ at $R$ by $X(R)$.

**Remark 1.9.** If Theorem E holds for $\mathbb{Q}$ and for $\mathbb{F}_p$ for all non-invertible primes in some $R \subseteq \mathbb{Q}$ containing $\frac{1}{6}$ and $H_*(\Omega Y; R)$ is torsion-free then by the Hilton-Serre-Baues Theorem (Theorem 4.12) $H_*(\Omega Y; R) \cong UL_Y$ as algebras and localized at $R$, $\Omega Y \in \prod S$.

**Theorem F.** Let $R \subseteq \mathbb{Q}$ be a subring containing $\frac{1}{6}$. Let $\bigvee_{j \in J} S^{n_j} \xrightarrow{\alpha_j} X$ be a cell attachment satisfying the hypotheses of Theorem D. Let $Y = X \cup \bigvee_{\alpha_j} (\vee e^{n_j+1})$. Furthermore assume that there exists a map $\sigma_X$ right inverse to the Hurewicz map $h_X$. Let $P_Y$
be the set of implicit primes and let $R' = \mathbb{Z}[P_Y^{-1}]$. Then

(i) $H_*(\Omega Y; R')$ is torsion-free and as algebras

$$H_*(\Omega Y; R') \cong UL_Y$$

where $\text{gr}(L_Y) \cong L_Y^X \times \mathbb{L}(H_L)_1$ as Lie algebras, and

(ii) localized away from $P_Y$, $\Omega Y \in \prod S$.

(iii) If in addition $f$ is semi-inert then localized away from $P_Y$, $L_Y \cong L_Y^X \mathbb{L}\mathbb{K}$ as Lie algebras for some $\mathbb{K} \subset F_1L_Y$, and there exists a map $\sigma_Y$ right inverse to $h_Y$.

**Remark 1.10.** By the Hilton-Serre-Baues Theorem (Theorem 4.12), the assumption that $H_*(\Omega X; R)$ is torsion-free and that $H_*(\Omega X; R) \cong UL_X$ as algebras, is equivalent to the assumption that localized at $R$, $\Omega X \in \prod S$. Therefore under the conditions of Theorem F, if $\Omega X(R) \in \prod S$ and there are no new invertible primes then $\Omega Y(R) \in \prod S$.

This theorem is important for the following reason. If we know that $\Omega Y \in \prod S$ then we can apply results about the homotopy groups of spheres, which have been extensively studied, to make some strong conclusions about $Y$. For example, localized away from $P_Y$, $Y$ satisfies the *Moore conjecture* (Conjecture 4.9) [Sel88] which relates the torsion of the homotopy groups of $Y$ to the torsion-free behaviour of the $\pi_*(Y)$. Furthermore, it creates the possibility that one might be able to demonstrate the Moore conjecture for other spaces by following Anick’s plan [Ani89] of embedding them in spaces for which $\Omega Y \in \prod S$.

The following two corollaries follow from Theorem F.

**Corollary 1.11.** If $Y$ is a finite complex constructed out of semi-inert attachments then there is a finite set of primes $P_Y$ such that localized away from $P_Y$, $\Omega Y$ is homotopy equivalent to a product of spheres and loops on spheres. Equivalently $H_*(\Omega Y; \mathbb{Z}[P_Y^{-1}]) \cong UL_Y$.

**Corollary 1.12.** If $f$ is a non-inert free cell attachment satisfying the conditions of Theorem F and $\dim(H_L)_1 \neq 1$ then $L_Y$ contains a free Lie algebra on two generators.
CHAPTER 1. INTRODUCTION AND SUMMARY OF RESULTS

To apply our results one needs to know that certain Lie ideals are free Lie algebras. This is difficult to check in general. However, it is a well-known result that over a field \( \mathbb{F} \), any Lie subalgebra of a free (graded) Lie algebra is a free Lie algebra [Sir53, Mik85, MZ95]. This is referred to as the Schreier property [MZ95].

Unfortunately our Lie ideals are in Lie algebras that are generally not free so we cannot use this property. By the results of this thesis, our ideals are often in Lie algebras whose associated graded object is a certain semi-direct product. For this more general situation we have succeeding in giving a simple condition from which one can conclude that a given Lie ideal is in fact a free Lie algebra.

**Theorem G.** Over a field \( \mathbb{F} \), let \( L \) be a finite-type graded Lie algebra with filtration \( \{ F_k L \} \) such that \( \text{gr}(L) \cong L_0 \ltimes \mathbb{L} V_1 \) as Lie algebras, where \( L_0 = F_0 L \) and \( V_1 = F_1 L / F_0 L \). Let \( J \subset L \) be a Lie subalgebra such that \( J \cap F_0 L = 0 \). Then \( J \) is a free Lie algebra.

As an application of our results we obtain an algebraic analogue of Ganea’s fiber-cofiber construction, which we discuss in Chapter 9. Like Ganea’s construction, our construction can be iterated.

We apply our results to various topological examples in Chapter 10. For example we illustrate our results by calculating the loop space homology of a particular spherical three-cone and show that its loop space is homotopy equivalent to a product of spheres and loop spaces on spheres. By induction we produce an infinite family of finite CW-complexes constructed out of semi-inert cell attachments. We also give a more abstract analysis of (spherical) two-cones, three-cones and \( N \)-cones, as well as some other examples.

1.3 Outline of the thesis

In Chapter 2 we review some algebraic constructions and introduce the differential graded algebra (dga) extension problem.
In Chapter 3 we review some topology, introduce Adams-Hilton models [AH56] and prove some of their properties which will be needed in Chapter 7.

In Chapter 4 we introduce the cell attachment problem and compare it to the previously discussed dga extension problem. We also review some topological theory.

In Chapter 5 we prove Theorems A and B, the main algebraic theorems.

In Chapter 6 we prove Theorems C and D, the topological analogues of Theorems A and B.

In Chapter 7 we prove Theorems E and F which identify Hurewicz images.

In Chapter 8 we prove Theorem G which will be useful in applying our results to examples.

In Chapter 9 we show that our results can be applied to give an algebraic construction analogous to Ganea’s Fiber-Cofiber construction [Gan65].

In Chapter 10 we apply our results to some topological examples and conclude with some open questions.
Chapter 2

Differential Graded Algebra

In this chapter we review some standard algebraic constructions and results. We also motivate and define our main algebraic objects of study: \textit{dga extensions} which are \textit{free} and \textit{semi-inert}. Furthermore we introduce the spectral sequence with which we study these objects.

Let \( R \) be a principal ideal domain containing \( \frac{1}{2} \) and \( \frac{1}{5} \). This is equivalent to saying that \( R \) is a principal ideal domain containing \( \frac{1}{6} \). This assumption is made so that the \textit{Hurewicz images} (defined in Section 3.1) have the structure of a \textit{(graded) Lie algebra} (defined in Section 2.1).

2.1 Filtered, differential and graded objects

A \textit{graded} \( R \)-module is an \( R \)-module \( M \) such that \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) where \( M_i \) is an \( R \)-module. All of our \( R \)-modules will be assumed to be graded. \( R \) itself is a graded \( R \)-module concentrated in degree 0. If \( a \in M_i \) let \( |a| = i \). \( M \) is said to have \textit{finite type} if for all \( i \), \( M_i \) is a finite dimensional \( R \)-module. \( M \) is said to be connected if \( M_i = 0 \) for \( i \leq 0 \). Since \( R \) is a principal ideal domain, \( M \) is a \textit{free} \( R \)-module if it is torsion-free. That is, for \( m \in M, r \in R, rm = 0 \implies r = 0 \). Equivalently we will also say that \( M \) is \textit{R-free}. We will always assume that our \( R \)-modules are free and have finite type. Given
a set \( S \), let \( RS \) be the free \( R \)-module with basis \( S \). A \textit{graded \( R \)-module map} preserves the gradation and the \( R \)-module structure. That is, \( f(r \cdot m) = r \cdot f(m) \) and \( |f(a)| = |a| \). All of our maps will be assumed to be graded \( R \)-module maps.

Though all of our definitions depend on \( R \), we will usually leave this dependence implicit.

For a torsion-free graded \( R \)-module \( M \), the \textit{Hilbert series} for \( M \) is the power series \( M(z) = \sum_{i \in \mathbb{Z}} (\text{Rank}_R M_i) z^i \).

A \textit{(graded) Lie algebra} is a graded \( R \)-module \( L \) together with a linear map \([ \ , \ ] : L_i \otimes L_j \to L_{i+j}\) (called the bracket) satisfying the following relations:

1. \( [x, y] = -(-1)^{|x||y|}[y, x] \) (anti-commutativity) and

2. \( [x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] \) (the Jacobi identity).

Note that since \( \frac{1}{2} \in R \), \( [x, x] = 0 \) when \( |x| \) is even and \( [[x, x], x] = 0 \) for all \( x \).

A \textit{Lie algebra map} is a map between Lie algebras which preserves the Lie algebra structure. That is, \( f([a, b]) = [f(a), f(b)] \). A Lie algebra is said to be \textit{connected} if it is connected as an \( R \)-module. A \textit{Lie subalgebra} \( J \subseteq L \) is an \( R \)-submodule which is closed under the bracket. That is, for \( a, b \in J \), \([a, b] \in J \). \( J \) is a \textit{Lie ideal} of \( L \) if furthermore for all \( a \in J \) and for all \( b \in L \), \([a, b] \in J \). If \( J \) is a Lie ideal then there is an induced Lie algebra structure on the quotient \( L/J \).

We will use the following notation for iterated brackets

\[
[[a_1, a_2, \ldots a_{n-1}, a_n] = [[\cdots [a_1, a_2], \ldots a_{n-1}], a_n].
\]

If \( J \) is the Lie ideal generated by \( a \in L \) we will sometime denote \( L/J \) by \( L/a \).

A \textit{(graded) algebra} is a graded \( R \)-module \( A \) together with \( R \)-module maps \( \mu : A_i \otimes A_j \to A_{i+j} \) (multiplication) and \( \nu : R \to A_0 \) (unit) such that:

1. the composite \( A \cong A \otimes R \xrightarrow{A \otimes \nu} A \otimes A \xrightarrow{\mu} A \) equals \( 1_A \);

2. the composite \( A \cong R \otimes A \xrightarrow{\nu \otimes A} A \otimes A \xrightarrow{\mu} A \) equals \( 1_A \); and
3. \( \mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu) : A \otimes A \otimes A \to A. \)

We remark that \( \mu(a \otimes b) \) is usually denoted by \( a \cdot b \) or just \( ab \). An algebra \( A \) is said to be connected if \( A_i = 0 \) for \( i < 0 \) and \( A_0 \cong R \). An algebra map is a map between algebras which preserves the algebra structure. That is, \( f(a \cdot b) = f(a) \cdot f(b) \).

We will assume that all of our \( R \)-modules, Lie algebras and algebras are connected, \( R \)-free and have finite type.

A subalgebra \( B \subset A \) is an \( R \)-submodule which is closed under multiplication (that is, for \( a, b \in B \), \( \mu(a \otimes b) \in B \)) and which contains the unit (that is, \( \nu(R) \subset B \)). A two-sided ideal of \( A \) is an \( R \)-submodule \( I \subset A \) such that for all \( a \in A \) and for all \( b \in I \), \( \mu(a \otimes b) \in I \) and \( \mu(b \otimes a) \in I \).

Let \( T \) denote the (twist) map \( V \otimes W \to W \otimes V \) given by \( v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \).

A (graded, cocommutative) Hopf algebra is an algebra \( H \) together with \( R \)-module maps \( \Delta : H \to H \otimes H \) (comultiplication) and \( \epsilon : H \to R \) (counit) such that

1. the composite \( H \xrightarrow{\Delta} H \otimes H \xrightarrow{H \otimes \epsilon} H \otimes R \cong H \) equals \( 1_H \);

2. the composite \( H \xrightarrow{\Delta} H \otimes H \xrightarrow{\epsilon \otimes H} R \otimes H \cong H \) equals \( 1_H \);

3. \( (\Delta \otimes 1_H) \circ \Delta = (1_H \otimes \Delta) \circ \Delta : H \to H \otimes H \otimes H \);

4. \( T \circ \Delta = \Delta : H \to H \otimes H \) and

5. \( \Delta \) is a homomorphism of algebras, where the multiplication on \( H \otimes H \) is given by

\[
H \otimes H \otimes H \otimes H \xrightarrow{1 \otimes T \otimes 1} H \otimes H \otimes H \otimes H \xrightarrow{\mu \otimes \mu} H \otimes H.
\]

A differential (graded) \( R \)-module (dgm) is a \( R \)-module \( M \) together with a linear map \( d : M_i \to M_{i-1} \) such that \( d^2 = 0 \), called the differential. Equivalently we say \( M \) is an \( R \)-module with a differential. A differential graded Lie algebra (dgL) is a Lie algebra \( L \) together with a differential \( d \) such that \( d[x, y] = [dx, y] + (-1)^{|x|}[x, dy] \) (the Leibniz rule).

A differential graded algebra (dga) is an algebra \( A \) together with a differential \( d \) such that \( d(xy) = (dx)y + (-1)^{|x|}x(dy) \) (also the Leibniz rule).
A filtered dgm is a dgm $M$ together with a filtration $\{F_iM\}_{i \in \mathbb{Z}}$ such that $\ldots \subset F_iM \subset F_{i+1}M \subset \ldots \subset M$ and $d(F_iM) \subset F_iM$. A filtered dgL is a dgL which has a dgm filtration such that $[F_iL, F_jL] \subset F_{i+j}L$. A filtered dga is a dga which has a dgm filtration such that $F_iA \cdot F_jA \subset F_{i+j}A$. A map of filtered objects is a map $f : M \to N$ such that $f(F_iM) \subset F_iN$. That is, $f$ preserves the filtration. Given a filtered dgm there exists an associated graded object
\[
\text{gr}(M) = \bigoplus_{i \in \mathbb{Z}} F_iM/F_{i-1}M,
\]
which becomes a dgm with the differential induced from the differential on $M$. If $L$ is a filtered dgL then the Lie bracket on $L$ induces a Lie algebra structure on $\text{gr}(L)$. Similarly if $A$ is a filtered dga then the product on $A$ induces an algebra structure on $\text{gr}(A)$. If $f : M \to N$ is a filtered map then there exists an induced map $\text{gr}(f) : \text{gr}(M) \to \text{gr}(N)$. We will sometimes refer to $\text{gr}(M)$ by $\text{gr}_*(M)$ where $\text{gr}_i(M) = F_iM/F_{i-1}M$.

An $R$-module $M$ is bigraded if it has a second grading. That is, $M = \bigoplus_{i,j \in \mathbb{Z}} M_{i,j}$ where $M_{i,j}$ is an $R$-module. A (graded) algebra is bigraded if it is bigraded as an $R$-module and $\mu : A_{i,j} \otimes A_{k,l} \to A_{i+j,k+l}$. In this thesis we will say that a (graded) Lie algebra is bigraded if it is bigraded as an $R$-module and $[\cdot, \cdot] : L_{i,j} \otimes L_{k,l} \to L_{i+j,k+l}$. Note that the anti-commutativity and Jacobi relations need not hold with respect to the second grading. A differential bigraded $R$-module is a bigraded $R$-module with a map $d : M_{i,j} \to M_{i-1,j-1}$ such that $d^2 = 0$. We will say that a dga/dgL is bigraded if it is a bigraded algebra/Lie algebra and a differential bigraded $R$-module. Note that the Leibniz rule need not hold for the second grading.

We will usually denote the first grading by dimension and the second grading by degree. The dimension and degree of $a$ will be denoted by $|a|$ and $\deg(a)$ respectively. The anti-commutativity and Jacobi relations and the Leibniz rule all use $| \cdot |$ and not $\deg(\cdot)$.

A short exact sequence of $R$-modules is a sequence of $R$-modules $A, B, C$ together
with \( R \)-module maps
\[
0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
\]
such that \( \ker(f) = 0 \), \( \ker(g) = \text{im}(f) \), and \( \text{im}(g) = 0 \). It is a short exact sequence of Lie algebras if each of the modules is a Lie algebra and each of the maps is a Lie algebra map. The short exact sequence of Lie algebras \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) is said to split if there exists a Lie algebra map \( h : C \to B \) such that \( g \circ h = \text{id}_C \).

## 2.2 Free algebras and Lie algebras and products

Let \( V \) be a free \( R \)-module. Let \( TV \) be the free algebra generated by \( V \). As an \( R \)-module \( TV \cong \bigoplus_{k=0}^{\infty} T^k V \) where \( T^0 V = R \) and for \( k \geq 1 \), \( T^k V = V \otimes \cdots \otimes V \). The unit is the inclusion \( R = T^0 V \hookrightarrow TV \). Multiplication \( T^i V \otimes T^j V \to T^{i+j} V \) is given by \( a \cdot b = a \otimes b \). Any algebra of this form is called a tensor algebra.

Given a free \( R \)-module \( V \) we define the free commutative algebra or symmetric algebra by \( SV = TV/I \), where \( I \) is the two-sided ideal generated by the commutator brackets \([a, b] = a \cdot b - (-1)^{|a||b|} b \cdot a\).

Recall that the commutator bracket gives any algebra the structure of a Lie algebra. In particular \( TV \) is a Lie algebra with the commutator bracket. Let \( LV \) be the Lie subalgebra of \( TV \) generated by \( V \). We call any Lie algebra of this form a free Lie algebra.

The coproduct between two algebras and the coproduct between two Lie algebras is defined categorically [HS97]. One can check that the following constructions satisfy the definitions. Given two tensor algebras \( TV \) and \( TW \) we define the coproduct or free product \( TV \amalg TW = T(V \oplus W) \). Similarly we define \( LV \amalg LW = L(V \oplus W) \). More generally given two algebras \( A \) and \( B \) we can define the coproduct \( A \amalg B \) to be the free \( R \)-module on words of the form \( a_1 b_1 a_2 b_2 \cdots a_n b_n, a_1 b_1 a_2 b_2 \cdots b_{n-1} a_n, b_1 a_1 \cdots a_n b_n \) and \( b_1 a_1 \cdots b_{n-1} a_n \) (where \( a_i \in A \) and \( b_i \in B \)) modulo the relations \( wa_k 1a_{k+1}w' = w(a_k a_{k+1})w' \) and \( wb_k 1b_{k+1}w' = w(b_k b_{k+1})w' \) where \( w, w' \) are words in the coproduct.
Multiplication is given by concatenation. Similarly we define $L_1 \ll L_2$ for two Lie algebras $L_1$ and $L_2$.

Given a Lie algebra $L_0$, an $R$-module $M$ is an $L_0$-module if there exists a map $[\cdot, \cdot] : L_0 \otimes M \to M$ such that $[x_1, [x_2, y]] = [[x_1, x_2], y] + (-1)^{|x_1||x_2|}[x_2, [x_1, y]]$.

**Remark 2.1.** Given a Lie algebra $L_0$ and a $L_0$-module $M$ then $\mathbb{L}M$ is also a $L_0$-module where the action of $L_0$ on $\mathbb{L}M$ is given inductively by the Jacobi identity and the action of $L_0$ on $M$. If $x \in L_0$ and $[[y_1, \ldots, y_{n-1}, y_n] \in M$ then

$$[x, [[[y_1, \ldots, y_{n-1}, y_n]]] = [[x, [[[y_1, \ldots, y_{n-1}], y_n]] + (-1)^{|x||y|}[[[y_1, \ldots, y_{n-1}], [x, y_n]].$$

Given a Lie algebra $L_0$ and a $L_0$-module $N$ we define the semi-direct product $L = L_0 \ltimes N$ to be the following Lie algebra. As an $R$-module $L \cong L_0 \times N$. For $a, b \in L_0$ (respectively $N$), $[a, b]$ is given by the Lie bracket in $L_0$ (respectively $N$). For $x \in L_0$ and $y \in N$, $[x, y]$ is given by the action of $L_0$ on $N$.

The direct product $L_0 \times N$ is the special case of the semi-direct product where $[x, a] = 0$ if $x \in L_0$ and $a \in N$.

**Lemma 2.2.** Let $L$ be a Lie algebra. Then $L$ is a semi-direct product $L_0 \ltimes N$ if and only if there exists a split short exact sequence of Lie algebras

$$0 \to N \xrightarrow{i} L \xrightarrow{\rho} L_0 \to 0.$$  

*Proof.* ($\Rightarrow$) Since $L_0$ acts on $N$. That is, $[L_0, N] \subset N$, $N$ is a Lie ideal. Thus there exists a quotient map $\rho : L \to L_0$. Hence the inclusion and projection give a short exact sequence of Lie algebras

$$0 \to N \to L \xrightarrow{\rho} L_0 \to 0. \quad (2.1)$$

Let $h : L_0 \to L$ be the inclusion map. Then $\rho \circ h = id_{L_0}$. So (2.1) is a split short exact sequence of Lie algebras.

($\Leftarrow$) Since $0 \to \mathbb{L} \xrightarrow{i} L \xrightarrow{\rho} L_0 \to 0$ is a short exact sequence of $R$-modules, $L \cong L_0 \times N$ as $R$-modules. Let $h : L_0 \to L$ be the splitting. Since $g \circ h = id_{L_0}$, $h$ is an injection.
Since \( f \) and \( h \) are injections of Lie algebras, for \( a, b \in L_0 \) (respectively \( N \)), \( [a, b] \) within \( L \) is given by the bracket in \( L_0 \) (respectively \( N \)). Since \( g \) is a Lie algebra map, \( \ker(g) \) is a Lie ideal. Since \( f \) is an injection \( N = \text{im}(f) = \ker(g) \) and \( N \) is a Lie ideal. Therefore \([L_0, N] \subset N\). That is, \( N \) is a \( L_0 \)-module. So \( L \cong L_0 \ltimes N \). 

\( \square \)

**Lemma 2.3.** Assume \( L_1 \) is a \( L_0 \)-module. Given a Lie algebra map \( u_0 : L_0 \to L \) and an \( L_0 \)-module map \( u_1 : L_1 \to L \) (e.g. for \( x \in L_0, y \in L_1 \), \( u_1([y, x]) = [u_1(y), u_0(x)] \)) then there exists an induced Lie algebra map \( u : L_0 \ltimes \mathbb{L}L_1 \to L \).

**Proof.** Since \( L_1 \) is a \( L_0 \)-module, by Remark 2.1 so is \( \mathbb{L}L_1 \). Since there exists a map \( u_1 : L_1 \to L \) there is a canonical induced Lie algebra map \( \tilde{u} : \mathbb{L}L_1 \to L \) (given by \( \tilde{u}|_{L_1} = u_1 \) and otherwise defined inductively by \( \tilde{u}([a, b]) = [\tilde{u}(a), \tilde{u}(b)] \)). It is easy to prove by induction that this is an \( L_0 \)-module map. That is, \( \tilde{u}([y, x]) = [\tilde{u}(y), u_0(x)] \).

Since \( L_0 \ltimes \mathbb{L}L_1 \cong L_0 \ltimes \mathbb{L}L_1 \) as \( R \)-modules this defines a map \( u : L_0 \ltimes \mathbb{L}L_1 \to L \). Finally for \( x \in L_0 \) and \( y \in \mathbb{L}L_1 \), \([x, y] \in \mathbb{L}L_1 \) so \( u([x, y]) = \tilde{u}([x, y]) = [u_0(x), \tilde{u}(x)] = [u(x), u(y)] \).

Thus \( u \) is a Lie algebra map. \( \square \)

### 2.3 Universal enveloping algebras

Let \( L \) be a Lie algebra. Define \( UL \) the universal enveloping algebra on \( L \) as follows. \( UL = \mathbb{T}L/I \) where \( I \) is the two-sided ideal generated by \( \{x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \} \).

If \( L \) is a dgL then \( \mathbb{T}L \) is a dga under the induced differential: \( d(ab) = (da)b + (-1)^{|a|}a(db) \).

This dga structure induces a dga structure on \( UL \). Similarly, if \( L \) is filtered there is an induced filtration on \( \mathbb{T}L \) which induces a filtration on \( UL \).

It is a basic result [Jac79, p.168; Theorem V.7] that \( ULV \cong TV \).

The commutator bracket \( [x, y] = xy - (-1)^{|y||x|}yx \) gives any algebra the structure of a Lie algebra.

The universal enveloping algebra is universal in the following sense. Given any algebra
(dga) $A$ and a Lie algebra (dgL) map $f$

\[
\begin{array}{c}
L \\ f \\ \downarrow \\
\mathbb{U}L \\
g \\
A
\end{array}
\]

there exists a unique map $g$ making the diagram commute. In fact $\mathbb{U}L$ can be defined as the unique object satisfying this universal property.

The Poincaré-Birkhoff-Witt Theorem is the central theorem in the study of universal enveloping algebras (see for example [Jac79, Sel97]). We give a weak form of this Theorem.

**Theorem 2.4 (Poincaré-Birkhoff-Witt Theorem).** Let $L$ be a Lie algebra. Let $SL$ denote the free commutative algebra (symmetric algebra) (see Section 2.2) on the $R$-module $L$. Then there is an $R$-module isomorphism $SL \cong \mathbb{U}L$ which restricts to the identity on $L$.

**Remark 2.5.** Although we will only need the weak form given above, the full theorem specifies that is isomorphism is an isomorphism of coalgebras.

**Lemma 2.6.** If $L$ is a filtered Lie algebra such that $\text{gr}(L) \cong L$ as $R$-modules then the canonical map $U \text{gr}(L) \to \text{gr}(\mathbb{U}L)$ is an isomorphism.

**Proof.** If $L$ is filtered then $\mathbb{U}L$ has an induced filtration and the inclusion $L \hookrightarrow \mathbb{U}L$ preserves the filtration. This induces a map $\text{gr}(L) \to \text{gr}(\mathbb{U}L)$ which induces a map $U \text{gr}(L) \to \text{gr}(\mathbb{U}L)$. Similarly there is an induced map $S\text{gr}(L) \to \text{gr}(SL)$.

Using the Poincaré-Birkhoff-Witt Theorem, these fit into the following commutative diagram of $R$-linear morphisms.

\[
\begin{array}{c}
S\text{gr}(L) \\ \text{gr}(L) \\ \downarrow \cong \\
U \text{gr}(L) \\
\end{array} \xymatrix{ & \text{gr}(SL) \\
S\text{gr}(L) \ar[ru]_{\phi} \ar[ld]_{\cong} & \\
U \text{gr}(L) \ar[ru]_{\cong} & \\
\text{gr}(UL) \ar[lu]_{\cong} & \\
}
\]
Since $SL$ depends only on the $R$-module structure on $L$, the $R$-module isomorphism $L \cong \text{gr}(L)$ induces an algebra isomorphism

$$SL \cong S \text{gr}(L). \quad (2.2)$$

From this we get the following canonical commutative diagram.

$$\begin{array}{ccc}
\text{gr}(L) & \longrightarrow & \text{gr}(SL) \\
\downarrow & & \downarrow \\
S \text{gr}(L) & \cong & \text{gr}(S \text{gr}(L))
\end{array}$$

Thus $\phi$ is an isomorphism, and therefore the canonical map $U \text{gr}(L) \to \text{gr}(UL)$ is an isomorphism. \hfill $\square$

Let $L$ be a differential graded Lie algebra. The canonical inclusion $\iota : L \to UL$ induces a map on homology $H(\iota) : HL \to HUL$ which induces the natural map

$$\psi : UHL \to HUL.$$ 

Since $H(\iota)$ is the restriction of $\psi$ to $HL$ we will usually also refer to this map as $\psi$.

If $R = \mathbb{Q}$ then Quillen [Qui69] showed that $\psi : UHL \to HUL$ is an isomorphism. We can also state this result as follows:

$$HUL \cong U\psi(HL), \text{ when } R = \mathbb{Q}.$$ 

In Theorems A and B we show that under certain hypotheses $HUL$ can be calculated from $HL$ for more general coefficient rings. See [Pop99] for other results of this type.

### 2.4 Inert, free and semi-inert dga extensions

Let $R = \mathbb{Q}$ or $\mathbb{F}_p$ where $p > 3$ or $R \subset \mathbb{Q}$ is a subring containing $\frac{1}{6}$. If $R \subset \mathbb{Q}$ let $P$ be the set of invertible primes in $R$. Let $\hat{P} = \{p \in \mathbb{Z} \mid p \text{ is prime and } p \not\in P\} \cup \{0\}$. Furthermore for an $R$-module $V$ and a $R$-module map $f$, for each $p \in \hat{P}$ denote $V \otimes \mathbb{F}_p$ by $\hat{V}$ and $f \otimes \mathbb{F}_p$ by $\hat{f}$. Note that the prime is omitted from the notation.
Chapter 2. Differential Graded Algebra

Let \((\hat{A}, \hat{d})\) be a dga such that \(H(\hat{A}, \hat{d})\) is \(R\)-free and that as algebras \(H(\hat{A}, \hat{d}) \cong UL_0\) for some Lie algebra \(L_0\). If \(H(\hat{A}, \hat{d})\) is \(R\)-free and \((\hat{A}, \hat{d}) \cong U(\bar{L}, \bar{d})\) then by [Hal92], \(H(\hat{A}, \hat{d}) \cong UL_0\) as algebras for some Lie algebra \(L_0\). Let \(Z\hat{A}\) denote the cycles in \(\hat{A}\). Let \(A = (\hat{A} \amalg TV, d)\) where \(d|_{\hat{A}} = \hat{d}\) and \(dV \subset Z\hat{A}\). Then there is an induced map

\[d' : V \xrightarrow{d} Z\hat{A} \rightarrow H(\hat{A}, \hat{d}) \xrightarrow{d} UL_0.\]

If for some choice of \(L_0\), \(d' : V \subset L_0\) then call \(A\) a \textit{dga extension of} \(((\hat{A}, \hat{d}), L_0)\). We will sometimes abbreviate this to a dga extension of \(\hat{A}\).

\textbf{Remark 2.7.} The technical condition of the existence of a choice of \(L_0\) such that \(d' : V \subset L_0\) is satisfied in all the examples in this thesis. We do not know if there are examples arising from cell attachments or such that \((\hat{A}, \hat{d}) \cong U(\bar{L}, \bar{d})\) for which there does not exist such a choice of \(L_0\).

If \(R = \mathbb{Q}\) or \(\mathbb{F}_p\) where \(p > 3\), call \(A\) an \textit{inert} extension if \(HA \cong U(L_0/[dV])\) [HL87, Ani82b] (Anick uses the terminology \textit{strongly-free} and he gives a nice combinatorial characterization in the case where \(L_0\) is a free Lie algebra). Recall that all of our \(R\)-modules have finite type. If \(R \subset \mathbb{Q}\) we will say that \(A\) is \textit{inert} if \(L_0/[dV]\) is torsion-free and \(HA \cong U(L_0/[dV])\).

\textbf{Remark 2.8.} Note that if \(R \subset \mathbb{Q}\) and \(A\) is inert then \(HA\) is \(R\)-free. Thus by the Universal Coefficient Theorem, for each \(p \in \hat{P}\), \(H(A \otimes \mathbb{F}_p) \cong U(\bar{L}_0/[\bar{d}V])\). That is \(A \otimes \mathbb{F}_p\) is inert over \(\mathbb{F}_p\) (\(p\)-inert).

\textbf{Example 2.9.} \((\hat{A}, \hat{d}) = U(\mathbb{L}(x, y), 0)\). \(A = (\hat{A} \amalg T\langle a \rangle, d)\), where \(da = [[x, y], y]\)

Using the results of [Ani82b], \(A\) is an inert dga extension (since in Anick’s terminology \(\{xyy\}\) is a \textit{combinatorially free set}). It follows that \(HA \cong U(\mathbb{L}(x, y)/[[x, y], y]). \)

The following theorem gives an equivalent characterization of the inert condition. We will use it as motivation to generalize the inert condition. Let \(J\) be the Lie ideal
[\mathcal{d}V] \subset L_0$. The following is proved by Halperin and Lemaire [HL87, Theorem 3.3] in the case when $R = \mathbb{Q}$, and by Felix and Thomas [FT89, Theorem 1] in the case when $R$ is a field of characteristic $\neq 2$.

**Theorem 2.10 ([HL87]).** If $R = \mathbb{Q}$ or $\mathbb{F}_p$ where $p > 3$ then $A$ is an inert extension if and only if

(i) $J$ is a free Lie algebra, and

(ii) $J/[J,J]$ is a free $U(L/J)$-module.

We will study extensions which in general do not satisfy the second condition. If $R = \mathbb{Q}$ or $\mathbb{F}_p$ where $p > 3$ we define $A$ to be a free dga extension if $[\mathcal{d}V] \subset L_0$ is a free Lie algebra. If $R \subset \mathbb{Q}$ we define the dga extension $A$ to be free if for each $p \in \hat{P}$, $[\mathcal{d}V] \subset \hat{L}_0$ is a free Lie algebra.

This condition is a broad generalization of the inert condition and we will prove results about $H\Lambda$ if $A$ is a free dga extension (see part (i) of Theorems A and B). However many free dga extensions satisfy a second condition. We define this condition below and call it the *semi-inert* condition. Under this condition we are able to determine $H\Lambda$ as an algebra (see part (ii) of Theorems A and B). We will show that it too is a generalization of the inert condition.

We will use the following filtration. $A$ is filtered by taking $F_{-1}A = 0$, $F_0A = \hat{A}$ and for $n \geq 0$, $F_{n+1}A = \sum_{i=0}^{n} F_iA \cdot V_1 \cdot F_{n-i}A$. Since $dF_nA \subset F_nA$ and $F_nA \cdot F_mA \subset F_{n+m}A$ this makes $A$ into a filtered dga. There is an induced dga filtration on $H\Lambda$. Let $gr_*(H\Lambda)$ be the associated graded object. For degree reasons $gr_1(H\Lambda)$ is a $gr_0(H\Lambda)$-bimodule.

Let $L = (L_0 \uplus \mathbb{L} V, d')$. Then $L$ and $H\Lambda L$ are bigraded. So for degree reasons $(H\Lambda L)_1$ is a $(H\Lambda L)_0$-module. Thus by Remark 2.1 we can form the semi-direct product $(H\Lambda L)_0 \ltimes \mathbb{L}(H\Lambda L)_1$.

We will use the following lemma to define the semi-inert condition. Since this condition is not used in the statements or the proofs of Theorem A(i) and Theorem B(i) we will take the liberty of using these results.
Lemma 2.11. Let $A$ be a free dga extension. Then the following conditions are equivalent:

(a) $(HL_0) \ltimes \mathbb{L}(HL_1) \cong (HL_0) \amalg \mathbb{L}K$ for some free $R$-module $K \subset (HL_1)_0$,

(b) $(HL_0)_1$ is a free $(HL_0)_0$-module, and

(c) $gr_1(HA)$ is a free $gr_0(HA)$-bimodule.

Proof. (b) $\implies$ (a) Let $K$ be a basis for $(HL_0)_1$ as a free $(HL_0)_0$-module. Then $(HL_0)_0 \ltimes \mathbb{L}(HL_0)_1 \cong (HL_0) \amalg \mathbb{L}K$.

(a) $\implies$ (c) Since $A$ is a free dga extension, by Theorem A(i) or Theorem B(i), $gr_* (HA) \cong U ((HL_0) \ltimes \mathbb{L}(HL_0)_1)$. So by (a),

$$gr_* (HA) \cong U ((HL_0) \amalg \mathbb{L}K) \cong gr_0 (HA) \amalg \mathbb{L}K,$$

for some free $R$-module $K \subset (HL_1)_0$. Therefore

$$gr_1 (HA) \cong [gr_0 (HA) \amalg \mathbb{L}K]_1 \cong gr_0 (HA) \otimes K \otimes gr_0 (HA).$$

(c) $\implies$ (b) Let $L' = (HL_0)_0 \ltimes \mathbb{L}(HL_0)_1$. Then by Theorem A(i) or Theorem B(i), $gr_* (HA) \cong UL'$ and $gr_1 (HA) \cong (UL')_1$. By (c), $(UL')_1$ is a free $(UL')_0$-bimodule. Then we claim that it follows that $L'_1$ is a free $L'_0$-module. Indeed, if there is a nontrivial degree one relation in $L'$ then there is a corresponding nontrivial degree one relation in $UL'$.

We say that a dga extension $A$ is semi-inert if $A$ is free and it satisfies the conditions of the previous lemma. We justify this terminology with the following.

Lemma 2.12. An inert dga extension is semi-inert.

Proof. Let $A$ be an inert dga extension. If $R = \mathbb{Q}$ or $\mathbb{F}_p$ then by Theorem 2.10 $[dV]$ is a free Lie algebra. If $R \subset \mathbb{Q}$ then by Remark 2.8 for each $p \in \bar{P}$, $A \otimes \mathbb{F}_p$ is $p$-inert. So by Theorem 2.10 for each $p \in \bar{P}$, $[dV]$ is a free Lie algebra. So in either case $A$ is a free dga extension.

Since $(HL_1)_0 = 0$, the semi-inert condition is trivially satisfied.
Example 2.13. A semi-inert dga extension which is not inert.

Let \( R = \mathbb{F}_p \) where \( p > 3 \) or \( R \subset \mathbb{Q} \) containing \( \frac{1}{6} \). Let \( L = (\mathbb{L}(x, y, a, b), d) \) where \( |x| = |y| = 2, \ |a| = |b| = 7, \ dx = dy = 0, \ da = [[x, y], x] \) and \( db = [[x, y], y] \). We will show that \( UL \) is semi-inert dga extension of \( \mathbb{T}(x, y) \) but not an inert dga extension of \( \mathbb{T}(x, y) \).

Let \((\hat{A}, \hat{d}) = U(\mathbb{L}(x, y), 0)\). Then \( H(\hat{A}, \hat{d}) \cong U\mathbb{L}(x, y), \) \( \hat{d} = d \) and \( UL \) is a dga extension of \( U\mathbb{L}(x, y) \). Since \( \mathbb{L}(x, y) \) is a free Lie algebra, if \( R \) is a field then by the Schreier property the Lie ideal \([R\{da, db\}] \subset \mathbb{L}(x, y)\) is automatically a free Lie algebra. If \( R \subset \mathbb{Q} \) then for each \( p \in \mathbb{P} \), the Lie ideal \([\mathbb{F}_p \{\tilde{da}, \tilde{db}\}] \subset \mathbb{L}_{\mathbb{F}_p}(x, y)\) is automatically a free Lie algebra. Thus in either case \( UL \) is a free dga extension.

Let \( w = [a, y] - \{b, x\} \). Then by anti-commutativity,

\[
dw = [[[x, y], x], y] - [[[x, y], y], x] \\
= [[x, y], [x, y]] \\
= 0.
\]

Since \( w \) is not a boundary \( 0 \neq [w] \in (HL)_1 \) and \( 0 \neq [w] \in (HUL)_1 \). Thus \( UL \) is not an inert dga extension. By the definition of homology

\[
(HL)_0 \cong \mathbb{L}(x, y) / [R\{[[x, y], x], [[x, y], y]]].
\]

One can check that \((HL)_1 \) is freely generated by the \((HL)_0\)-action on \([w]\). Thus \((HL)_0 \ltimes \mathbb{L}(HL)_1 \cong (HL)_0 \Pi \mathbb{L}(\{w\}) \). That is, \( UL \) is a semi-inert extension. Therefore by part (ii) of either Theorem A or Theorem B,

\[
HUL \cong U((HL)_0 \Pi \mathbb{L}(\{w\}))
\]
as algebras.

Example 2.14. Another semi-inert dga extension which is not inert.
Let \( R = \mathbb{F}_p \), where \( p > 3 \) or \( R \subset \mathbb{Q} \) containing \( \frac{1}{6} \). Let \( L = (\mathbb{L}(x, y, z, a, b, c), d) \) where \( |x| = |y| = |z| = 2, |a| = |b| = |c| = 5 \), \( dx = dy = dz = 0 \), \( da = [y, z] \), \( db = [z, x] \), and \( dc = [x, y] \). \( UL \) is a free dga extension of \( U\mathbb{L}(x, y, z) \). As in the previous example, we will show that \( UL \) is a semi-inert extension but not an inert extension.

Let \( w = [x, a] + [y, b] + [z, c] \). By the Jacobi identity \( dw = 0 \). Since \( w \) is not a boundary \( 0 \neq [w] \in (HL)_1 \) and \( 0 \neq [w] \in (HUL)_1 \). Thus \( UL \) is not an inert dga extension. By the definition of homology \( (HL)_0 \cong \mathbb{L}_{ab}(x, y, z) \), where \( \mathbb{L}_{ab} \) denotes the free abelian Lie algebra (that is all brackets are zero).

One can check that \( (HL)_1 \) is freely generated by the \( (HL)_0 \)-action on \([w] \). Therefore \( UL \) is a semi-inert extension and by part (ii) of either Theorem A or Theorem B,

\[ HUL \cong U((HL)_0 \amalg \mathbb{L}([w])) \]

as algebras.

\[ \square \]

**Example 2.15.** A free extension which is not semi-inert.

Let \( R = \mathbb{F}_p \), where \( p > 3 \) or \( R \subset \mathbb{Q} \) containing \( \frac{1}{6} \). Let \( L = (\mathbb{L}(x, a), d) \), where \( |x| \) is odd, \( dx = 0 \) and \( da = [x, x] \). Then \( UL \) is a free dga extension of \( U\mathbb{L}(x) \). Let \( u = [x, a] \) Then \( du = [x, [x, x]] = 0 \) by the Jacobi identity. \( u \) is not a boundary so \( 0 \neq [u] \in (HL)_1 \) and \( 0 \neq [u] \in (HUL)_1 \). Thus \( UL \) is not an inert extension. By the definition of homology \( (HL)_0 \cong \mathbb{L}_{ab}([x]) \), where \( \mathbb{L}_{ab} \) denotes the free abelian Lie algebra (where all brackets are zero). However \( d(\frac{1}{3}[a, a]) = [u, x] \), so \( [[u], [x]] = 0 \) and \( HUL \neq U(\mathbb{L}_{ab}([x]) \amalg \mathbb{L}([u])) \).

Since the cycles of \( L \) in degree one are just \( R\{u, [u, x]\} \) and \( [u, x] \) is a boundary, by the definition of homology \( (HL)_1 = R\{[u]\} \). So by part (i) of either Theorem A or Theorem B, \( HUL \cong U(\mathbb{L}_{ab}([x]) \ltimes \mathbb{L}([u])) \) as algebras, where the semi-direct product is given by \( [[x], [u]] = 0 \). Thus \( HUL \cong U(\mathbb{L}_{ab}([x]) \ltimes \mathbb{L}([u])) \cong U(\mathbb{L}_{ab}([x], [u])) \) as algebras.

\[ \square \]
2.5 The spectral sequence of a dga extension

Let $A$ be a dga extension of $((\tilde{A}, \tilde{d}), L_0)$. Then by definition $A = (\tilde{A} \amalg TV_1, d)$ where $d|_A = \tilde{d}$ and $dV_1 \subset Z \tilde{A}$. Also by definition $H(\tilde{A}, \tilde{d}) \cong UL_0$ and $d'V_1 \subset L_0$ where $d'$ is the induced differential $d' : V_1 \xrightarrow{d} Z \tilde{A} \to H(\tilde{A}, \tilde{d}) \xrightarrow{\cong} UL_0$.

$A$ is filtered by taking $F_{-1}A = 0$, $F_0A = \tilde{A}$ and for $n \geq 0$, $F_{n+1}A = \sum_{i=0}^{n} F_iA \cdot V_1 \cdot F_{n-i}A$. Since $dF_nA \subset F_nA$ and $F_nA \cdot F_mA \subset F_{n+m}A$ this makes $A$ into a filtered dga. There is an induced dga filtration on $HA$.

From the filtration of $A$ there is an associated first quadrant spectral sequence

$$\text{gr}(A) \Rightarrow \text{gr}(HA)$$

which thus converges [McC01, Sel97]. Anick studied this spectral sequence [Ani82a] and showed that it collapses under certain conditions. In Chapter 5 we will show that Anick’s conditions are satisfied if the dga extension is free.

The $E^0$ term is given by $E^0_{p,q} = F_pA_{p+q}/F_{p-1}A_{p+q}$, where $A_k$ denotes the component of $A$ in dimension $k$. The differential $d^0$ is the induced differential (from $d$) on $\text{gr}(A)$. Since $\tilde{d}$ does not lower filtration but $d|_{V_1}$ does, $d^0|_A = \tilde{d}$ and $d^0|_{V_1} = 0$. In fact $(E^0A, d^0) = (\tilde{A} \amalg TV_1, \tilde{d})$, where $d|_{TV_1} = 0$. Therefore

$$E^1A = H(E^0A, d^0) \cong H(\tilde{A}, \tilde{d}) \amalg TV_1 \cong U(L_0 \amalg LV_1).$$

One can check that the induced differential $d^1$ is just the induced differential $d'$. Therefore

$$E^2A \cong HUL$$

where $L = (L_0 \amalg LV_1, d')$.

Since the spectral sequence converges, $E^\infty \cong \text{gr} H A$. Our main algebraic result (see Chapter 5) will involve showing that when the dga extension is free, the associated spectral sequence collapses at the $E^2$-term. That is, $E^\infty = E^2$.

Unfortunately since $A$ is not bigraded as a dga, it is not necessarily the case that $\text{gr}(HA) \cong HA$ as algebras. The following dga extension (which is not free) illustrates this.
**Example 2.16.** \((L, d) = (L(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2), d)\) where for \(i = 1, 2\), \(|x_i| = |y_i| = 2\), \(dx_i = dy_i = 0\), \(du_i = [[x_1, x_2], y_i]\) and \(dv_i = [[y_1, y_2], x_i]\). Let \((\tilde{A}, \tilde{d}) = U(L, d)\) and let \(A = (\tilde{A} \oplus T(a, b), d)\) where \(da = x_1\) and \(db = [y_1, y_2]\).

Let \(\{F_iA\}\) be the usual filtration. Let \(L_0 = H(\tilde{L}, \tilde{d})\). Abusing notation we will refer to the homology classes represented by \(x_i\) and \(y_i\) by \(x_i\) and \(y_i\).

For \(i = 1, 2\) let \(\alpha_i = [[a, x_2], y_i] - u_i\) and \(\beta_i = [b, x_i] - v_i\). One can check that these are cycles in \(F_1A\) which are not boundaries in \(A\).

Let \(\gamma = -[\beta_1, x_2] - [\alpha_1, y_2] + [\alpha_2, y_1] \in ZF_1A\). Let \(\epsilon = [v_1, x_2] - [v_2, x_1] + [u_1, y_2] - [u_2, y_1] \in ZF_0A\). Then one can check \(\gamma\) is not a boundary in \(A\). Therefore \([\gamma] \neq 0 \in F_1HA\). Thus

\[
-[[\beta_1], [x_2]] - [[\alpha_1], [y_2]] + [[\alpha_2], [y_1]] \neq 0 \in HA. \tag{2.3}
\]

However

\[
d([[a, b], x_2] - [a, v_2]) = \gamma - \epsilon.
\]

As a result \([\gamma] = [\epsilon] \in F_0HA\). For a cycle \(\bar{z}\) in \(A\) let \([\bar{z}]\) denote the corresponding homology class in \(\text{gr}(HA)\). Therefore

\[
-[[\bar{\beta}_1], [\bar{x}_2]] - [[\bar{\alpha}_1], [\bar{y}_2]] + [[\bar{\alpha}_2], [\bar{y}_1]] = 0 \in \text{gr}(HA). \tag{2.4}
\]

Comparing (2.3) and (2.4) we see that the multiplicative structure in \(HA\) and \(\text{gr}(HA)\) is not the same.
Chapter 3

Basic Topology and Adams-Hilton Models

In this chapter we review some basic topological results and Adams-Hilton models. We also prove some results on Adams-Hilton models which we will need in Chapter 7.

3.1 Basic topology

We will work in the usual category studied in algebraic topology, that of compactly generated topological spaces [May99, Sel97, Spa93, Whi78]. We will assume that all of our spaces $X$ are simply-connected. That is, $X$ is path-connected and has trivial fundamental group ($\pi_1(X) = 0$) and that our spaces are pointed (also called based). That is, they come with a chosen point called the basepoint and usually denoted *. The pointed space $(X, \ast)$ will usually be referred to as just $X$. All our maps are assumed to be continuous and pointed. That is, $f : (X, \ast) \rightarrow (Y, \ast)$ satisfies $f(\ast) = \ast$. Furthermore we will assume that all of our spaces have ‘the weak homotopy type of a finite-type CW complex.’ We explain this statement below.

Whenever we take products of infinitely many spaces we will always mean the weak infinite product. That is, $x \in \prod_i X_i$ implies that $x_i = \ast$ for all but finitely many $i$. A
weak product is a finite product or a weak infinite product.

Given spaces $X$ and $Y$ define the wedge $X \vee Y$ to be the space obtained by attaching $X$ and $Y$ at the basepoints. That is, $X \vee Y = \{(x, y) \in X \times Y \mid x = * \text{ or } y = *\}$ with basepoint $(*, *)$. Let an $(n + 1)$-cell be the unit disc $e^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ and let the $n$-sphere be its boundary $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Let $W$ be a subspace of $Z$ and $f : W \to X$. Then the attaching map construction builds a new space $Y = (Z \amalg X)/\sim$ where $f(w) \sim w$, $\forall w \in W$. This space is called an adjunction space and we denote\footnote{In the literature this space is sometimes also denoted by $Z \cup_f X$.} it by $X \cup_f Z$ and call $f$ the attaching map. For example if $Z = \bigcup_{j \in J} e^{n_j+1}$ and $W = \bigcup_{j \in J} S^{n_j}$ then we say that the space $Y$ is obtained from $X$ by attaching cells along $W$. In this case $W \to X \to Y$ is a homotopy cofibration \cite{May99, Sel97, Spa93, Whi78}. A cofibration $W \to X$ is an inclusion and given any homotopy $H : W \times I \to A$ and an extension $g : X \to A$ of $H|_{W \times \{0\}}$ there exists an extension $H' : X \times I \to A$ of $H$ such that $H'|_{X \times \{0\}} = g$.

A finite-type CW complex is any space $X$ that can be constructed by the following inductive procedure. Let $X^{(0)}$ be a finite set of points. Obtain $X^{(n+1)}$ from $X^{(n)}$ by attaching a finite number of $(n + 1)$-cells along their boundary. Let $X = \cup_n X^{(n)}$. Call $X^{(n)}$ the $n$-skeleton of $X$. Call a map $f : X \to Y$ between CW-complexes cellular if $f : X^{(n)} \to Y^{(n)}$, $\forall n$.

Let $I$ denote the unit interval $[0, 1]$. Two (pointed) maps $f, g : X \to Y$ are said to be homotopy equivalent, written $f \simeq g$ if there is a map (called a homotopy) $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(*, t) = *$, $\forall x \in X$ and $\forall t \in I$. This is an equivalence relation. Let $X$ and $Y$ be two topological spaces. If there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$ then we say $X$ is homotopy equivalent to $Y$ or $X$ has the same homotopy type as $Y$, denoted $X \approx Y$. This is also an equivalence relation.

Let $\pi_n(X)$ be the set of homotopy classes of pointed maps from $S^n \to X$. For $n \geq 1$
this is a group [Sel97, May99, Spa93, Whi78].

A map $f : X \to Y$ is called a weak homotopy equivalence if $f_{\#} : \pi_n X \to \pi_n(Y)$ is an isomorphism for all $n$. Topological spaces $X$ and $Y$ have the same weak homotopy type if there exists a chain of weak homotopy equivalences $X \leftrightarrow Z_1 \to \cdots \leftrightarrow Z_n \to Y$. The following theorem shows that any topological space has the weak homotopy type of a CW-complex.

**Theorem 3.1 (The CW Approximation Theorem [Whi78, Theorem V.2.2]).**

Given a topological space $Y$ there is a CW-complex $X$ and a map $f : X \to Y$ such that $f_{\#} : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n$.

If $Y$ is simply-connected then one can choose $X$ such that $X^{(1)} = \ast$.

In addition to the CW Approximation Theorem, the following two theorems show why we can work in the category of CW-complexes and why it is convenient to do so.

**Theorem 3.2 (The Cellular Approx. Theorem [Whi78, Theorem II.4.5]).** Let $f : X \to Y$ be a map between CW-complexes. Then there is a cellular map $g : X \to Y$ such that $f \simeq g$.

**Theorem 3.3 (The Whitehead Theorem for simply-connected CW-complexes [Whi78, Theorems IV.7.13 and V.3.5]).** Let $X$ and $Y$ be simply-connected CW-complexes. Then $X$ and $Y$ are homotopy equivalent if and only $X$ and $Y$ are weak homotopy equivalent if and only if $\exists f : X \to Y$ such that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

Given a (based) topological space $(X, \ast)$ let $\Omega X$ denote the set of (based) loops in $X$. That is, $\Omega X = \text{Map}_*(S^1, \ast) \to (X, \ast))$, where $\text{Map}_*(X, Y)$ denotes the pointed maps from $X$ to $Y$ with the compact-open topology (see [Sel97, May99, Spa93, Whi78]). The constant map $(S^1, \ast) \to (\ast, \ast)$ is the basepoint of $\Omega X$. $\Omega X$ is called the loop space on $X$ and $H_*(\Omega X; R)$ is called the loop space homology of $X$. A key property of $\Omega X$ is the following.
Lemma 3.4 ([Whi78, Corollary IV.8.6]). \(\forall n \geq 1, \pi_n(\Omega X) \cong \pi_{n+1}(X)\).

Thus we can study the homotopy groups of \(X\) by studying \(\Omega X\). From our point of view the main advantage of studying \(\Omega X\) is that the homology groups of \(\Omega X\) have the structure of a Hopf algebra whereas the homology groups of \(X\) do not have the structure of an algebra. The algebra structure of \(H_*(\Omega X)\) can reveal important information about the homotopy type of \(X\).

There is a natural map

\[
h_X : \pi_*(\Omega X) \otimes R \to H_*(\Omega X; R)
\]

called the Hurewicz map and defined as follows. For \(\alpha \in \pi_n(\Omega X) \otimes R\) choose a representative \(a : S^n \to \Omega X\). Then there is an induced map \(a_* : H_*(S^n; R) \to H_*(\Omega X; R)\). Let \(h_X(\alpha) = a_*(\iota_n)\), where \(\iota_n\) is a generator for \(H_n(S^n)\). One can check that this map is well defined and is in fact a homomorphism.

Concatenation of loops induces an algebra structure on \(H_*(\Omega X; R)\) [Sel97] called the Pontrjagin product. If \(f : X \to Y\) then using this algebra structure on \(H_*(\Omega X; R)\) and \(H_*(\Omega Y; R)\) the induced map

\[
(\Omega f)_* : H_*(\Omega X; R) \to H_*(\Omega Y; R)
\]

is an algebra map. Using this algebra structure, the commutator bracket \([a, b] = ab - (-1)^{|a||b|}ba\) gives \(H_*(\Omega X; R)\) the structure of a Lie algebra.

Given \(\alpha \in \pi_m(\Omega X) \otimes R\) and \(\beta \in \pi_n(\Omega X) \otimes R\) define \([\alpha, \beta] \in \pi_{m+n}(\Omega X) \otimes R\), called the Samelson product, as follows. Let \(f : S^m \to \Omega X\) and \(g : S^n \to \Omega X\) be representatives of \(\alpha\) and \(\beta\). Define \(h : S^m \times S^n \to \Omega X\) by letting \(h(x, y)\) be the loop obtained by concatenating the loops \(f(x), g(y), -f(x)\) and \(-g(y)\), where the negative of a loop is the loop traced in reverse. Since \(h|_{S^m \vee S^n}\) is contractible one can show that there is an induced map \(S^{m+n} \to \Omega X\). Let \([\alpha, \beta]\) be the homotopy class of this map. One can check that the Samelson product gives \(\pi_*(\Omega X) \otimes R\) the structure of a Lie algebra [Sel97].
Using these products the Hurewicz map \( h_X : \pi_*(\Omega X) \otimes R \to H_*(\Omega X; R) \) is a Lie algebra map [Sam53]. Let \( L_X \) denote the image of \( h_X \) and call this Lie subalgebra of the loop space homology the Hurewicz images. A map \( f : W \to X \) induces a map \( L_W \to L_X \).

Denote the image of this map by \( L^W_X \), and let \([L^W_X]\) be the Lie ideal in \( L_X \) generated by \( L^W_X \).

Given \( \alpha \in \pi_m(X) \) and \( \beta \in \pi_n(X) \) there is a Whitehead product \( [\alpha, \beta] \in \pi_{m+n-1}(X) \) which can be defined as follows. Let \( \omega_{m,n} : S^{m+n-1} \to S^m \vee S^n \) be the attaching map of the top cell in \( S^m \times S^n \). An explicit description is given in [FHT01]. \([\alpha, \beta]\) is then the homotopy class represented by the composite \( S^{m+n-1} \xrightarrow{\omega_{m,n}} S^m \vee S^n \xrightarrow{ab} X \) where \( a \) and \( b \) are representatives of \( \alpha \) and \( \beta \). Up to sign, the Whitehead product can also be defined using the adjoint (defined below) of the Samelson product [Sel97].

Assume \( R \subset \mathbb{Q} \) is a subring containing \( \frac{1}{6} \) and assume \( H_*(\Omega X; R) \) is torsion-free. Let \( P \) be the set of invertible primes in \( R \), and let \( \tilde{P} = \{ p \in \mathbb{Z} \mid p \text{ is prime and } p \notin P \} \cup \{ 0 \} \).

Let \( F_0 = \mathbb{Q} \). Then \( \forall p \in \tilde{P} \) we have the commutative diagram

\[
\begin{array}{ccc}
\pi_*(\Omega X) \otimes R & \xrightarrow{h_X} & H_*(\Omega X; R) \\
\downarrow \otimes F_p & & \downarrow \otimes F_p \\
\pi_*(\Omega X) \otimes F_p & \xrightarrow{h_X \otimes F_p} & H_*(\Omega X; R) \otimes F_p \xrightarrow{\cong} H_*(\Omega X; F_p)
\end{array}
\]

where the bottom right map is the isomorphism given by the Universal Coefficient Theorem. Abusing notation we will refer to the composition of the bottom two maps as \( h_X \otimes F_p \) and refer to its image as \( L_X \otimes F_p \). It is easy to check that this is the same as the map \( h_X : \pi_*(\Omega X) \otimes F_p \to H_*(\Omega X; F_p) \) defined in (3.1) when \( R = F_p \). We will sometimes denote \( h_X \otimes F_p \) and \( L_X \otimes F_p \) by \( \tilde{h}_X \) and \( \tilde{L}_X \) omitting \( p \) from the notation.

If \( f : W \to X \) and \( H_*(\Omega W; R) \) and \( H_*(\Omega X; R) \) are \( R \)-free then \( \forall p \in \tilde{P} \) there is an induced map \( \tilde{L}_W \to \tilde{L}_X \). Denote the image of this map by \( \tilde{L}^W_X \), and let \([\tilde{L}^W_X]\) be the Lie ideal in \( \tilde{L}_X \) generated by \( \tilde{L}^W_X \).

When \( R = \mathbb{Q} \) Milnor and Moore [MM65] proved the following major result about the rational Hurewicz map.
Theorem 3.5 (The Milnor-Moore Theorem [MM65]). The rational Hurewicz homomorphism $h_X : \pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X; \mathbb{Q})$ is an injection, and furthermore, as algebras $H_*(\Omega X; \mathbb{Q}) \cong UL_X$.

In Theorems E and F we prove versions of this theorem for more general coefficient rings under certain hypotheses. See [Sco] for another extension of the Milnor-Moore Theorem.

Given a space $X$ define the (reduced) suspension $\Sigma X$, as follows. Let $\Sigma X = (X \times I)/\sim$ where $(a, 0) \sim (b, 0)$, $(a, 1) \sim (b, 1)$ and $(*, s) \sim (*, t)$, $\forall a, b \in X$ and $\forall s, t \in I$. The basepoint is the equivalence class of $(*, 0)$. For example $\Sigma S^n \cong S^{n+1}$ and $\Sigma \left( \bigvee_j S^{n_j} \right) \cong \bigvee_j S^{n_j+1}$.

Given a map $g : \Sigma X \to Y$ there is an adjoint map $\hat{g} : X \to \Omega Y$ defined by $\hat{g}(x)(t) = g(x, t)$ for $x \in X$ and $t \in I$. Let $\alpha : X \to \Omega \Sigma X$ be the adjoint of $id_{\Sigma X}$.

Theorem 3.6 (The Bott-Samelson Theorem [BS53]). Let $R$ be a principal ideal domain and let $X$ be a connected space such that $H_*(X; R)$ is torsion-free. Then

$$H_*(\Omega \Sigma X; R) \cong \mathbb{T}(\tilde{H}_*(X; R)).$$

Furthermore $\alpha : X \to \Omega \Sigma X$ induces the canonical inclusion $\tilde{H}_*(X; R) \hookrightarrow \mathbb{T}(\tilde{H}_*(X; R))$.

Example 3.7. $H_*(\Omega S^{n+1}; R) \cong \mathbb{T}(x) \cong U\mathbb{L}(x)$ where $|x| = n$. Furthermore $x = h_{S^{n+1}}([\alpha])$ where $\alpha : S^n \to \Omega S^{n+1}$ is defined above. Therefore $H_*(\Omega S^{n+1}; R) \cong UL_{S^{n+1}}$.

Example 3.8. $H_*(\Omega(\bigvee_{j \in J} S^{n_j+1}); R) \cong TV \cong U\mathbb{L}V$ where $V$ is a free $R$-module with basis $\{x_j\}_{j \in J}$ and $|x_j| = n_j$. Let $\iota_j : S^{n_j} \to \bigvee_{j \in J} S^{n_j}$ denote the inclusion of one of the spheres. Then $x_j = h_{\bigvee S^{n_j+1}}([\alpha \circ \iota_j])$ and hence $H_*(\Omega(\bigvee_{j \in J} S^{n_j+1}); R) \cong UL_{\bigvee S^{n_j+1}}$.

In Section 4.3 we will need the following infinite mapping telescope construction for the localization of CW-complexes. Let

$$X_1 \xrightarrow{j_1} X_2 \xrightarrow{j_2} \ldots X_n \xrightarrow{j_n} \ldots$$
be a sequence of maps between topological spaces. The infinite mapping telescope of this sequence is the space
\[
T = \left( \prod_{n=1}^{\infty} (X_n \times [n - 1, n]) \right) \sim \text{ where } (x_n \times \{n\}) \sim (j_n(x_n) \times \{n\}).
\]

3.2 Adams-Hilton models

Let \( R \) be a principal ideal domain containing \( \frac{1}{s} \). Given a differential graded algebra (dga) \( A \), a dga morphism \( A' \to A \) is called a model for \( A \) if it induces an isomorphism on homology. If \( A' \cong (TV, d) \) then it is called a free model. A model for a simply-connected topological space \( Z \) means a model for \( CU_\ast^1(\Omega Z) \) where \( CU_\ast^1(\cdot) \) is the first Eilenberg subcomplex of the cubical singular chain complex. If such a model is free it is called an Adams-Hilton model (AH-model) [AH56].

Every simply-connected topological space \( Z \) has a (non-unique) AH-model
\[
\theta_Z : A(Z) \xrightarrow{\simeq} CU_\ast^1(\Omega Z).
\]

This induces an isomorphism of algebras \( H_\ast(A(Z)) \xrightarrow{\simeq} H_\ast(\Omega Z; R) \). We will usually denote an Adams-Hilton model by just \( A(Z) \). We state some basic properties of these models.

For any map \( f : X \to Y \) between simply-connected CW-complexes and any choice of Adams-Hilton models \( (A(X), \theta_X) \) and \( (A(Y), \theta_Y) \) there is a dga homomorphism \( A(f) : A(X) \to A(Y) \) which comes with a dga homotopy \( \psi_f \) from \( CU_\ast^1(\Omega f) \circ \theta_X \) to \( \theta_Y \circ A(f) \).

If \( X \) is a finite-type CW-complex then a CW-structure on \( X \) determines an Adams-Hilton model on \( X \) as follows. The CW-structure gives a sequence of cofibrations
\[
\bigvee_{j \in J_n} S^j \xrightarrow{\alpha_{n,j}} X^{(n)} \to X^{(n+1)}
\]
where \( X^{(n)} \) is the \( n \)-skeleton of \( X \) and each index set \( J_n \) is finite. Let \( \{x_{n,j}\}_{j \in J_n} \) be a set of graded elements with \( |x_{n,j}| = n \). Let \( V_n = R\{x_{n,j}\}_{j \in J_n} \) and let \( V = \bigoplus_n V_n \). One can
choose \((\mathbb{T}V,d)\) as an AH-model for \(X\), where the differential is defined inductively and satisfies \(dx_{n,j} \subset Z(\mathbb{T}(\bigoplus_{i=1}^{n-1} V_i),d)\).

Let \(f : W \to X\) be a map between finite-type simply-connected CW-complexes where \(W = \bigvee_{j \in J} S^{n_j}, f = \bigvee_{j \in J} \alpha_j\), \(H_*(\Omega X; R)\) is torsion-free and \(H_*(\Omega X; R) \cong UL_X\) (recall \(L_X\) from Section 3.1). Let \(Y\) be the adjunction space \(X \cup_f \left( \bigvee_{j \in J} e^{n_j+1} \right)\). Then one can take

\[
A(Y) = A(X) \amalg \mathbb{T}\langle y_j \rangle_{j \in J},
\]

where \(\mathbb{T}\langle y_j \rangle_{j \in J}\) is the tensor algebra on the free \(R\)-module \(R\langle y_j \rangle_{j \in J}\) and \(A \amalg B\) is the coproduct or free product of \(A\) and \(B\). The differential on \(A(Y)\) is an extension of the differential on \(A(X)\) satisfying \(dy_j \in A(X)\). Furthermore if \(d'\) is the induced map

\[
d' : R\langle y_j \rangle \to ZA(X) \to HA(X) \xrightarrow{\cong} H_*(\Omega X; R) \xrightarrow{\cong} UL_X
\]

then \(d'y_j = h_X(\alpha_j) \in L_X\). That is, \(A(Y)\) is a dga extension (see Section 2.4) of \((A(X), L_X)\).

We can filter \(A(Y)\) as a dga by taking \(F_{-1}A(Y) = 0\), \(F_0A(Y) = A(X)\) and for \(i \geq 0\),

\[
F_{i+1}A(Y) = \sum_{j=0}^{i} F_j A(Y) \cdot R\langle y_j \rangle_{j \in J} \cdot F_{i-j} A(Y).
\]

This filtration induces a filtration on \(HA(Y)\). From the filtration on \(A(Y)\) we get a first quadrant spectral sequence \(\text{gr}(A(Y)) \Longrightarrow \text{gr}(HA(Y))\) (see Section 2.5).

### 3.3 Some lemmas

We will prove some results using Adams-Hilton models which we will need in Chapter 7. Let \(R\) be a principal ideal domain containing \(1/6\).

First we need the following Adams-Hilton models of some standard spaces. Recall from Section 3.2 that there is an Adams-Hilton model corresponding to a CW structure. Consider \(S^m\) with the CW structure \(S^m_0 \cong * \cup e^m\). Its corresponding AH model is \((\mathbb{T}(a),0)\) where \(|a| = m - 1\). Attach another cell to get the disk \(D^{m+1}_0 \cong S^m_0 \cup e^{m+1}\). Its
corresponding AH model is \((T(a, b), d)\) where \(|a| = m - 1, |b| = m, da = 0\) and \(db = a\).

We fix the AH model for the inclusion \(i_0 : S^m_0 \hookrightarrow D^{m+1}_0\) to be the homomorphism defined by \(A(i_0)(a) = a\).

For any \(s\), the sphere also has the following more complicated CW structure

\[
S^{m+1}_s \cong \left( \bigvee_{k=1}^s D^{m+1}_0 \right) \cup \bigcup_{k=1}^s \mathcal{C}^{m+1}
\]

where the attaching map for the last cell is given by the inclusions of the spheres into \(\bigvee_{k=1}^s S^m_0\). Its corresponding AH model is \((T(a_1, \ldots, a_s, b_0, \ldots, b_s), d)\) where \(|a_i| = m - 1 |b_i| = m, da_i = 0, db_i = a_i\) for \(1 \leq i \leq s\), and \(db_0 = a_1 + \ldots + a_s\). Furthermore there is a homeomorphism

\[
\Psi : S^{m+1}_0 \cong S^{m+1}_s
\]

which has an Adams-Hilton model that sends \(a\) to \(b_1 + \ldots + b_s - b_0\).

Recall from Section 3.1 that the Whitehead product is defined using a map \(\omega_{m,n} : S^{m+n-1}_o \to S^m_0 \vee S^n_0\). If we take \((T(a_{m+n-1}), 0)\) and \((T(a_m, a_n), 0)\) to be the Adams-Hilton models of those spaces then by [AH56, Corollary 2.4] we may take \(A(\omega_{m,n})(a_{m+n-1}) = \pm[a_m, a_n]\). We will make use of the following fact.

**Lemma 3.9.** [Ani89, Lemma 5.2] Let \((T(a, b), d)\) with \(db = a\) be an AH model for \(D^{m+n}_0\). Let \((T(a', b', c'), d)\) be an AH model for \(D^{m+1}_0 \vee S^n_0\) where \(c'\) corresponds to the \(S^n_0\) and \(d(b') = a'\). Then there is an extension of \(\pm \omega_{m,n} : S^{m+n-1}_0 \to S^m_0 \vee S^n_0\) to a map \(f : D^{m+n}_0 \to D^{m+1}_0 \vee S^{n+1}_0\) whose AH model may be chosen so as to satisfy

\[
A(f)(a) = [a', c'], \quad A(f)(b) = [b', c'], \quad \text{and} \quad \psi_f(a) \in CU_*(\Omega(S^n_0 \vee S^n_0)).
\]

From this we prove the following lemma which we will need in Chapter 7 to construct a map \(S^{m+1} \to Y\). It is a slight generalization of [Ani89, Lemma 5.3] and we copy Anick’s proof.

**Lemma 3.10.** Suppose we are given a simply-connected space \(X\) for which there exists a Lie algebra map \(\sigma_X\) right inverse to \(h_X\). In addition we are given \(c \in R\), a map
\( \alpha : S_0^{n+1} \to X \), and \( x_i \in ZA(X) \), for \( i = 1, \ldots, t \) such that \( [x_i] \in L_X \) and \( \beta_i : S^{n_i+1} \to X \) is the adjoint of \( \sigma_X([x_i]) \). Choose an Adams-Hilton model \( A(\alpha) \). Let \( z = A(\alpha)(a) \in A(X) \) and let \( Y = X \cup_\alpha e^{n+2} \). Then we can choose \( A(Y) = A(X) \amalg \mathbb{T}(y) \) with \( dy = z \). In addition there exists a map \( g : (D_0^{m+1}, S_0^m) \to (Y, X) \) such that \( g|_{S_0^m} = cc[\alpha, \beta_1, \ldots, \beta_t] \), where \( c = \pm 1 \). Furthermore \( A(g) \) may be chosen so that \( A(g)(a) = c[z, x_1, \ldots, x_i] \) and \( A(g)(b) = c[y, x_1, \ldots, x_i] \) and the dga homotopy \( \psi_g(a) \) lies in the submodule \( CU_*(\Omega X) \subset CU_*(\Omega Y) \).

**Proof.** The proof is by induction on \( t \), the length of the list of indices. When \( t = 0 \), [AH56, Theorem 3.2] tells us that the AH model \( A(\alpha) \) may be extended over \( A(D_0^{m+2}) \) such that the generators \( a \) and \( b \) are sent to \( z \) and \( y \). Composing this with the degree \( c \) map from \( D_0^{m+2} \) to itself gives the map \( g \) for which \( A(g)(a) = cz \) and \( A(g)(b) = cy \).

For the inductive step, suppose the result to be true when the list has \( t-1 \) elements. Let \( m' = n + 1 + n_1 + \ldots + n_{t-1} \). Then the inductive hypothesis gives us a map

\[
g'_i : (D_0^{m'+1}, S_0^{m'}) \to (Y, X)
\]

satisfying \( A(g'_i)(a') = c[z, x_1, \ldots, x_{t-1}] \), \( A(g'_i)(b') = c[y, x_1, \ldots, x_{t-1}] \) and furthermore \( \psi_{g'_i}(a') \in CU_*(\Omega X) \). Define a map

\[
g'' = g'_i \cup \beta_i : (D_0^{m'+1} \cup S_0^{n_{t-1}}, S_0^{m'} \cup S_0^{n_{t-1}}) \to (Y, X)
\]

with \( A(g'') \) an extension of \( A(g'_i) \) such that \( A(g'')(c') = x_i \) where \( c' \) is the generator corresponding to \( S_0^{n_{t-1}} \). Let \( f \) denote the map of Lemma 3.9, where \( m = m' \) and \( n = n_{t-1} + 1 \), and set \( g = g'' \circ f \). Take \( A(g) = A(g'') \circ A(f) \). Then \( g \) has the desired properties. \( \square \)
Chapter 4

The Cell Attachment Problem

In this chapter we review various topological constructions and results. We also motivate and define our main topological objects of study: cell attachments which are free and semi-inert.

4.1 Whitehead’s cell-attachment problem

One of the oldest questions in homotopy theory asks what effect attaching one or more cells has on the homotopy groups and loop space homology groups of a space. This questions was perhaps first considered by J.H.C. Whitehead [Whi41], [Whi39, Section 6], around 1940.

The cell attachment problem: Given a simply-connected topological space $X$ and a cofibration

\[
\bigvee_{j \in J} S^{n_j} \xrightarrow{f=\bigvee \alpha_j} X \xrightarrow{i} Y
\]

how is $H_*(\Omega Y; R)$ related to $H_*(\Omega X; R)$ and how is $\pi_*(Y)$ related to $\pi_*(X)$?

We assume that $H_*(\Omega X; R)$ is torsion-free. This condition is trivial if $R$ is a field. For $R \subset \mathbb{Q}$ one can often reduce to this case by localizing (see Section 4.3) away from a finite set of primes. Even if the loop space homology of a given space has torsion at infinitely
many primes [Ani86, Avr86] one might be able to study the given space by including into a space \( X \) such that \( H_*(\Omega X; R) \) is torsion-free [Ani89].

The inclusion \( i \) induces an algebra map \((\Omega i)_* : H_*(\Omega X; R) \to H_*(\Omega Y; R)\) (see (3.2)). Recall (from Section 3.1) that \( h_X \) denotes the Hurewicz map and that \( \hat{\alpha}_j \) is the adjoint of \( \alpha_j \). Let \( W = \bigvee_{j \in J} S^{n_j} \), let \( a_j = h_X(\hat{\alpha}_j) \) and let \( V_1 = R\{y_j\}_{j \in J} \) where \( |y_j| = |a_j| + 1 \). Using the notation of Section 3.1, \( L^W_X \) is the Lie subalgebra of \( L_X \subset H_*(\Omega X; R) \) generated by \( \{a_j\}_{j \in J} \). Let \( (L^W_X) \subset H_*(\Omega X; R) \) be the two-sided ideal generated by \( L^W_X \). Since \((\Omega i)_* \) is an algebra map \((\Omega i)_*((L^W_X)) = 0 \) and hence \((\Omega i)_* \) factors through the quotient map.

\[
\begin{array}{ccc}
H_*(\Omega X; R) & \xrightarrow{(\Omega i)_*} & H_*(\Omega Y; R) \\
\downarrow g & & \downarrow\text{inert} \\
H_*(\Omega X; R)/(L^W_X) & & 
\end{array}
\]

We will say that the attaching map \( f \) is inert [HL87] if \( H_*(\Omega X; R)/(L^W_X) \) is torsion-free and \( g \) is an isomorphism.

**Remark 4.1.** Note that if \( R \subset \mathbb{Q} \) and \( f \) is inert then \( H_*(\Omega Y; R) \) is \( R \)-free. Thus by the Universal Coefficient Theorem \( \forall p \in \mathcal{P}, H_*(\Omega Y; \mathbb{F}_p) \cong H_*(\Omega X; \mathbb{F}_p)/(L^W_X) \). That is, \( f \) is inert over \( \mathbb{F}_p \) (\( p \)-inert in [HL96]).

**Example 4.2.** \( X = S^3_a \cup S^3_b \) and \( Y = X \cup f e^8 \), where \( f \) is the iterated Whitehead product \([[[\iota_a, \iota_b], \iota_b]]\) with \( \iota_a \) and \( \iota_b \) the inclusions of \( S^3_a \) and \( S^3_b \) in \( X \).

By the Bott-Samelson Theorem (Theorem 3.6) \( H_*(\Omega X; R) \cong \mathbb{T}(x, y) \). Let \( I \) be the two-sided ideal in \( H_*(\Omega X; R) \) generated by the image of \( f \). That is, \( I \) is the two-sided ideal generated by \([x, y], y\] \).

For an Adams-Hilton model of \( Y \) we can take (see Section 3.2) \( U(L(Y)) \) where \( L(Y) = (L(x, y, a), d), \ dx = dy = 0 \) and \( da = [x, y], y] \). So \( H_*(\Omega Y; R) \cong HU(L(Y)) \) as algebras. By Example 2.9 \( HU(L(Y)) \cong U(L(x, y)/[[x, y], y]) \) \( \cong U(L(x, y)/([x, y], y]) \) as algebras. Thus \( H_*(\Omega Y; R) \cong H_*(\Omega X; R)/I \) as algebras and \( f \) is an inert attaching map. \( \square \)
If \( R \) is a field then Halperin and Lemaire [HL87] showed that \( g \) is surjective if and only if it is an isomorphism. In general the map \( g \) in (4.1) need not be injective or surjective. The attaching map \( f \) is said to be \textit{nice} [HL96] if the map \( g \) (4.1) is injective. As we will see in Example 4.8 \( f \) can be nice but not inert.

The following theorem gives an equivalent characterization of the inert condition. We will use it as motivation to generalize the inert condition. It is proved by Halperin and Lemaire [HL87, Theorem 3.3] if \( R = \mathbb{Q} \), and Félix and Thomas [FT89, Theorem 1] if \( R \) is a field of characteristic \( \neq 2 \).

\textbf{Theorem 4.3 ([HL87, FT89])}. Let \( J = [L_X^W] \subset L_X \) be the Lie ideal of \( L_X \) generated by \( L_X^W \). If \( R = \mathbb{Q} \) or \( \mathbb{F}_p \) where \( p > 3 \) then \( f \) is an inert attaching map if and only if

(i) \( J \) is a free Lie algebra, and

(ii) \( J/[J,J] \) is a free \( U(L_X/J) \)-module.

We will study attaching maps which in general do not satisfy the second condition. If \( R = \mathbb{Q} \) or \( \mathbb{F}_p \) we define \( f \) to be a \textit{free} attaching map if \( [L_X^W] \subset L_X \) is a free Lie algebra. If \( R \subset \mathbb{Q} \) we define the attaching map \( f \) to be \textit{free} if for each \( p \in \mathbb{P} \), \( [L_X^W] \subset L_X \) is a free Lie algebra.

This condition is a broad generalization of the inert condition and we will prove results about \( H_*(\Omega Y; R) \) if \( f \) is a free cell attachment (see part (i) of Theorems C and D). However many free cell attachments satisfy a second condition. We define this condition below and call it the \textit{semi-inert} condition. Under this condition we are able to determine \( H_*(\Omega Y; R) \) as an algebra (see part (ii) of Theorems C and D). We will show that it too is a generalization of the inert condition.

We will use the following filtration on \( H_*(\Omega Y; R) \). Recall from Section 3.2 that \( X \) and \( Y \) have Adams-Hilton models \( A(X) \) and \( A(Y) \) satisfying \( A(Y) = A(X) \amalg TV_1 \) where \( V_1 = R\{y_j\}_{j \in J} \). \( A(Y) \) is filtered by taking \( F_{-1}A(Y) = 0 \), \( F_0A(Y) = A(X) \) and for \( n \geq 0 \), \( F_{n+1}A(Y) = \sum_{i=0}^{n} F_iA(Y) \cdot V_1 \cdot F_{n-i}A(Y) \). Since \( dF_nA(Y) \subset F_nA(Y) \) and \( F_nA(Y) \cdot F_mA(Y) \subset F_{n+m}A(Y) \) this makes \( A(Y) \) into a filtered dga. There is an induced
dga filtration on $H \mathbf{A}(Y)$. Since $H_*(\Omega Y; R) \cong H \mathbf{A}(Y)$ there is a corresponding filtration on $H_*(\Omega Y; R)$. Let $\text{gr}_*(H_*(\Omega Y; R))$ be the associated graded object. For degree reasons $\text{gr}_1(H_*(\Omega Y; R))$ is a $\text{gr}_0(H_*(\Omega Y; R))$-bimodule.

Let $L = (L_X \amalg L V_1, d')$ where $d'y_j = h_X(\alpha_j)$. Note that $[dV_1] = [L^W_X]$. If we let $L_X$ be in degree 0 and $V_1$ be in degree 1 then $L$ is a bigraded dgL and $HL$ is a bigraded Lie algebra. Let $(HL)_i$ denote the component of $HL$ in degree $i$. Then for degree reasons $(HL)_1$ is a $(HL)_0$-module. Thus by Remark 2.1 there exists a semi-direct product $(HL)_0 \ltimes \mathbb{L}(HL)_1$.

We will use the following lemma to define the semi-inert condition. Since this condition is not used in the statements or the proofs of Theorem C(i) and Theorem D(i) we will take the liberty of using these results.

**Lemma 4.4.** Let $f : W \to X$ be a free cell attachment and let $Y$ be the corresponding adjunction space. Then the following conditions are equivalent:

(a) $(HL)_0 \ltimes \mathbb{L}(HL)_1 \cong (HL)_0 \amalg \mathbb{L}K$ for some free $R$-module $K \subset (HL)_1$,

(b) $(HL)_1$ is a free $(HL)_0$-module, and

(c) $\text{gr}_1(H_*(\Omega Y; R))$ is a free $\text{gr}_0(H_*(\Omega Y; R))$-bimodule.

**Proof.** The proof of this lemma is the same as the proof of Lemma 2.11.

(b) $\implies$ (a) Let $K$ be a basis for $(HL)_1$ as a free $(HL)_0$-module. Then $(HL)_0 \ltimes \mathbb{L}(HL)_1 \cong (HL)_0 \amalg \mathbb{L}K$.

(a) $\implies$ (c) Since $f$ is a free cell attachment, by Theorem C(i) or Theorem D(i),

$$\text{gr}_*(H_*(\Omega Y; R)) \cong U ((HL)_0 \ltimes \mathbb{L}(HL)_1).$$

So by (a),

$$\text{gr}_* (H_*(\Omega Y; R)) \cong U ((HL)_0 \amalg \mathbb{L}K) \cong \text{gr}_0(H_*(\Omega Y; R)) \amalg \mathbb{L}K,$$

for some free $R$-module $K \subset (HL)_1$. Therefore

$$\text{gr}_1(H_*(\Omega Y; R)) \cong [\text{gr}_0(H_*(\Omega Y; R)) \amalg \mathbb{L}K]_1 \cong \text{gr}_0(H_*(\Omega Y; R)) \otimes K \otimes \text{gr}_0(H_*(\Omega Y; R)).$$

(c) $\implies$ (b) Let $L' = (HL)_0 \ltimes \mathbb{L}(HL)_1$. Then by Theorem C(i) or Theorem D(i),

$$\text{gr}_* (H_*(\Omega Y; R)) \cong UL'$ and $\text{gr}_1(H_*(\Omega Y; R)) \cong (UL')_1$. By (c), $(UL')_1$ is a free $(UL')_0$-
bimodule. We claim that it follows that $L'_1$ is a free $L'_0$-module. Indeed, if there is a nontrivial degree one relation in $L'$ then there is a corresponding nontrivial degree one relation in $UL'$.

We say that a cell attachment $f$ is semi-inert if $f$ is free and it satisfies the conditions of the previous lemma. We justify this terminology with the following lemma.

**Lemma 4.5.** An inert attaching map is semi-inert.

**Proof.** Let $f$ be an inert attaching map. If $R = \mathbb{Q}$ or $\mathbb{F}_p$ then by Theorem 4.3 $[dV_1]$ is a free Lie algebra. If $R \subset \mathbb{Q}$ then by Remark 4.1 $\forall p \in \tilde{P}$, $f \otimes \mathbb{F}_p$ is $p$-inert. So by Theorem 4.3 for each $p \in \tilde{P}$, $[dV_1]$ is a free Lie algebra. So in either case $f$ is free.

Since $(HL)_1 = 0$, the semi-inert condition is trivially satisfied.

**Remark 4.6.** For two-cones there is a nice equivalent condition to the semi-inert condition. It is known that the attaching map of a two cone $Y$ is inert iff $\text{gldim} H_\ast(\Omega Y) \leq 2$. Using the main result of [FL92] it follows directly that the attaching map of a two cone $Y$ is semi-inert iff $\text{gldim} H_\ast(\Omega Y) \leq 3$.

**Example 4.7.** Let $X = S^3 \vee S^3$ and let $\iota_a, \iota_b$ denote the inclusions of the spheres into $X$. Let $Y = X \cup_{\alpha_1 \vee \alpha_2} (e^8 \vee e^8)$ where the attaching maps are given by the iterated Whitehead products $\alpha_1 = [\iota_a, \iota_b], \iota_a]$ and $\alpha_2 = [\iota_a, \iota_b], \iota_b]$.

Let $W = S^7 \vee S^7$ and for $i = 1, 2$ let $\hat{\alpha}_i : S^6 \to \Omega X$ denote the adjoint of $\alpha_i$. Then $W \overset{\alpha_1 \vee \alpha_2}{\longrightarrow} X \to Y$ is a homotopy cofibration and $[L^W_X] = [h_X(\hat{\alpha}_1), h_X(\hat{\alpha}_2)]$.

$Y$ has an Adams-Hilton model (see Section 3.2) $U(L, d)$ where $L = \mathbb{L}(x, y, a, b)$, $|x| = |y| = 2$, $dx = dy = 0$, $da = [x, y], x]$ and $db = [x, y], y]$. Furthermore $h_X(\hat{\alpha}_1) = [da]$ and $h_X(\hat{\alpha}_2) = [db]$.

By Example 2.13, as algebras

$$HU(L, d) \cong \left( H_\ast(\Omega X; R)/(L^W_X) \right) \otimes \mathbb{T}([w]),$$
where \( w = [a, y] - [b, x] \). Thus \( \alpha_1 \vee \alpha_2 \) is nice but not inert. From Example 3.8
\[ H_*(\Omega X; R) \cong UL_X. \]
Thus as algebras
\[ H_*(\Omega Y; R) \cong U \left( L_X/[L_X^W] \amalg \mathbb{L}[w] \right). \]
Therefore \( \alpha_1 \vee \alpha_2 \) is a semi-inert attaching map.

\[ \square \]

**Example 4.8.** The 6-skeleton of \( S^3 \times S^3 \times S^3 \).

This space \( Y \) is also known as the fat wedge [Sel97] \( FW(S^3, S^3, S^3) \). Let \( X = S^3_a \cup S^3_b \cup S^3_c \). Let \( \iota_a, \iota_b \) and \( \iota_c \) be the inclusions of the respective spheres. Let \( W = \bigvee_{j=1}^3 S^5 \).

Then \( Y = X \cup_f \left( \bigvee_{j=1}^3 e_j^6 \right) \) where \( f : W \to X \) is given by \( \bigvee_{j=1}^3 \alpha_j \) with \( \alpha_1 = [\iota_b, \iota_c] \), \( \alpha_2 = [\iota_c, \iota_a] \) and \( \alpha_3 = [\iota_a, \iota_b] \). Let \( \hat{\alpha}_j : S^2 \to \Omega X \) denote the adjoint of \( \alpha_j \). Let \( I \) denote the two-sided ideal of \( H_*(\Omega X; R) \) generated by \( \{ h_X(\hat{\alpha}_j) \}_{j \in J} \).

Then \( Y \) has Adams-Hilton model (see Section 3.2) \( U(L, d) \) where \( L = \mathbb{L}(x, y, z, a, b, c), \)
\( |x| = |y| = |z| = 2, \ dx = dy = dz = 0, \ da = [y, z], \ db = [z, x] \) and \( dc = [x, y] \). By Example 2.14, as algebras \( HU(L, d) \cong U(\mathbb{L}_{ab}([x], [y], [z]) \amalg \mathbb{L}[w])) \) where \( w = [a, x] + [b, y] + [c, z] \). Therefore, as algebras
\[ H_*(\Omega Y; R) \cong (H_*(\Omega X; R)/I) \amalg \mathbb{L}[w]). \]

Thus \( f \) is nice but not inert.

From Example 3.8 \( H_*(\Omega X; R) \cong UL_X \). Let \( [L_X^W] \subset L_X \) be the Lie ideal generated by \( L_X^W \). Then as algebras
\[ H_*(\Omega Y; R) \cong U \left( (L_X/[L_X^W]) \amalg \mathbb{L}[w] \right). \]

Therefore \( f \) is a semi-inert attaching map. \[ \square \]

### 4.2 Correspondence between topology and algebra

Let \( R = \mathbb{Q} \) or \( \mathbb{F}_p \) where \( p > 3 \) or \( R \subset \mathbb{Q} \) be a subring containing \( \frac{1}{6} \). Let \( W \to X \to Y \) be the cofibration in the previous section where \( W = \bigvee_{j \in J} S^{n_j}, f = \bigvee_{j \in J} \alpha_j \) and \( H_*(\Omega X; R) \)
is torsion-free. Let $\hat{\alpha}_j$ denote the adjoint of $\alpha_j$. Let $A(X)$ and $A(Y)$ be the corresponding AH-models given in Section 3.2. Recall that $A(Y) = A(X) \amalg T(y_j)_{j \in J}$.

Recall from Section 3.1 that $L_X$ is the image of the Hurewicz map. Assume that $H_*(\Omega X; R) \cong UL_X$ as algebras. Let $d' : R\{y_j\} \to ZA(X) \to HA(X) \cong UL_X$ be the induced map. Recall from Section 3.2 that $d'y_j = h_X(\hat{\alpha}_j) \in L_X$. That is, $A(Y)$ is a dga extension of $(A(X), L_X)$ (see Section 2.4). Therefore the notation $L = (L_X \amalg T(y_j)_{j \in J}, d')$ defined in Section 4.1 is consistent with the notation defined in Section 2.4.

In fact if $V_1 = R\{y_j\}_{j \in J}$ then $d'V_1 \subset L^W_X$ and the Lie subalgebra of $L_X$ generated by $d'V_1$ is $L^W_X$. Recall that $[V] \subset L_X$ denotes the Lie ideal generated by $V \subset L_X$. Therefore $[L^W_X] = [d'V_1]$. Similarly if $R \subset \mathbb{Q}$ then for each $p \in \tilde{P}$, $[\tilde{L}^W_X] = [\tilde{d}'V_1]$.

Recall from Section 4.1 that if $R = \mathbb{Q}$ or $\mathbb{F}_p$ then $f$ is free if $[L^W_X] \subset L_X$ is a free Lie algebra, and if $R \subset \mathbb{Q}$ then $f$ is free if for each $p \in \tilde{P}$, $[\tilde{L}^W_X] \subset \tilde{L}_X$ is a free Lie algebra. Also recall from Section 2.4 that if $R = \mathbb{Q}$ or $\mathbb{F}_p$ then $A(Y)$ is a free dga extension of $A(X)$ if $[d'V_1] \subset L_X$ is a free Lie algebra and if $R \subset \mathbb{Q}$ then $A(Y)$ is a free dga extension of $A(X)$ if for each $p \in \tilde{P}$, $[\tilde{d}'V_1] \subset \tilde{L}_X$ is a free Lie algebra.

Since $[L^W_X] = [d'V_1]$ and $[\tilde{L}^W_X] = [\tilde{d}'V_1]$, the conditions for $f$ being free and $A(Y)$ being a free dga extension of $A(X)$ coincide. Likewise the inert and semi-inert conditions (see Sections 4.1 and 2.4) coincide.

### 4.3 Loop space decomposition theory

Let $X$ denote a simply-connected CW complex of finite type. Let $R \subset \mathbb{Q}$ be a subring of the rationals containing $1/2$ and $1/3$. Let $P$ be the set of primes invertible in $R$. Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$, where $p \notin P$.

An important idea in homotopy theory is to show that a given loop space is homotopy equivalent to a product of other spaces [CMN79a, Hus80, Coh95]. In particular, one can try and show that a space is homotopy equivalent to a product of simpler atomic spaces.
(defined below). These simpler spaces will have the property that their loop spaces are not homotopy equivalent to any nontrivial product. For example, if one can show that \( \Omega Y \approx \prod_i Y_i \) then \( \pi_n(Y) \cong \pi_{n-1}(\Omega Y) \cong \prod_i \pi_{n-1}(Y_i) \).

A space \( Y \) is called atomic if it is \( r \)-connected, \( \pi_{r+1}(Y) \) is a cyclic abelian group, and any self-map \( f : Y \rightarrow Y \) inducing an isomorphism on \( \pi_{r+1}(Y) \) is a homotopy equivalence. Let \( \mathcal{S} = \{ S^{2m-1}, \Omega S^{2m+1} | m \geq 1 \} \). The spaces in \( \mathcal{S} \) are atomic. Let \( \prod \mathcal{S} \) be the collection of spaces homotopy equivalent to a weak product of spaces in \( \mathcal{S} \). The properties of the homotopy groups of spaces in \( \prod \mathcal{S} \) are determined by the properties of the homotopy groups of spheres. For example if \( \Omega X \in \prod \mathcal{S} \) then \( X \) ‘satisfies the statement of the Moore conjecture’ [Sel88]. We make a brief digression explaining this statement.

Let \( X \) be a finite simply-connected CW-complex. The homotopy groups of \( X \), \( \{ \pi_n(X) \}_{n \geq 2} \), are finitely-generated abelian groups. As such, each has a torsion-free subgroup \( \mathbb{Z}^n \), and for each prime \( p \), a finite subgroup \( G_{n,p} \) of the elements with order \( p^t \) for some \( t \). Since each \( p \)-torsion subgroup \( G_{n,p} \) is finite, there is a number \( m \) such that \( p^m \cdot G_{n,p} = 0 \). After tensoring \( \pi_n(X) \) with the rational numbers \( \mathbb{Q} \), only the torsion-free subgroup remains.

Call an integer \( M \) an exponent for a group \( G \) if \( M \cdot x = 0 \\forall x \in G \). Call \( p^M \) a homotopy exponent for \( X \) at the prime \( p \) if \( \forall n, p^M \cdot G_{n,p} = 0 \). Call \( p^M \) an eventual \( H \)-space exponent for \( X \) at the prime \( p \) if \( p^M \) is an exponent for \( \Omega^N X \) for sufficiently large \( N \). The existence of an eventual \( H \)-space exponent implies the existence of a homotopy exponent.

We say that \( X \) is rationally elliptic if \( \pi_*(X) \otimes \mathbb{Q} \) is finite. Otherwise \( X \) is said to be rationally hyperbolic. The Rational Dichotomy Theorem of Félix, Halperin and Thomas [FHT82, Fé89, FHT01] states that if \( X \) is rationally hyperbolic then \( \pi_*(X) \otimes \mathbb{Q} \) grows exponentially. Similarly \( H_*(\Omega X; \mathbb{Q}) \) grows polynomially for rationally elliptic spaces and exponentially for rationally hyperbolic spaces.

In the late 1970’s, J.C. Moore conjectured a deep connection between the rational homotopy groups \( \pi_*(X) \otimes \mathbb{Q} \) and the \( p \)-torsion subgroups for each prime \( p \) [Sel88].
Conjecture 4.9 (The Moore conjecture). Let $X$ be a simply-connected finite CW-complex.

(a) if $X$ is rationally elliptic then $X$ has an eventual $H$-space exponent at every prime $p$, and

(b) if $X$ is rationally hyperbolic then $X$ does not have a homotopy exponent at any prime $p$.

This conjecture has been verified to hold for all but finitely many primes for elliptic spaces [MW86] and for 2-cones [Ani89], but there have been only sparse results for spaces outside these two classes [NS82, Sel83].

Lemma 4.10. If $\Omega X \in \prod S$ then for all primes $p$, $X$ is rationally elliptic if and only $X$ has a homotopy exponent at $p$ if and only if $X$ has an eventual $H$-space exponent at $p$.

Proof. By assumption $\Omega X \approx \prod_{i \in I} T_i$ for some $T_i \in S$. Then $X$ is rationally elliptic if and only if $I$ is finite. By [CMN79c, CMN79b] and [Gra69] this coincides with the existence of a homotopy exponent and an eventual $H$-space exponent.

From this it follows immediately:

Corollary 4.11. Let $R \subset \mathbb{Q}$ be a subring containing $\frac{1}{6}$. Let $W \rightarrow X \rightarrow Y$ be a cell attachment satisfying the assumptions of Theorem F. Then $\forall p \notin P_Y$ (see Section 4.4), $Y$ is rationally elliptic iff $Y$ has a homotopy exponent at $p$ iff $Y$ has an eventual $H$-space exponent at $p$.

Let $R \subset \mathbb{Q}$ be a subring with invertible primes $P$ and let $X$ be a simply-connected topological space. Then there exists a topological analogue to the algebraic localization of a $\mathbb{Z}$-module at $R$. We will denote the localization at $R$ by $X_R$ which we will also call the localization away from $P$.

There are many localization constructions. Perhaps the most widely used is that of Bousfield and Kan [BK72] (outlined in [Sel97]). We present a simple construction for the
localization of CW-complexes [FHT01]. One can construct $S^n_R$ to be the infinite mapping
telecope (see Section 3.1) of the sequence of maps

$$S^n \xrightarrow{j_1} S^n \xrightarrow{j_2} S^n \xrightarrow{j_3} \ldots S^n \xrightarrow{j_n} \ldots$$

where $j_k$ is the degree $m_k$ map where $m_k$ is the product of the first $k$ primes in $P$. 
Note that $e^{n+1} \cong (S^n \times I)/(S^n \times \{0\})$. We can localize the $(n + 1)$-cell by letting

$$e^{n+1}_R = (S^n_R \times I)/(S^n_R \times \{0\}) \text{.}$$

Since we can localize cells and spheres we can localize any CW-complex. One can check that this construction is functorial [FHT01]. Furthermore

$$\pi_*(X_R) = \pi_*(X) \otimes R \text{ and } H_*(X_R) = H_*(X) \otimes R \text{.}$$

The following is an expanded version of the Hilton-Serre-Baues Theorem [Bau81, 
Ani92]. Recall from Section 3.1 that if $H_*(\Omega X; R)$ is torsion-free, we have maps $h_X$ 
and $h_X \otimes \mathbb{F}$. Since $h_X$, $h_X \otimes \mathbb{F}$ are maps of Lie algebras taking Samelson products to 
commutator brackets [Sam53], they induce maps $\tilde{h}_X : U\pi_*(\Omega X) \otimes R \to H_*(\Omega X; R)$ and 
$\tilde{h}_X \otimes \mathbb{F} : U\pi_*(\Omega X) \otimes \mathbb{F} \to H_*(\Omega X; \mathbb{F})$.

**Theorem 4.12 (The Hilton-Serre-Baues Theorem).** The following are equivalent.

(a) There exists an $R$-equivalence $\tilde{\lambda} : \prod_i T_i \to \Omega X$ where $T_i \in S$ and the factors 
correspond\(^1\) to an $R$-basis of the image of $h_X$. That is, localized at $R$, $\Omega X \in \prod S$.

(b) $H_*(\Omega X; R)$ is $R$-free and the map $\tilde{h}_X$ is surjective. That is, $H_*(\Omega X; R)$ is generated 
as an algebra by the image of $h_X$.

(c) $H_*(\Omega X; R)$ is torsion-free and equal to a universal enveloping algebra generated 
by $\text{im}(h_X)$.

(d) $H_*(\Omega X; R)$ is torsion-free and $\forall p \notin P$ the map $\tilde{h}_X \otimes \mathbb{F}_p$ is surjective. That is, $H_*(\Omega X; \mathbb{F}_p)$ is generated as an algebra by the image of $h_X \otimes \mathbb{F}_p$.

(e) $H_*(\Omega X; R)$ is torsion-free and for all $p \notin P$, $H_*(\Omega X; \mathbb{F}_p)$ is equal to a universal 
enveloping algebra generated by $\text{im}(h_X \otimes \mathbb{F}_p)$.

\(^1\) For $\lambda_i \in \pi_*(X) \otimes R$, let $\tilde{\lambda}$ be the adjoint and $\{h(\tilde{\lambda}_i)\}$ a $R$-basis for $L_X$. If $n_i$ is even then $T_i = S^{n_i-1}$ 
and $\tilde{\lambda}_i = \lambda_i : T_i \to \Omega X$. If $n_i$ is odd then $T_i = \Omega S^{n_i}$ and $\tilde{\lambda}_i = \Omega \lambda_i : T_i \to \Omega X$. 


Proof. (a) $\Leftrightarrow$ (b) is the usual statement of the theorem ([Bau81, Lemma V.3.10], see also [Ani89, Lemma 3.1]).

(a) $\implies$ (c) By assumption, we have that $H_*(\Omega X; R) \cong H_*(\prod T_i; R)$. Use the Eilenberg-Zilber theorem and the Künneth theorem to write this as $\otimes_i H_*(T_i; R)$. Let $m$ be an odd number. Then by the Bott-Samelson Theorem (Theorem 3.6) $H_*(\Omega S^m; R) \cong U\mathbb{L}(x_{m-1})$.

In addition $H_*(S^m; R) \cong U\mathbb{L}_{ab}\langle x_m \rangle$, where $\mathbb{L}_{ab}$ denotes the abelian Lie algebra. Therefore $H_*(\Omega X; R) \cong \otimes_i U L_i \cong U(\prod L_i)$, where $L_i$ is generated by a Hurewicz image.

(c) $\implies$ (b) is trivial.

Also (c) $\implies$ (e) $\implies$ (d) is trivial.

(d) $\implies$ (b) $\tilde{h}_X \otimes \mathbb{Q}$ is surjective by the Milnor-Moore theorem. Since $\tilde{h}_X \otimes \mathbb{F}_p$ is surjective for all $p \notin P$, it follows that $\tilde{h}_X$ is surjective.

\[\square\]

### 4.4 Implicit primes

Let $R \subset \mathbb{Q}$ be a subring with invertible primes $P \supset \{2, 3\}$.

Consider the cofibration $W \xrightarrow{f=\bigvee \alpha_j} X \to Y$, where $W = \bigvee_{j \in J} S^{n_j}$ is a finite-type wedge of spheres, $H_*(\Omega X; R)$ is torsion-free and as algebras $H_*(\Omega X; R) \cong U L_X$.

We may need to exclude those primes $p$ for which an attaching map $\alpha_j \in \pi_*(X)$ includes a term with $p$-torsion. We will define this set of \textit{implicit primes} below.

Recall from Section 3.1 that $h_X : \pi_*(\Omega X) \otimes R \to L_X \subset H_*(\Omega X; R)$ is a Lie algebra map. Assume that there exists a Lie algebra map $\sigma_X : L_X \to \pi_*(\Omega X) \otimes R$ such that $h_X \circ \sigma_X = id_{L_X}$.

By the Milnor-Moore theorem [MM65], $h_X, \sigma_X$ are rational isomorphisms, so $\text{im}(\sigma_X \circ h_X - id)$ is a torsion element of $\pi_*(\Omega X) \otimes R$. Let $\gamma_j = \sigma_X h_X(\tilde{\alpha}_j) - \tilde{\alpha}_j$ where $\tilde{\alpha}_j : S^{n_j-1} \to \Omega X$ is the adjoint of $\alpha_j$. Then $t_j \gamma_j = 0 \in \pi_*(\Omega X) \otimes R$ for some $t_j > 0$. Let $t_j$ be the smallest such integer.

Define $P_Y$, the set of \textit{implicit primes} of $Y$ as follows. A prime $p$ is in $P_Y$ if and only
if \( p \in P \) or \( p \mid t_j \) for some \( j \in J \). We can invert these primes in \( R \) to get a new ring \( R' = \mathbb{Z}[P^{-1}] \) with invertible primes \( P' \). Localized away from \( P' \), \( \hat{\alpha}_j = \sigma_X h_X(\hat{\alpha}_j) = \sigma_X(dy_j) \).

The implicit primes have the following properties.

**Lemma 4.13.** (a) Let \( \{x_i\} \) be a set of Lie algebra generators for \( L_X \) and let \( \beta_i = \sigma_X x_i \). If all of the attaching maps are \( R \)-linear combinations of iterated Whitehead products of the maps \( \beta_i \), then \( P_Y = P \).

(b) If \( P = \{2, 3\} \) and \( n = \dim(Y) \) then the implicit primes are bounded by \( \max(3, n/2) \).

**Proof.** (a) Since \( \sigma_X \) is a Lie algebra map, in this case each \( \hat{\alpha}_j \in \text{im}(\sigma_X) \). So \( \hat{\alpha}_j = \sigma_X a_j \) for some \( a_j \in L_X \). Then \( \sigma_X h_X \hat{\alpha}_j = \sigma_X h_X \sigma_X a = \sigma_X a = \hat{\alpha}_j \). Therefore \( \forall j, \gamma_j = 0 \) and \( P_Y = P \).

(b) We assumed that \( H_*(\Omega X; R) \cong U L_X \) as algebras. By the Hilton-Serre-Baues Theorem (Theorem 4.12) \( \Omega X \in \prod S \). So \( \Omega X \cong \prod S^{2m_i-1} \times \prod \Omega S^{2m_k+1} \) where \( m_i, m_k \geq 1 \).

Take \( \hat{\alpha}_j \) to be the map \( \hat{\alpha}_j : S^{n_j} \to \prod S^{2m_i-1} \times \prod \Omega S^{2m_k+1} \). Thus let \( \hat{\alpha}_j = (g_i, g'_k) \) where \( g_i : S^{n_i} \to S^{2m_i-1} \) and \( g'_k : S^{n_k} \to \Omega S^{2m_k+1} \). So \( g_i \in \pi_{n_i} S^{2m_i-1} \) and \( g'_k \in \pi_{n_j+1} S^{2m_k+1} \).

Now in \( \pi_*(S^n) \) the first \( p \)-torsion element occurs in \( \pi_{n+2p-3}(S^n) \). Since \( \pi_l(S^1) = 0 \) for \( l > 1 \) so we can assume \( n \geq 3 \). So there exists \( p \)-torsion in \( \pi_l(S^n) \) only if \( l \geq n + 2p - 3 \iff p \leq (l - n + 3)/2 \leq l/2 \) (since \( n \geq 3 \)). Hence \( \hat{\alpha}_j \) contains \( p \)-torsion only if \( p \leq (n_j + 1)/2 \).

Since \( n_j \leq n - 1 \) the lemma follows. \( \square \)

### 4.5 Ganea’s fiber-cofiber construction

We review Ganea’s construction [Gan65]. Let

\[
F \xrightarrow{i} X \xrightarrow{\rho} Y
\]

be a fibration. Let \( X' \) be the cofiber of \( F \xrightarrow{i} X \). Since (4.2) is a fibration \( \rho \circ i = * \). Therefore \( \rho \) extends to a map \( \rho' : X' \to Y \). Let \( F' \) be the homotopy fiber of \( \rho' \).
Theorem 4.14 (Ganea’s Theorem [Gan65]). \( F' \) is weakly homotopy equivalent to \( F \ast \Omega Y \) (where \( A \ast B \) is the join of \( A \) and \( B \) [Sel97]).

Rutter [Rut71] strengthened this theorem to the following.

**Theorem 4.15 ([Rut71]).** \( F' \approx F \ast \Omega Y \).

**Remark 4.16.** If \( F \to X \to Y \) is a fibration of CW-complexes then by Whitehead’s Theorem (Theorem 3.3) these two theorems are equivalent.

Mather [Mat75] generalized Ganea’s construction as follows. Given the fibration (4.2) above and a map \( W \xrightarrow{f} X \) such that \( \rho \circ f \simeq * \), let \( X' \) be the cofiber of \( f \). Since \( \rho \circ f \simeq * \), \( \rho \) can be extended to a map \( \rho' : X' \to Y \). Let \( F' \) be the homotopy fiber of \( \rho' \). This construction is also used in [FT88]

Since \( \rho \circ f \simeq * \) there is a lifting \( g : W \to F \). Let \( K \) be the cofiber of \( g \).

**Theorem 4.17 ([Mat75]).** There is a cofibration \( K \to F' \to F \ast \Omega Y \).

Since Ganea’s construction begin and ends with a fibration we can iterate the construction to get the following commutative diagram of fibrations.

\[
\begin{array}{cccccc}
X & \xrightarrow{\rho} & X_1 & \xrightarrow{\rho_1} & X_2 & \cdots & \xrightarrow{\rho_{n-1}} & X_n & \xrightarrow{\rho_n} & \cdots \\
\downarrow{\rho} & & \downarrow{\rho_1} & & \downarrow{\rho_2} & & \cdots & & \downarrow{\rho_n} & & \\
Y & & & & & & & & \\
\end{array}
\]

Let \( F_n \) be the homotopy fiber of \( \rho_n \).

For example starting with the path-space fibration \( \Omega X \to PX \to X \) the iterated Ganea construction together with Theorem 4.15 gives that \( F_n \approx (\Omega X)^n \) (the \( n \)-th fold join of \( \Omega X \)).

The iterated Ganea construction is particularly useful for studying Lusternik-Schnirelmann category [Gan65, Jam95, FHT01, Sel97].
Part II

Homology of Differential Graded Algebras and Loop Spaces
Chapter 5

The Homology of DGA’s

In this chapter we prove our main algebraic results: Theorems A and B.

Let $R = \mathbb{Q}$ or $\mathbb{F}_p$ where $p > 3$ or $R$ is a subring of $\mathbb{Q}$ containing $\frac{1}{6}$. Recall that if $R \subset \mathbb{Q}$ then $P$ is the set of invertible primes in $R$ and $\hat{P} = \{ p \in \mathbb{Z} \mid p \text{ is prime and } p \notin P \} \cup \{0\}$. Let $(\hat{A}, \hat{d})$ be a connected finite-type dga over $R$ which is $R$-free. Let $Z\hat{A}$ denote the subalgebra of cycles of $\hat{A}$. Let $\mathbf{A} = (\hat{A} \oplus TV_1, d)$ be a connected finite-type dga over $R$ where $V_1$ is a free $R$-module and $d|_{\hat{A}} = \hat{d}$ and $dV_1 \subset Z\hat{A}$.

Assume that as algebras $H(\hat{A}, \hat{d}) \cong UL_0$ as algebras for some Lie algebra $L_0$ which is a free $R$-module. There is an induced map $d' : V_1 \xrightarrow{d} Z\hat{A} \rightarrow H\hat{A} \xrightarrow{\cong} UL_0$. Assume that $L_0$ can be chosen such that $d'V_1 \subset L_0$. In other words assume that $\mathbf{A}$ is a dga extension of $((\hat{A}, \hat{d}), L_0)$ (see Section 2.4).

Let $\mathbf{L} = (L_0 \oplus TV_1, d'')$. Then $\mathbf{L}$ is a bigraded dgL where the Lie algebra $L_0$ and the free $R$-module $V_1$ are in degrees 0 and 1 respectively and the differential $d'$ has bidegree $(-1, -1)$. Subscripts of bigraded objects will denote degree, eg. $M_0$ is the component of $M$ in degree 0.

The following lemma is a well-known fact, and the subsequent two lemmas are parts of lemmas from [Ani89]. We remind the reader that all of our $R$-modules have finite type.

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Lemma 5.1. Let $R \subset \mathbb{Q}$. A homomorphism $f : M \to N$ is an isomorphism if and only if for each $p \in \mathcal{P}$, $f \otimes \mathbb{F}_p$ is an isomorphism.

Let $L$ be a connected bigraded dgL. The inclusion $L \hookrightarrow UL$ induces a natural map

$$
\psi : UHL \to HUL. \quad (5.1)
$$

Lemma 5.2 ([Ani89, Lemma 4.1]). Let $R$ be a field of characteristic $k$ where $k = 0$ or $k > 3$. Then the map $\psi$ in (5.1) is an isomorphism in degrees 0 and 1.

Lemma 5.3 ([Ani89, Lemma 4.3]). Let $R$ be a subring of $\mathbb{Q}$ containing $\frac{1}{N}$. Suppose that $HUL$ is $R$-free in degree 0 and 1. Then $HL$ is $R$-free in degrees 0 and 1 and the map $\psi$ in (5.1) is an isomorphism in degrees 0 and 1.

Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. The Hilbert series of an $\mathbb{F}$-module is given by the power series $A(z) = \sum_{n=0}^{\infty} (\text{Rank}_F A_n) z^n$. Assuming that $A_0 \neq 0$, the notation $(A(z))^{-1}$ denotes the power series $1/(A(z))$.

Recall from Section 2.4 that $A$ is a filtered dga under the increasing filtration given by $F_{-1}A = 0$, $F_0A = \hat{A}$, and for $i \geq 0$, $F_{i+1}A = \sum_{j=0}^{i} F_jA \cdot V_1 \cdot F_{i-j}A$. We showed that this gives a first quadrant spectral sequence of algebras:

$$(E^0(A), d^0) = \text{gr}(A) \implies E^\infty = \text{gr}(HA)$$

where $E^0_{p,q}(A) = [F_p(A)/F_{p-1}(A)]_{p+q}$.

It is easy to check (see Section 2.5) that $(E^1, d^1) \cong UL$ and hence $E^2 \cong HUL$. The following theorem follows from the main result of Anick’s thesis [Ani82a, Theorem 3.7]. Anick’s theorem holds under either of two hypotheses. We will use only one of these.

Theorem 5.4. Let $R = \mathbb{F}$. If the two-sided ideal $(dV_1) \subset UL_0$ is a free $UL_0$-module then the above spectral sequence collapses at the $E^2$ term. That is, $\text{gr}(HA) \cong HUL$ as algebras. Furthermore the multiplication map

$$
\nu : \mathbb{T}(\psi(HL)_1) \otimes (HUL)_0 \to HUL
$$
is an isomorphism and \((HUL)_0 \cong UL_0/(d' V_1)\). In addition,
\[
HA(z)^{-1} = HUL(z)^{-1} = (1 + z)(HUL)_0(z)^{-1} - z(UL_0)(z)^{-1} - V_1(z).
\]
(5.2)

Proof. [Ani82a, Theorem 3.7] shows that the spectral sequence collapses as claimed and that the multiplication map \(TW \otimes (HUL)_0 \to HUL\) is an isomorphism where \(W\) is a basis for \((HUL)_1\) as a right \((HUL)_0\)-module. By Lemma 5.2 and the Poincaré-Birkhoff-Witt Theorem the homomorphism \(\psi(HL)_1 \otimes (HUL)_0 \to (HUL)_1\) induced by multiplication in \(HUL\) is an isomorphism. So we can let \(W = \psi(HL)_1\).

The remainder of the theorem follows directly from [Ani82a, Theorem 3.7].

Corollary 5.5. If \(R \subset Q\) and for each \(p \in \tilde{P}\), \((dV_1) \subset U\bar{L}_0\) is a free \(U\bar{L}_0\)-module then \(HA\) is R-free iff \(HUL\) is R-free iff \(L_0/[d' V_1]\) is R-free.

Proof. First \((HUL)_0 \cong UL_0/(d' V_1) \cong U(L_0/[d' V_1])\). So \((HUL)_0\) is R-free if and only if \(L_0/[d' V_1]\) is R-free.

Let \(A_p(z), B_p(z)\) and \(C_p(z)\) be the left, middle and right parts of (5.2) for \(F = F_p\) with \(F_0 = Q\). Then \(HA\) is R-free iff \(\forall p \in \tilde{P}, A_p(z) = A_0(z)\) and \(HUL\) is R-free iff \(\forall p \in \tilde{P}, B_p(z) = B_0(z)\). Since \(UL_0\) and \(V_1\) are R-free, \((HUL)_0\) is R-free iff \(\forall p \in \tilde{P}, C_p(z) = C_0(z)\). Since \(\forall p \in \tilde{P}, A_p(z) = B_p(z) = C_p(z)\) this proves the corollary.

We now prove a version of Theorem 5.4 for subrings of Q.

Theorem 5.6. Let \(R \subset Q\). If \(L_0/[d' V_1]\) is R-free and for each \(p \in \tilde{P}\), \((dV_1) \subset U\bar{L}_0\) is a free \(U\bar{L}_0\)-module, then \(HA\) is R-free and the multiplication map
\[
\nu : \mathbb{T}(\psi(HL)_1) \otimes (HUL)_0 \to HUL
\]
is an isomorphism. Also \(\text{gr}(HA) \cong HUL\) as algebras and \((HUL)_0 \cong UL_0/(d' V_1)\).

Proof. Since \(L_0/[d' V_1]\) is R-free, by Corollary 5.5 so are \(HUL\) and \(HA\). It follows from the Universal Coefficient Theorem that \(\forall p \in \tilde{P}, HA \otimes F_p \cong H(A \otimes F_p)\) and \(HUL \otimes F_p \cong HU(L \otimes F_p)\). Hence \(\forall p \in \tilde{P}, (HUL)_0 \otimes F_p \cong (HU(L \otimes F_p))_0\). Using Lemmas 5.3 and 5.2
\[
\psi(HL)_1 \otimes F_p \cong (HL)_1 \otimes F_p \cong H(L \otimes F_p)_1 \cong \psi(H(L \otimes F_p)_1).
\]
Thus \( \forall p \in \tilde{P} \)

\[
\nu \otimes \mathbb{F}_p : T(\psi(\mathcal{H}L_1) \otimes \mathbb{F}_p) \otimes (\mathcal{H}U \mathcal{L})_0 \otimes \mathbb{F}_p \rightarrow HA \otimes \mathbb{F}_p
\]
corresponds under these isomorphisms to the multiplication map

\[
T(\psi(\mathcal{H}(L \otimes \mathbb{F}_p)))_1 \otimes (\mathcal{H}U(L \otimes \mathbb{F}_p))_0 \rightarrow H(A \otimes \mathbb{F}_p).
\]

But this is an isomorphism by Theorem 5.4. Therefore \( \nu \) is an isomorphism by Lemma 5.1.

The last two isomorphisms also follow from Theorem 5.4. \( \Box \)

The next lemma will prove that if the Lie ideal \([d'V_1] \subset L_0\) is a free Lie algebra then the hypothesis in Anick’s Theorem (Theorem 5.4) holds. That is, \((d'V_1)\) is a free \(UL_0\)-module.

**Lemma 5.7.** Let \( A = UL \) over a field \( \mathbb{F} \). Assume \( I \) is a two-sided ideal of \( A \) generated by a Lie ideal \( J \) of \( L \) which is a free Lie algebra, \( \mathbb{L}W \). Then the multiplication maps \( A \otimes W \rightarrow I \) and \( W \otimes A \rightarrow I \) are isomorphisms of left and right \( A \)-modules respectively.

**Proof.** From the short exact sequence of Lie algebras

\[
0 \rightarrow J \rightarrow L \rightarrow L/J \rightarrow 0
\]

we get the short exact of sequence of Hopf algebras [Sel97, Theorem 10.5.3]

\[
\mathbb{F} \rightarrow U(J) \rightarrow U(L) \rightarrow U(L/J) \rightarrow \mathbb{F}
\]

and so \( UL \cong UJ \otimes U(L/J) \) as \( \mathbb{F} \)-modules. Since \( J \) is a free Lie algebra \( \mathbb{L}W \), \( UJ \cong TW \).

It is also a basic fact that \( U(L/J) \cong UL/I \). Hence we have that

\[
A \cong TW \otimes A/I \quad (5.3)
\]

as \( \mathbb{F} \)-modules. Furthermore

\[
A \cong I \oplus A/I \quad (5.4)
\]
as \( \mathbb{F} \)-modules.

Let \( M(z) \) denote the Hilbert series for the \( \mathbb{F} \)-module \( M \), and to simplify the notation let \( B = A/I \). Then from equations (5.3) and (5.4) we have the following (using \( (TW)(z) = 1/(1 - W(z)) \)).

\[
B(z) = A(z)(1 - W(z)), \quad I(z) = A(z) - B(z).
\]

Combining these we have \( I(z) = A(z)W(z) \). That is, \( I \cong A \otimes W \) as \( \mathbb{F} \)-modules.

Let \( \mu : A \otimes W \to I \) be the multiplication map. To show that it is an isomorphism it remains to show that it is either injective or surjective.

We claim that \( \mu \) is surjective. Since \( I \) is the ideal in \( A \) generated by \( W \), any \( x \in I \) can be written as

\[
x = \sum_i a_i w_i b_i \cdots b_{i_m}, \text{ where } a_i \in A, \ w_i \in W \text{ and } b_{i_k} \in L. \tag{5.5}
\]

Each such expression gives a sequence of numbers \( \{m_i\} \). Let \( M(x) = \min \max_i(m_i) \), where the minimum is taken over all possible ways of writing \( x \) as in (5.5). We claim that \( M(x) = 0 \).

Assume that \( M(x) = t > 0 \). Then \( x = x' + \sum_i a_i w_i b_i \cdots b_i \), where \( M(x') < t \). Now \( w_i b_i = [w_i, b_i] \pm b_i w_i \). Furthermore since \( J \) is a Lie ideal \( [w_i, b_i] \in J \cong \mathbb{F} W \), so

\[
[w_i, b_i] = \sum_j c_j [w_{j_1}, \ldots, w_{j_n}] = \sum_k d_k w_{k_1} \cdots w_{k_{N_k}} = \sum l a_l w_l,
\]

where \( a_l \in A \) and \( w_l \in W \). So \( x = x' + \sum_i \sum_k a_i a_k w_i b_2 \cdots b_{i_i} \). But this is of the form in (5.5) and shows that \( M(x) \leq t - 1 \) which is a contradiction.

Therefore for \( x \in I \) \( M(x) = 0 \) and we can write \( x = \sum_i a_i w_i \) where \( a_i \in A \) and \( w_i \in W \). Then \( x \in \text{im}(\mu) \) and hence \( \mu \) is an isomorphism.

Since \( A \) is associative \( \mu \) is a map of left \( A \)-modules.

The second isomorphism follows similarly. \( \square \)

We are now almost ready to prove Theorems A and B. Recall that \( A = (\bar{A} \oplus TV_1, d) \) where \( d\bar{A} \subset \bar{A} \) and \( dV_1 \subset Z\bar{A} \). Also \( H(\bar{A}, d) \cong UL_0 \) as algebras and if \( d' : V_1 \to UL_0 \) is
the induced map then $d'(V_1) \subset L_0$. Let $L = (L_0 \amalg LV_1, d')$ with $d'L_0 = 0$. Recall that there is a map $\psi : UHL \to HUL$. $A$ is a filtered dga under the increasing filtration given by $F_{-1}A = 0$, $F_0A = \tilde{A}$, and for $i \geq 0$, $F_{i+1} = \sum_{k=0}^{i} F_kA \cdot V_1 \cdot F_{i-k}A$. We prove one last lemma.

**Lemma 5.8.** There exists a quotient map

$$f : F_1HA \to (HUL)_1.$$  \hspace{1cm} (5.6)

Given $R \{\bar{w}_i\} \subset (HL)_1$ there exist cycles $\{w_i\} \subset F_1A$ such that $f([w_i]) = \bar{w}_i$.

**Proof.** By Theorem 5.4 $(\text{gr}(HA))_1 \cong (HUL)_1$. So there is a quotient map

$$f : F_1HA \to (\text{gr}(HA))_1 \cong (HUL)_1.$$ 

By Lemma 5.2 $(HL)_1 \cong (\varphi HL)_1 \subset (HUL)_1$. So for each $\bar{w}_i$ one can choose a representative cycle $w_i \in ZF_1A$ such that $f([w_i]) = \bar{w}_i$. \hfill \Box

We are now ready to prove Theorem A. We prove a slightly more detailed form of the theorem which we now state.

**Theorem 5.9 (Theorem A).** Let $R = F$ where $F = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $(\tilde{A}, \tilde{d})$ be a connected finite-type dga and let $V_1$ be a connected finite-type $\mathbb{F}$-module with a map $d : V_1 \to \tilde{A}$. Let $A = (\tilde{A} \amalg TV_1, d)$. Assume that there exists a Lie algebra $L_0$ such that $H(\tilde{A}, \tilde{d}) \cong UL_0$ as algebras and $d'V_1 \subset L_0$ where $d'$ is the induced map. Also assume that $[d'V_1] \subset L_0$ is a free Lie algebra. That is, $A$ is a dga extension of $((\tilde{A}, \tilde{d}), L_0)$ which is free. Let $L = (L_0 \amalg LV_1, d')$.

(a) Then as algebras

$$\text{gr}(HA) \cong U((HL)_0 \ltimes LL(L)_1)$$

with $(HL)_0 \cong L_0/[d'V_1]$ as Lie algebras.

(b) Furthermore if $A$ is semi-inert (that is, there is a free $R$-module $K$ such that $(HL)_0 \ltimes LL(L)_1 \cong (HL)_0 \amalg \mathbb{L}K$) then as algebras

$$HA \cong U((HL)_0 \amalg \mathbb{L}K').$$
for some $K' \subset F_1HA$ such that $f : K' \overset{\cong}{\to} K$, where $f$ is the quotient map in Lemma 5.8.

**Proof.** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Assume $[d'V_1] \subset L_0$ is a free Lie algebra.

(a) By Lemma 5.7 $(d'V_1) \subset UL_0$ is a free $UL_0$-module. So we can apply Theorem 5.4 to show that $\text{gr}(HA) \cong HUL$ as algebras and that the multiplication map

$$\nu : \mathbb{T}(\psi(HL)_1) \otimes (HUL)_0 \to HUL$$

is an isomorphism.

By Lemma 5.2 $(HUL)_0 \cong U(HL)_0$ and $\psi(HL)_1 \cong (HL)_1$. By the definition of homology $(HL)_0 \cong L_0/[d'V_1]$.

$(HL)_0$ acts on $(HL)_1$ via the adjoint action. Let $L' = (HL)_0 \ltimes (\psi(HL)_1)$. There is an induced Lie algebra map $u : L' \to HUL$ which gives an induced algebra map $\tilde{u} : UL' \to HUL$. To simplify the notation we will refer to $(HL)_0$ and $\psi(HL)_1$ by $L'_0$ and $L'_1$ respectively.

Recall that as $R$-modules, $L' \cong L'_0 \times L'_1$. The Poincaré-Birkhoff-Witt Theorem shows that the multiplication map

$$\phi : \mathbb{T}L'_1 \otimes (HUL)_0 \cong U\mathbb{L}L'_1 \otimes UL'_0 \to UL'$$

is an isomorphism. Since $\tilde{u}$ is an algebra map $\nu = \tilde{u} \circ \phi$. Therefore $\tilde{u} : UL' \to HUL$ is an isomorphism. Hence $\text{gr}(HA) \cong UL'$ as algebras. This finishes the first part of the Theorem.

(b) Recall that $L'_0$ acts on $L'_1 = (\psi HL)_1 \cong (HL)_1$ via the adjoint action. Assume that $A$ is semi-inert. That is, $\exists \{w_i\} \subset L'_1$ such that $L' \cong L'_0 \mathbb{I} LK$, where $K = R\{w_i\}$. Recall from (a) that $HUL \cong \text{gr}(HA)$ and that the inclusions $L'_0 \subset HUL$ and $w_i \in HUL$ induce a Lie algebra map

$$u : L' \to \text{gr}(HA).$$
By Lemma 5.8 \( \exists w_i \in F_1 A \) such that \( f([w_i]) = \bar{w}_i \) where \( f \) is the map in (5.6). Let \( K' = R\{[w_i]\} \subset F_1 HA \), and let \( L'' = L_0' \amalg L' \). Then \( f : K' \to K \) and \( f \) induces an isomorphism \( L'' \overset{\cong}{\to} L' \).

By part (a) \( L_0' \subset (\text{gr} HA)_0 \). Since \( F_{-1} HA = 0 \), \( (\text{gr} HA)_0 \subset HA \), so \( L_0' \subset HA \). Since \( L_0', K' \subset HA \) there are induced maps

\[
\begin{array}{ccc}
L'' & \overset{\eta}{\longrightarrow} & HA \\
\downarrow & & \downarrow \\
UL'' & \overset{\theta}{\longrightarrow} & \end{array}
\]

where \( \theta \) is an algebra map.

Grade \( L'' \) by letting \( L_0' \) be in degree 0 and \( K' \) be in degree 1. This also filters \( L'' \). Then \( \eta \) is a map of filtered objects.

From this we get the following commutative diagram

\[
\begin{array}{ccc}
\text{gr}(L'') & \overset{\text{gr}(\eta)}{\longrightarrow} & \text{gr}(HA) \\
\downarrow & & \downarrow \\
\text{gr}(UL'') & \overset{\text{gr}(\theta)}{\longrightarrow} & \text{gr}(L'') \\
\cong & & \downarrow \rho \\
U \text{gr}(L'') & & \\
\end{array}
\]

Now \( \text{gr}(L'') \cong L'' \cong L' \) and \( \text{gr}(\eta) \) corresponds to \( u \) under this isomorphism. So \( \rho \) corresponds to \( \tilde{u} \) which is an isomorphism. Thus \( \text{gr}(\theta) \) is an isomorphism, and hence \( \theta \) is an isomorphism. Therefore \( HA \cong UL'' \) which finishes the proof. \qed

We will prove a slightly more detailed form of Theorem B which we now state.

**Theorem 5.10 (Theorem B).** Let \( R \subset \mathbb{Q} \) be a subring containing \( \frac{1}{6} \). Let \( (\bar{A}, \bar{d}) \) be a connected finite-type dga and let \( V_1 \) be a connected finite-type free \( R \)-module with a map \( d : V_1 \to \bar{A} \). Let \( A = (\bar{A} \amalg TV_1, d) \). Assume that \( H(\bar{A}, \bar{d}) \) is \( R \)-free and that there exists a Lie algebra \( L_0 \) such that \( H(\bar{A}, \bar{d}) \cong UL_0 \) as algebras and \( d'V_1 \subset L_0 \) where \( d' \) is the induced map. Also assume that \( L_0/[d'V_1] \) is a free \( R \)-module and that for any \( p \in \bar{P} \),
$[dV_1] \subset L_0$ is a free Lie algebra. That is, $A$ is a dga extension of $((\tilde{A}, \tilde{d}), L_0)$ which is free. Let $\underline{L} = (L_0 \amalg V_1, d')$.

(a) Then $HA$ and $\text{gr}(HA)$ are $R$-free and as algebras

$$\text{gr}(HA) \cong U((HL)_0 \ltimes \underline{(HL)}_1)$$

with $(HL)_0 \cong L_0/[dV_1]$ as Lie algebras. Additionally $(HL)_0 \ltimes \underline{(HL)}_1 \cong \underline{H(L)}$ as Lie algebras.

(b) Furthermore if $A$ is semi-inert (that is, there is a free $R$-module $K$ such that $(HL)_0 \ltimes \underline{(HL)}_1 \cong (HL)_0 \amalg \underline{L}K$) then as algebras

$$HA \cong U((HL)_0 \amalg \underline{L}K')$$

for some $K' \subset F_1HA$ such that $f : K' \cong \tilde{K}$, where $f$ is the quotient map in Lemma 5.8.

Proof. We follow the argument in the proof of Theorem A to prove Theorem B, but add an additional argument to show that $\underline{H(L)} \cong (HL)_0 \ltimes \underline{(HL)}_1$ as Lie algebras.

Let $R$ be a subring of $\mathbb{Q}$ containing $\frac{1}{6}$. Assume $L_0/[dV_1]$ is $R$-free and that for each $p \in \tilde{P}$, $[dV_1] \subset L_0$ is a free Lie algebra.

(a) By Corollary 5.5, $HA$ and $HUL$ are $R$-free and by Lemma 5.7, for each $p \in \tilde{P}$, $(dV_1) \subset UL_0$ is a free $UL_0$-module. So we can apply theorem 5.6 to show that $\text{gr}(HA) \cong HUL$ as algebras and that the multiplication map

$$\nu : \mathbb{T}(\underline{H(L)}_1) \otimes (HL)_0 \to HUL$$

is an isomorphism.

By Lemma 5.3 $(HUL)_0 \cong U(HL)_0$ and by the definition of homology $(HL)_0 \cong L_0/[dV_1]$.

Let $N = \underline{H(L)}$ and define

$$L' = N_0 \ltimes \underline{L}N_1.$$

Note that $L_0 = N_0$ and $L_1 = N_1$. There is a Lie algebra map $u : L' \to N$ and an induced algebra map $\tilde{u} : UL' \overset{u}{\longrightarrow} UN \to HUL$. 


Again the Poincaré-Birkhoff-Witt Theorem shows that the multiplication map

\[ \phi : \mathcal{T}N_1 \otimes (HUL)_0 \cong U_1 N_1 \otimes U N_0 \rightarrow U L' \]

is an isomorphism. Since \( \bar{u} \) is an algebra map \( \nu = \bar{u} \circ \phi \). Thus \( \bar{u} \) is an isomorphism. Therefore \( HUL \cong UL' \) as algebras, as in the proof of Theorem A.

Unlike the proof of Theorem A, we will show that \( u : L' \rightarrow N \) is an isomorphism. Let \( \iota : N \hookrightarrow HUL \) be the inclusion. Since the composition \( L' \xrightarrow{u} N \xrightarrow{\iota} HUL \cong UL' \) is the inclusion \( L' \hookrightarrow UL' \), \( u \) is injective. So as \( R \)-modules \( N \cong L' \oplus N/L' \). Since \( L' \) and \( N \) are \( R \)-free, so is \( N/L' \).

Recall that the composition \( \iota \circ u \) induces the isomorphism \( \bar{u} : UL' \xrightarrow{Uu} UN \rightarrow HUL \).

Tensor these maps with \( \mathbb{Q} \) to get the commutative diagram

\[
\begin{array}{ccc}
UL' \otimes \mathbb{Q} & \xrightarrow{Uu \otimes \mathbb{Q}} & UN \otimes \mathbb{Q} \\
\downarrow \cong & & \downarrow \\
HUL \otimes \mathbb{Q} & & \\
\end{array}
\]

(5.7)

It is a classical result that the natural map

\[
\psi_{\mathbb{Q}} : UH(L \otimes \mathbb{Q}) \xrightarrow{\cong} HU(L \otimes \mathbb{Q})
\]

is an isomorphism (see [FHT01, Theorem 21.7(i)] for example). Notice that

\[
N \otimes \mathbb{Q} = (\psi_{HUL}) \otimes \mathbb{Q} \cong \psi_{\mathbb{Q}} H(L \otimes \mathbb{Q}) \cong H(L \otimes \mathbb{Q})
\]

and \( HUL \otimes \mathbb{Q} \cong HU(L \otimes \mathbb{Q}) \). Under these isomorphisms the vertical map in (5.7) corresponds to the isomorphism in (5.8).

Therefore \( Uu \otimes \mathbb{Q} \) is an isomorphism and hence \( u \otimes \mathbb{Q} \) is surjective. As a result \( \operatorname{coker} u = N/L' \) is a torsion \( R \)-module. But we have already shown that \( N/L' \) is \( R \)-free. Thus \( N/L' = 0 \) and \( N \cong L' \). Hence \( HUL \cong UN \).

If \( A \) is semi-inert then the proof of (b) is the same as for Theorem A(b). \( \square \)

As claimed in the introduction, we prove that the following corollary follows from Theorems A and B.
Corollary 5.11 (Corollary 1.3). Let $A$ be a semi-inert dga extension satisfying the hypotheses of either Theorems A or B. Then

$$K'(z) = V_1(z) + z[U L_0(z)^{-1} - U(HL)_0(z)^{-1}].$$

Proof. By Theorem A or Theorem B,

$$HA \cong U((HL)_0 \star K') \cong U(HL)_0 \star TK'.$$

Since $(A \star B)(z)^{-1} = A(z)^{-1} + B(z)^{-1} - 1$ [Lem74, Lemma 5.1.10] it follows that

$$HA(z)^{-1} = U(HL)_0(z)^{-1} - K'(z)$$

Therefore, using Anick’s formula 5.2 we have that

$$K'(z) = U(HL)_0(z)^{-1} - HA(z)^{-1}
= U(HL)_0(z)^{-1} - (1 + z)(HU)_0(z)^{-1} + zUL_0(z)^{-1} + V_1(z)
= V_1(z) + z[U(HL)_0(z)^{-1} - UL_0(z)^{-1}]$$

$\square$
Chapter 6

Application to Cell Attachments

In this chapter we prove two of our main topological results: Theorems C and D.

Let \( f : W \xrightarrow{f} X \) be a map of finite-type simply-connected CW-complexes where \( W = \bigvee_{j \in J} S^{m_j} \) and \( f = \bigvee_{j \in J} \alpha_j \). Let \( \hat{\alpha}_j : S^{m_j-1} \rightarrow \Omega X \) denote the adjoint of \( \alpha_j \). Let \( Y \) be the adjunction space

\[
Y = W \cup_f \left( \bigvee_{j \in J} e^{m_j+1} \right).
\]

Let \( R = \mathbb{Q} \) or \( \mathbb{F}_p \) where \( p > 3 \) or \( R \) is a subring of \( \mathbb{Q} \) containing \( \frac{1}{6} \). Recall that \( P \) is the set of primes invertible in \( R \), \( \tilde{P} = \{ p \in \mathbb{Z} \mid p \text{ is prime and } p \notin P \} \cup \{0\} \) and \( \mathbb{F}_0 = \mathbb{Q} \).

Recall from Section 3.2 that we can choose Adams-Hilton models \( A(X) \) and \( A(Y) \) for \( X \) and \( Y \) such that

\[
A(Y) = A(X) \amalg T(y_j)_{j \in J}.
\] (6.1)

These come with algebra isomorphisms \( H_*(\Omega X; R) \cong HA(X) \) and \( H_*(\Omega Y; R) \cong HA(Y) \).

Filter \( A(Y) \) by taking \( F_{-1}A(Y) = 0 \), \( F_0A(Y) = A(X) \) and for \( i \geq 0 \), \( F_iA(Y) = \sum_{k=0}^i F_kA(Y) \cdot R\{y_j\}_{j \in J} \cdot F_{i-k}A(Y) \). This filtration makes \( A(Y) \) a filtered dga.

Recall from Section 2.5 that this filtration induces a first quadrant multiplicative spectral sequence converging from \( \text{gr}(A(Y)) \) to \( \text{gr}(HA(Y)) \).

Recall that \( L_X = \text{im}(h_X : \pi_*(\Omega X) \otimes R \rightarrow H_*(\Omega X; R)) \). Assume that \( H_*(\Omega X; R) \cong UL_X \) as algebras and that it is \( R \)-free. Then \( (E^1, d^1) \cong (UL_X \amalg T(y_j)_{j \in J}, d') \) where
$d'$ is determined by the induced map $d' : R\{y_j\} \to ZA(X) \xrightarrow{\cong} H_* (\Omega X; R) \xrightarrow{\cong} UL_X$. Recall from Section 3.2 that $d' y_j = h_X(\tilde{\alpha}_j) \in L_X$. Therefore $A(Y)$ is a dga extension (see Section 2.4) of $(A(X), L_X)$.

Furthermore, $L^W_X$ is the Lie subalgebra of $L_X$ generated by $R\{d' y_j\}_{j \in J}$. Therefore $[L^W_X] = [R\{d' y_j\}]$. Let $L = (L_X \amalg L(y_j), d')$. Then $(E^1, d^1) \cong UL_L$.

We can prove Theorem C by applying Theorem A to $A(Y)$ and $L$. For convenience we restate Theorem C here.

**Theorem 6.1 (Theorem C).** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $X$ be a finite-type simply-connected CW-complex such that $H_* (\Omega X; R)$ is torsion-free and as algebras $H_* (\Omega X; R) \cong UL_X$ where $L_X$ is the Lie algebra of Hurewicz images. Let $W = \bigvee_{j \in J} S^{n_j}$ be a finite-type wedge of spheres and let $f : W \to X$. Let $Y = X \cup_f \left( \bigvee_{j \in J} e^{n_j} \right)$. Assume that $[L^W_X] \subset L_X$ is a free Lie algebra. That is, $f$ is free.

(a) Then as algebras

$$\text{gr}(H_* (\Omega Y; \mathbb{F})) \cong U(L^X_Y \ltimes \mathbb{L}(H\mathbb{L}))_1$$

with $L^X_Y \cong L_X/[L^W_X]$ as Lie algebras.

(b) Furthermore if $f$ is semi-inert then as algebras

$$H_* (\Omega Y; \mathbb{F}) \cong U(L^X_Y \amalg K')$$

for some $K' \subset F_1 H_* (\Omega Y; \mathbb{F})$.

**Proof.** (a) Let $A(Y)$ be the Adams-Hilton model given above. Therefore $H_* (\Omega Y; \mathbb{F}) \cong A(Y)$ as algebras. $A(Y)$ is a dga extension of $(A(X), L_X)$. Since $[L^W_X] \subset L_X$ is a free Lie algebra, $A(Y)$ is a free dga extension of $(A(X), L_X)$. Thus by Theorem A we have the algebra isomorphism

$$\text{gr}(HA(Y)) \cong U(((H\mathbb{L})_0 \ltimes \mathbb{L}(H\mathbb{L}))_1)$$

with $(H\mathbb{L})_0 \cong L_X/[L^W_X]$. Therefore

$$F_0 HA(Y) \cong (\text{gr}(HA(Y)))_0 \cong U(H\mathbb{L})_0 \cong U(L_X/[L^W_X]).$$

(6.2)
It remains to show that $(H\mathbf{L})_0 \cong L_Y^X$.

The inclusion $i : A(X) \xrightarrow{\cong} F_0A(Y)$ induces a map $H(i) : HA(X) \to F_0HA(Y)$. Now under the isomorphism (6.2) and $UL_X \xrightarrow{\cong} HA(X)$ the map $H(i)$ corresponds to a map $UL_X \to U(L_X/[L_X^W])$ where $U(L_X/[L_X^W]) \subset UL_Y$. It is easy to check that this is the canonical map. In other words $L_Y^X \cong L_X/[L_X^W]$. Therefore $(H\mathbf{L})_0 \cong L_Y^X$.

(b) Assume that $f$ is semi-inert. That is, there exists an $\mathbb{F}$-module $K$ such that $L_Y^X \times \mathbb{L}(H\mathbf{L})_1 \cong L_Y^X \, \mathbb{L}K$. Then by Theorem A(b), $H_*(\Omega Y; \mathbb{F}) \cong U(L_Y^X \, \mathbb{L}K')$ for some $K' \subset F_1HUL = F_1H_*(\Omega Y; \mathbb{F})$. 

We recall the statement of Theorem D.

**Theorem 6.2 (Theorem D).** Let $R \subset \mathbb{Q}$ be a subring containing $\frac{1}{5}$. Let $X$ be a finite-type simply-connected CW-complex such that $H_*(\Omega X; R)$ is torsion-free and as algebras $H_*(\Omega X; R) \cong UL_X$ where $L_X$ is the Lie algebra of Hurewicz images. Let $W = \bigvee_{j \in J} S^{n_j}$ be a finite-type wedge of spheres and let $f : W \to X$. Let $Y = X \cup_f \left( \bigvee_{j \in J} e^{n_j + 1} \right)$. Assume that $L_X/[L_X^W]$ is $R$-free and that for each $p \in \hat{P}$, $[L_X^W] \subset L_X$ is a free Lie algebra. That is, $f$ is free.

(a) Then $H_*(\Omega Y; R)$ and $\text{gr}(H_*(\Omega Y; R))$ are torsion-free and as algebras

\[
\text{gr}(H_*(\Omega Y; R)) \cong U(L_Y^X \times \mathbb{L}(H\mathbf{L})_1)
\]

with $L_Y^X \cong L_X/[L_X^W]$ as Lie algebras.

(b) If in addition $f$ is semi-inert then as algebras

\[
H_*(\Omega Y; R) \cong U(L_Y^X \, \mathbb{L}K')
\]

for some $K' \subset F_1H_*(\Omega Y; R)$.

**Proof.** The proof Theorem D is exactly the same as the proof of Theorem C, except that it uses Theorem B instead of Theorem A. 

In the introduction we claimed that the following corollary follows from Theorems C and D.
Corollary 6.3 (Corollary 1.8). Let $f$ be a semi-inert attaching map satisfying the hypotheses of either Theorems C or D. Then

$$K'(z) = \tilde{H}_*(W)(z) + z[U L_X(z)^{-1} - U (L_Y^X)(z)^{-1}].$$ (6.3)

Proof. By Theorem C or Theorem D,

$$H_*(\Omega Y; R) \cong U(L_Y^X \ll K') \cong U L_Y^X \uplus K'.$$

Since $(A \ll B)(z)^{-1} = A(z)^{-1} + B(z)^{-1} - 1$ [Lem74, Lemma 5.1.10] it follows that

$$H_*(\Omega Y; R)(z)^{-1} = U L_Y^X(z)^{-1} - K'(z)$$

Let $V_1 = R\{y_j\}_{j \in J}$, where $y_j$ is given in (6.1). Then $V_1(z) = \tilde{H}_*(W)(z)$. Recall that $(HL)_0 \cong L_Y^X$. Therefore, using Anick’s formula (5.2) (with $L_0 = L_X$) we have that

$$K'(z) = U L_Y^X(z)^{-1} - H_*(\Omega Y; R)(z)^{-1}$$

$$= U L_Y^X(z)^{-1} - (1 + z)U (HL)_0(z)^{-1} + z U L_X(z)^{-1} + V_1(z)$$

$$= \tilde{H}_*(W)(z) + z[U L_Y^X(z)^{-1} - U L_X(z)^{-1}]$$
Chapter 7

Constructing Spherical Hurewicz Maps

In this chapter we prove our other two main topological results: Theorems E and F.

Let $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ be a subring containing $\frac{1}{6}$.

Consider the map $W \rightarrow X$ where $W = \bigvee_{j \in J} S^{m_j}$ is a finite-type wedge of spheres, $f = \bigvee_{j \in J} \alpha_j$, $H_*(\Omega X; R)$ is torsion-free and as algebras $H_*(\Omega X; R) \cong U L_X$. Let $Y$ be the adjunction space

$$Y = X \cup_f \left( \bigvee_{j \in J} e^{m_j+1} \right).$$

Assume that $f$ is free. That is, $[L_X]$ is a free Lie algebra. Recall that $h_X : \pi_*(\Omega X) \otimes R \rightarrow L_X \subset H_*(\Omega X; R)$ is a Lie algebra map. Assume that there exists a Lie algebra map $\sigma_X : L_X \rightarrow \pi_*(\Omega X) \otimes R$ such that $h_X \circ \sigma_X = id_{L_X}$.

If $R \subset \mathbb{Q}$ recall from Section 4.4 that there is a set of implicit primes $P_Y$ containing the invertible primes in $R$. By replacing $R$ with $R' = \mathbb{Z}[P_Y^{-1}]$ if necessary, we may assume that there are no non-invertible implicit primes. This implies that $\forall j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$.

If $R = \mathbb{F}_p$ assume that $\forall j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$.

By Theorems C and D, $H_*(\Omega Y; R)$ is torsion-free and $\text{gr}(H_*(\Omega Y; R)) \cong U(L_Y^N \ltimes \mathbb{L}(HL_1))$ as algebras. From this we want to show that $H_*(\Omega Y; R) \cong UL_Y$ as algebras.
Chapter 7. Constructing Spherical Hurewicz Maps

This situation closely resembles that of torsion-free spherical two-cones, and we will generalize Anick’s proof for that situation [Ani89].

Recall that \( L = L_X \amalg \mathbb{L}(y_j)_{j \in J}, \) \( \psi : H_\mathbb{L} \to H(U_\mathbb{L}), (\psi H_\mathbb{L})_1 \cong (H_\mathbb{L})_1, \) and \( H(U_\mathbb{L}) \cong \text{gr}(HA(Y)). \) Let \( K' = (H_\mathbb{L})_1 \) and \( \{ \bar{w}_i \} \) be a \( R \)-module basis for \( K' \). Then \( \bar{w}_i \in (H_\mathbb{L})_1 = (H(L_X \amalg \mathbb{L}(y_j)_{j \in J}, d'))_1. \) So each \( \bar{w}_i \) is represented by a sum of brackets each with one \( y_j \) and other elements in \( L_X. \) Using Jacobi identities we can write

\[
\bar{w}_i = \left[ \sum_{k=1}^{s} c_k v_k \right] \text{ where } c_k \in R \text{ and } v_k = \left[ [y_{j_k}, [x_{k_1}], \ldots, [x_{k_{n_k}}]] \right] \tag{7.1}
\]

where \( x_{k_1} \in A(X), [x_{k_1}] \in L_X \) and the bracket is defined inductively by \( [[a, \ldots b], c] = [[[a, \ldots b], c]. \)

Define \( \gamma_i \in F_1 A(Y) \) by

\[
\gamma_i = \sum_{k=1}^{s} c_k u_k \text{ where } u_k = \left[ [y_{j_k}, x_{k_1}, \ldots, x_{k_{n_k}}] \right]. \tag{7.2}
\]

We will use \( \gamma_i \) and Adams-Hilton models to construct a map \( g_i : S^{m+1} \to Y, \) whose Hurewicz image ‘modulo lower filtration’ is \( \bar{w}_i. \)

The following geometric construction is a slight generalization of [Ani89, Proposition 5.4] whose proof is essentially the same. It is the central construction of this chapter and will give us the desired map \( S^{m+1} \to Y. \)

**Proposition 7.1.** Let \( W \to X \to Y \) be as above. Let \( \gamma = \sum_{k=1}^{s} c_k u_k \) as in (7.2) with \( [x_{k_1}] \in L_X. \) Then there exists a map

\[
g : S^{m+1}_s \to Y
\]

which has an AH model \( A(g) \) satisfying \( A(g)(b_k) = c_k u_k \) for \( 1 \leq k \leq s, \) and \( A(g)(b_0) \in A(X). \)

**Proof.** By Lemma 3.10 there exist maps \( g_k : (D^{m+1}_0, S^m_0) \to (Y, X) \) for \( 1 \leq k \leq s \) with models \( A(g_k)(ii) = c_k u_k. \) In addition

\[
g_k|S^m_0 = \epsilon_k c_k [[\alpha_{j_k}, \beta_{k_1}, \ldots, \beta_{k_{n_k}}]]
\]
where \( \epsilon_k = \pm 1 \) and \( \beta_k : S^{m+1} \to X \) is the adjoint of \( \sigma([x_i]) : S^n \to \Omega X \) for \( 1 \leq k \leq s \), and \( \psi_{g_k}(i) \in CU_\ast(\Omega X) \) for \( 1 \leq k \leq s \). Let

\[
g' = g_1 \lor \ldots \lor g_s : \left( \bigvee_{k=1}^s D_0^{m+1}, \bigvee_{k=1}^s S_0^m \right) \to (Y, X)
\]

for which one can choose \( A(g')(b_k) = c_k u_k \) for \( 1 \leq k \leq s \).

Restricting \( g' \) we get a map \( g_0 : \left( \bigvee_{k=1}^s S_0^m \right) \to X \), and \( (A(g'), \psi_{g'}) \) extends a valid AH model \( (A(g_0), \psi_{g_0}) \) for \( g_0 \).

We will show that \( g_0 \) can also be extended to a map

\[
g'' : \left( \bigvee_{k=1}^s S_0^m \right) \cup \Sigma_{k=1}^s \iota_k E^{m+1} \to X,
\]

where \( \iota_k \) is the inclusion of the \( k \)th sphere into the wedge. It will follow at once that there is an AH model for \( g'' \) which extends \( (A(g_0), \psi_{g_0}) \) and has \( A(g'')(b_0) \in A(X) \), \( b_0 \) denoting the \( m \)-dimensional generator of \( A(\bigvee_{k=1}^s S_0^m \cup \Sigma_{k=1}^s \iota_k E^{m+1}) \).

To prove the existence of \( g'' \), it suffices to show that \( g_0\#(\Sigma_{k=1}^s \iota_k) \) vanishes in \( \pi_m(X) \).

\[
g_0\#(\Sigma_{k=1}^s \iota_k) = \Sigma_{k=1}^s \epsilon_k c_k [\alpha_{j_k}, \beta_{k_1}, \ldots \beta_{k_{n_k}}]
\]

Recall from (7.1) that \( \bar{w}_i = [\Sigma_{k=1}^s c_k v_k] \). Since \( \Sigma_{k=1}^s c_k v_k \) is a cycle in \( L \),

\[
0 = d \Sigma_{k=1}^s c_k v_k = \Sigma_{k=1}^s c_k \left[ \frac{d}{y_{j_k}}, [x_{k_1}], \ldots [x_{k_{n_k}}] \right]
\]

\[
= \Sigma_{k=1}^s c_k \left[ \frac{\partial X(\partial_{j_k})}{x_{k_1}}, [x_{k_1}], \ldots [x_{k_{n_k}}] \right].
\]

Since \( \sigma_X h_X \partial_{j_k} = \partial_{j_k} \) applying the map \( \sigma_X \) gives

\[
0 = \Sigma_{k=1}^s c_k \left[ \partial_{j_k}, \sigma_X([x_{k_1}]), \ldots, \sigma_X([x_{k_{n_k}}]) \right]
\]

and adjoining gives that \( g_0\#(\Sigma_{k=1}^s \iota_k) = 0 \).

Now \( g' \) and \( g'' \) are compatible extensions of \( g_0 \), so together they define a map \( g : S^{m+1} \to Y \). Furthermore the corresponding AH models are compatible so they give a valid AH model for \( g \) with the desired properties. \( \square \)
We review and continue our construction. Starting with $\bar{w}_i \in (\text{HL})_1$ we choose $\gamma_i$ as above and use Proposition 7.1 to construct $g_i : S^{m+1}_a \rightarrow Y$. We then have the following map
\[
\phi_i : S^{m+1}_0 \xrightarrow{\Psi} S^{m+1}_a \xrightarrow{g_i} Y
\]
where $\Psi : S^{m+1}_0 \rightarrow S^{m+1}_a$ is the homeomorphism from (3.4). Taking $A(\phi_i) = A(g_i) \circ A(\Psi)$ we have $A(\phi_i)(i) = \Sigma_{k=1}^s c_k u_k - A(g_i)b_0$. Letting $\lambda_i = A(g_i)(b_0)$ we have $A(\phi_i)(i) = \gamma_i - \lambda_i$ and hence $h_Y(\phi_i) = [\gamma_i - \lambda_i] \in L_Y$.

Furthermore $\gamma_i \in F_1 A(Y)$ and $\lambda_i \in A(X) = F_0 A(Y)$. Since $L_Y$ inherits a filtration from $HA(Y)$ which in turn induced by the filtration on $A(Y)$, $[\gamma_i - \lambda_i] \in F_1 L_Y$. Recall the quotient map (5.6) $f : F_1(HA(Y)) \rightarrow \text{gr}(HA(Y))_1 \xrightarrow{\cong} (HL)_1$. By construction $f([\gamma_i - \lambda_i]) = \bar{w}_i$.

Recall that $K' = \{\bar{w}_i\}$. Therefore we have an injection $K' \hookrightarrow (\text{gr}(L_Y))_1$. Furthermore by construction this is a map of $L^X_Y$-modules. Thus we have proved the following.

**Proposition 7.2.** Let $W \rightarrow X \rightarrow Y$ and $K'$ be as above. Then there exists an injection of $L^X_Y$-modules
\[
u_1 : K' \hookrightarrow (\text{gr}(L_Y))_1.
\]

We are now ready to prove Theorems E and F. For convenience we restate the theorems.

**Theorem 7.3 (Theorem E).** Let $R = \mathbb{F}$ where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 3$. Let $\bigvee_{j \in J} S^{n_j} \xrightarrow{\bigvee_{a_j}} X$ be a cell attachment satisfying the hypotheses of Theorem C. Let $Y = X \cup_{\bigvee a_j} (\bigvee e^{n_j+1})$ and let $\hat{\alpha}_j$ denote the adjoint of $\alpha_j$. In addition assume that there exists a map $\sigma_X$ right inverse to $h_X$ and that $\forall j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$. Then the canonical algebra map
\[
UL_Y \rightarrow H_*(\Omega Y; \mathbb{F})
\]
is a surjection. That is, $H_*(\Omega Y; \mathbb{F})$ is generated as an algebra by Hurewicz images.
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\textbf{Proof.} Let \( R = \mathbb{F}_p \) where \( p > 3 \). Let \( g : \text{gr}(H_*(\Omega Y; R)) \xrightarrow{\cong} UL' \) be the algebra isomorphism given by Theorem C(i) where \( L' = L'_Y \ltimes \mathbb{L}K' \) with \( K' = (H_0)_{11} \). Note that \( L'_0 = L'_Y \) and that \( L'_1 = K' \). We will show that the canonical map \( UL_Y \to H_*(\Omega Y; R) \) is surjective.

We have an injection of Lie algebras

\[ u_0 : L'_Y \hookrightarrow F_0 L_Y \xrightarrow{\cong} (\text{gr}(L_Y))_0. \]  

(7.4)

Since \( \forall j \ h_\chi \sigma \chi \hat{\alpha}_j = \hat{\alpha}_j \), by Proposition 7.2 and get an injection of \( L'_Y \)-modules \( u_1 : K' \hookrightarrow (\text{gr}(L_Y))_1 \). So for \( x \in L'_Y \) and \( y \in K' \), \( u_1([y, x]) = [u_1(y), u_0(x)] \). Thus by Lemma 2.3 \( u_0 \) and \( u_1 \) can be extended to a Lie algebra map \( u : L' \to \text{gr}(L_Y) \).

The inclusion \( L_Y \hookrightarrow H_*(\Omega Y; R) \) induces a map between the corresponding graded objects, \( \chi : \text{gr}(L_Y) \to \text{gr}(H_*(\Omega Y; R)) \).

We claim that for \( j = 0 \) and \( 1 \), \( g \circ \chi \circ u_j \) is the ordinary inclusion of \( L'_j \) in \( UL' \).

By (7.4) \( u_0 \) is an injection. In addition, restricted to degree 0, \( g \) is the identity and \( \chi \) is just the ordinary inclusion. For \( j = 1 \), \( g \chi u_1 \tilde{w}_i = g \text{gr}([\gamma_i - \lambda_i]) = f([w_i]) = \tilde{w}_i \). It follows that \( g \circ \chi \circ u \) is the standard inclusion \( L' \hookrightarrow UL' \). Since \( g \circ \chi \circ u \) is an injection, so is \( u \).

By Lemma 2.6 the canonical map \( U \text{gr}(L_Y) \xrightarrow{\cong} \text{gr}(UL_Y) \) is an algebra isomorphism. Now \( u \) and \( \chi \) induce the maps \( U u \) and \( \tilde{\chi} \) in the following diagram.

\[ \begin{array}{ccc}
UL' & \xrightarrow{Uu} & U \text{gr}(L_Y) \\
\downarrow{g^{-1}} & & \downarrow{\tilde{\chi}} \\
\text{gr}(H_*(\Omega Y; R)) & & \text{gr}(UL_Y)
\end{array} \]

Since we showed that \( g\chi u \) is the ordinary inclusion \( L' \hookrightarrow UL' \) the diagram commutes. Since \( g^{-1} \) is surjective, the induced map \( \tilde{\chi} \) is surjective. Since \( \tilde{\chi} \) is the associated graded map to the canonical map \( UL_Y \to H_*(\Omega Y; R) \) and the filtrations are bicomplete, the associated ungraded map is also surjective. So the canonical map \( UL_Y \to H_*(\Omega Y; R) \) is surjective which finishes the proof.

The proof of Theorem E is still valid in the case where \( R \subset \mathbb{Q} \) containing \( \frac{1}{6} \). However
in this case we can tensor with \( \mathbb{Q} \) and make use of results from rational homotopy theory. As a result Theorem F is a stronger version of Theorem E.

**Theorem 7.4 (Theorem F).** Let \( R \subset \mathbb{Q} \) be a subring containing \( \frac{1}{6} \). Let \( \bigvee_{j \in J} S_{n_j}^{\mathbb{Q}} \xrightarrow{\alpha_j} X \) be a cell attachment satisfying the hypotheses of Theorem D. Let \( Y = X \cup \bigvee_{j \in J} \left( \bigvee e_{n_j}^{n_j+1} \right) \). Furthermore assume that there exists a map \( \sigma_X \) right inverse to \( h_X \). Let \( P_Y \) be the set of implicit primes and let \( R' = \mathbb{Z}[P_Y^{-1}] \). Then

(i) \( H_*(\Omega Y; R') \) is torsion-free and as algebras

\[
H_*(\Omega Y; R') \cong UL_Y
\]

where \( \text{gr}(L_Y) \cong L_Y^X \ltimes \mathbb{L}(H_L)_1 \) as Lie algebras, and

(ii) localized away from \( P_Y \), \( \Omega Y \in \prod \mathcal{S} \).

(iii) If in addition \( f \) is semi-inert then localized away from \( P_Y \), \( L_Y \cong L_Y^X \ltimes \mathbb{L}K \) as Lie algebras for some \( K \subset F_1L_Y \), and there exists a map \( \sigma_Y \) right inverse to \( h_Y \).

**Remark 7.5.** Using Remark 1.9 we could have tried proving Theorem F(i) using Theorem E and the Hilton-Serre-Baues Theorem (Theorem 4.12), however this is not the approach we use here.

**Proof of Theorem F.** Let \( R \subset \mathbb{Q} \) containing \( \frac{1}{6} \).

(i) Since there are no non-invertible implicit primes we have that \( \sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j \). As a result we can copy the proof of Theorem E verbatim, except we use Theorem D(i) instead of Theorem C(i).

Recall that \( \text{gr}(H_*(\Omega Y; R)) \cong UL' \) and that we constructed a Lie algebra map \( u : L' \rightarrow \text{gr}(L_Y) \) and showed that it is an injection. We claim that for \( R \subset \mathbb{Q} \), \( u \) is an isomorphism. \( H_*(\Omega Y; R) \) and \( \text{gr}(H_*(\Omega Y; R)) \) have the same Hilbert series. Also since \( H_*(\Omega Y; R) \) is torsion-free, it has the same Hilbert series as \( H_*(\Omega Y; \mathbb{Q}) \). Let \( S \) be the image of \( h_Y \otimes \mathbb{Q} \). Then \( S, L_Y \) and \( \text{gr}(L_Y) \) have the same Hilbert series. By the Milnor-Moore Theorem (Theorem 3.5), \( H_*(\Omega Y; \mathbb{Q}) \cong US \). So by the Poincaré-Birkhoff-Witt
Theorem, $S$ has the same Hilbert series as $L'$, and hence $\text{gr}(L_Y) \cong L'$ as $R$-modules. Since $u : L' \to \text{gr}(L_Y)$ is an injection it follows that it is an isomorphism.

By Lemma 2.6 the canonical map $U \text{gr}(L_Y) \xrightarrow{\cong} \text{gr}(UL_Y)$ is an algebra isomorphism. Now $u$ and $\chi$ induce the maps $Uu$ and $\bar{\chi}$ in the following diagram.

\[
\begin{array}{ccc}
UL' & U \text{gr}(L_Y) & \text{gr}(UL_Y) \\
\downarrow{Uu} & \downarrow{\cong} & \\
\text{gr}(H_*(\Omega Y; R)) & \chi & \bar{\chi}
\end{array}
\]

Since we showed that $g\chi u$ is the ordinary inclusion $L' \hookrightarrow UL'$ the diagram commutes. Since $u$ is an isomorphism, so is $Uu$. Hence the induced map $\bar{\chi}$ is an isomorphism. Since $\bar{\chi}$ is the associated graded map to the canonical map $UL_Y \to H_*(\Omega Y; R)$ and the filtrations are bicomplete, the associated ungraded map is also an isomorphism. So $H_*(\Omega Y; R) \cong UL_Y$ as algebras.

The map $u$ gives the desired Lie algebra isomorphism $\text{gr}(L_Y) \cong L' = L_Y^X \ltimes \mathbb{L}(H_\mathcal{L})_1$, which finishes the proof of (i).

(ii) is equivalent to (i) by the Hilton-Serre-Baues Theorem (Theorem 4.12). It remains to prove (iii).

(iii) Before we construct the desired map $\sigma_Y$ we strengthen the result in (i) in the semi-inert case.

Assume that $f$ is semi-inert. Recall the situation from Theorem D(ii). We have that $\text{gr}(H_*(\Omega Y; R)) \cong UL'$ where $L' \cong L_Y^X \amalg K'$ where $K' \cong R\{\bar{w}_i\} \subset (\text{gr}(H_*(\Omega Y; R)))_1$. For each $\bar{w}_i$ let $[w_i]$ be an inverse image under the quotient map $H_*(\Omega Y; R) \to \text{gr}(H_*(\Omega Y; R))$. Let $K'' = \{[w_i]\}$. By Theorem D(ii) as algebras $H_*(\Omega Y; R) \cong UL''$ where $L'' = L_Y^X \amalg K''$ (see Theorem 5.10(b)). Since $K'' \cong K'$, there is an induced Lie algebra isomorphism $L'' \cong L'$. So $\text{gr}(H_*(\Omega Y; R)) \cong H_*(\Omega Y; R)$ as algebras.

Let

$$\hat{L} = L_Y^X \amalg \hat{K}$$

where $\hat{K} = R\{[\gamma_i - \lambda_i]\} \subset F_1 L_Y$.

Recall that $[w_i] - [\gamma_i - \lambda_i] \in F_0(H_*(\Omega Y; R))$ and that $f([\gamma_i - \lambda_i]) = \bar{w}_i$ where $f$ is the
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quotient map from (5.6). So \( f : \hat{K} \overset{\cong}{\rightarrow} K' \), which induces a Lie algebra isomorphism \( \hat{L} \overset{\cong}{\rightarrow} L' \). This in turn induces the algebra isomorphism \( U\hat{L} \overset{\cong}{\rightarrow} UL' \cong H_*(\Omega Y; R) \).

Since \( U\hat{L} \cong H_*(\Omega Y; R) \) there is an injection \( L_Y \hookrightarrow U\hat{L} \). Also, since \( \hat{K} \subset L_Y \) there is a canonical Lie algebra map \( u : \hat{L} \rightarrow L_Y \). These fit into the following commutative diagram.

\[
\begin{array}{ccc}
\hat{L} & \xrightarrow{u} & U\hat{L} \\
\downarrow & & \downarrow \\
L_Y & \xrightarrow{} & U\hat{L}
\end{array}
\]

It follows that \( u \) is an injection.

We claim that \( u \) is an isomorphism. Since \( H_*(\Omega Y; R) \) is torsion-free, it has the same Hilbert series as \( H_*(\Omega Y; Q) \). Let \( S \) be the image of \( h_Y \otimes Q \). Then \( S \) and \( L_Y \) have the same Hilbert series. By the Milnor-Moore Theorem (Theorem 3.5), \( H_*(\Omega Y; Q) \cong US \). So by the Poincaré-Birkhoff-Witt Theorem (Theorem 2.4), \( S \) has the same Hilbert series as \( \hat{L} \), and hence \( L_Y \cong \hat{L} \) as \( R \)-modules. Since \( u : \hat{L} \rightarrow L_Y \) is an injection it follows that it is an isomorphism.

Therefore \( L_Y \cong L_Y^X \Pi \hat{K} \) as Lie algebras, with \( \hat{K} \subset F_1L_Y \) and \( H_*(\Omega Y; R) \cong UL_Y \)
as algebras.

Remark 7.6. If we generalize a conjecture of Anick [Ani89, Conjecture 4.9] from spherical two-cones to our adjunction space \( Y \) then the Lie algebra isomorphism \( L_Y \cong L' \) proves the conjecture in the semi-inert case.

We are now in a position to construct a map \( \sigma_Y \) right inverse to \( h_Y \). Let \( i \) denote the inclusion \( X \rightarrow Y \). Consider the composite map

\[
F : [L_X^W] \hookrightarrow L_X \xrightarrow{\sigma_X} \pi_*(\Omega X) \otimes R \xrightarrow{(\Omega i)_\#} \pi_*(\Omega Y) \otimes R.
\]

We claim that \( F = 0 \). Since \( F \) is a Lie algebra map it is sufficient to show that \( F(L_X^W) = 0 \).

\[
F(L_X^W) = (\Omega i)_\# \sigma_X(R\{h_X(\tilde{a}_j)\}).
\]
Since there are no implicit primes \( \sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j \). By the construction of \( Y \), \( \Omega \circ \hat{\alpha}_j \simeq 0 \). So \( F = 0 \) as claimed.

Therefore there is an induced map \( G : L^X_Y \simeq L_X /[L^W_X] \rightarrow \pi_*(\Omega Y) \otimes R \). That \( h_Y \circ G \) is the inclusion map can be seen from the following commutative diagram.

\[
\begin{array}{ccc}
L_X & \xrightarrow{\pi_*} & \pi_* (\Omega X) \otimes R \\
\downarrow{h_X} & & \downarrow{G} \\
L^X_Y & \xrightarrow{\Omega \circ} & L^X_Y \\
\downarrow{G} & & \downarrow{\pi_* (\Omega Y) \otimes R} \\
L_Y & \xrightarrow{h_Y} & \pi_* (\Omega Y) \otimes R
\end{array}
\]

Now construct \( \sigma_Y : L_Y \rightarrow \pi_* (\Omega Y) \otimes R \) as follows. We have shown that \( L_Y \simeq L^X_Y \Pi L K \) for some \( K \subset F_1 L_Y \). Since \( F_1 L_Y \simeq \hat{K} = R[\{\gamma_i - \lambda_i\}] \) it follows that \( K = R[\{\gamma_i + \lambda_i\}]_{i \in L} \) for some \( L \). Recall that \( \exists \phi_i \in \pi_* (\Omega Y) \otimes R \) such that \( h_Y (\phi_i) = [\gamma_i + \lambda_i] \). Let \( \sigma_Y |_{L^X_Y} = \sigma \) and let \( \sigma_Y ([\gamma_i + \lambda_i]) = \phi_i \). Now extend \( \sigma_Y \) canonically to a Lie algebra map on \( L_Y \).

We finally claim that \( h_Y \sigma_Y = id_{L_Y} \). Since \( h_Y \sigma_Y \) is a Lie algebra map it suffices to check that it is the identity for the generators.

\[
h_Y \sigma_Y L^X_Y = h_Y GL^X_Y = L^X_Y, \quad h_Y \sigma_Y [\gamma_i + \lambda_i] = h_Y \phi_i = [\gamma_i + \lambda_i]
\]

Therefore \( \sigma_Y \) is the desired Lie algebra map right inverse to \( h_Y \). \( \square \)
Part III

Applications
Chapter 8

Free Subalgebras

In this chapter we prove Theorem G and give a second proof of a special case of this theorem.

8.1 A generalized Schreier property

In chapter 5 we calculated the homology of $U \mathfrak{L} = U(L_0 \oplus \mathcal{L} V_1, d)$, a two-level universal enveloping algebra over a field $\mathbb{F}$ in which the Lie ideal $[d V_1] \subset L_0$ was a free Lie algebra. The result was a universal enveloping algebra on a Lie algebra

$$L = L_0 \ltimes \mathcal{L} L_1.$$  

We remark that an equivalent description of $L$ is the fact that it has homogeneous generators and relations in degrees 0 and 1.

In this chapter we give a simple criterion which we prove guarantees that a Lie subalgebra of $L$ is free.

If $R \subset \mathbb{Q}$ is a subring containing $\frac{1}{6}$, Theorem F shows that under certain conditions, $H_*(\Omega Y; R')$ is torsion-free, as algebras $H_*(\Omega Y; R') \cong UL_Y$ and as Lie algebras $\text{gr}(L_Y) \cong L^\mathcal{X}_Y \ltimes \mathcal{L} V$ for some free $R'$-module $V$. Let $p$ be a noninvertible prime in $R'$ and recall the notation $\bar{M} = M \otimes \mathbb{F}_p$ for any $R'$-module $M$. It follows that $\text{gr}(\bar{L}_Y) \cong L^\mathcal{X}_Y \ltimes \mathcal{L} \bar{V}$. 

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We will give a simple criterion guaranteeing that a Lie subalgebra \( J \subset \mathcal{L}_V \) is a free Lie algebra.

It is a well-known fact that any (graded) Lie subalgebra of a (graded) Lie algebra is a free Lie algebra [Mik85, Šir53, MZ95] (this is referred to as the Schreier property). In this chapter we generalize this result to the following.

**Theorem 8.1 (Theorem G).** Over a field \( \mathbb{F} \), let \( L \) be a finite-type graded Lie algebra with filtration \( \{ F_kL \} \) such that \( \text{gr}(L) \cong L_0 \ltimes \mathbb{L}V_1 \) as Lie algebras, where \( L_0 = F_0L \) and \( V_1 = F_1L/F_0L \). Let \( J \subset L \) be a Lie subalgebra such that \( J \cap F_0L = 0 \). Then \( J \) is a free Lie algebra.

Before proving this theorem, we prove the following lemma.

**Lemma 8.2.** Let \( J \) be a finite-type filtered Lie algebra such that \( \text{gr}(J) \) is a free Lie algebra. Then \( J \) is a free Lie algebra.

**Proof.** By assumption there is an \( \mathbb{F} \)-module \( \bar{W} \) such that \( \text{gr}(J) \cong \mathbb{L}\bar{W} \).

Let \( \{ \bar{w}_i \}_{i \in I} \subset \text{gr}(J) \) be an \( \mathbb{F} \)-module basis for \( \bar{W} \). Let \( m_i = \deg(\bar{w}_i) \). That is, \( \bar{w}_i \in F_{m_i}J/F_{m_i-1}L \). For each \( \bar{w}_i \) choose a representative \( w_i \in F_{m_i}J \). Let \( W = \mathbb{F}\{w_i\}_{i \in I} \subset J \).

Then there is a canonical map \( \phi : \mathbb{L}W \rightarrow J \). Grade \( \mathbb{L}W \) by letting \( w_i \in W \) be in degree \( m_i \). Then \( \phi \) is a map of filtered objects and there is an induced map \( \theta : \mathbb{L}W \rightarrow \text{gr}(J) \). However the composite map

\[
\mathbb{L}W \xrightarrow{\theta} \text{gr}(J) \cong \mathbb{L}\bar{W}
\]

is just the canonical isomorphism \( \mathbb{L}W \xrightarrow{\cong} \mathbb{L}\bar{W} \). So \( \theta \) is an isomorphism.

Therefore \( \phi \) is an isomorphism and \( J \) is a free Lie algebra. \( \square \)

**Proof of Theorem G.** The filtration on \( L \) filters \( J \) by letting

\[
F_kJ = J \cap F_kL.
\]
From this definition it follows that the inclusion \( J \hookrightarrow L \) induces an inclusion \( \text{gr}(J) \hookrightarrow \text{gr}(L) \). So \( \text{gr}(J) \hookrightarrow \text{gr}(L) \cong L_0 \ltimes \mathbb{L}V_1 \). Since \( J \cap F_0 L = 0 \) it follows that \( (\text{gr} J)_0 = 0 \) and \( \text{gr}(J) \hookrightarrow (\text{gr} J)_{\geq 1} \cong \mathbb{L}V_1 \). By the Schreier property \( \text{gr}(J) \) is a free Lie algebra.

Thus by Lemma 8.2, \( J \) is a free Lie algebra. \( \square \)

The following corollary is a special case of this theorem.

**Corollary 8.3.** Over a field \( \mathbb{F} \), if \( J \subset L_0 \ltimes \mathbb{L}(L_1) \) is a Lie subalgebra such that \( J \cap L_0 = 0 \) then \( J \) is a free Lie algebra.

Note that since \( J \) is not necessarily homogeneous with respect to degree \( J \cap L_0 = 0 \) does not imply that \( J \subset \mathbb{L}L_1 \).

### 8.2 A second proof

In this section we give an independent proof Corollary 8.3 which does not use the Schreier property. As such, it proves the Schreier property as a special case.

**Theorem 8.4.** Let \( L = L_0 \ltimes \mathbb{L}L_1 \) be a finite-type bigraded Lie algebra over a field \( \mathbb{F} \) with generators and relations in degrees 0 and 1. Let \( J \) be a graded subalgebra of \( L \) not necessarily graded with respect to degree, such that \( J \cap L_0 = 0 \). Then \( J \) is a free Lie algebra.

Let \( J \) be a Lie subalgebra of \( L \) generated by elements which are homogeneous with respect to dimension, but not necessarily with respect to degree. So \( J \) is graded by dimension and filtered by degree. Let \( J_m \) be the subspace of \( J \) in degrees \( \leq m \) and \( J_{m,n} \) the component of \( J_m \) in dimension \( n \). Let \( J_{m,n} = \bigoplus_{k<n} J_{m,k} \).

Let \( \{x_i\} \) be a set of generators homogeneous with respect to dimension for a filtered Lie algebra \( J \). Let \( J^j \) denote the Lie subalgebra of \( J \) generated by \( \{x_i\}_{i \neq j} \). The set \( \{x_i\} \) is called reduced if for each generator \( x_i \in J_{m,n} \setminus J_{m-1,n} \),

\[
\nexists y \in J^j_{*,n} \mid x_i - y \in J_{m-1,n}.
\]

(8.1)
Remark 8.5. Note that reduced $\implies$ minimal. That is, for each $x_i$, $\exists y \in J'$ such that $x_i - y = 0$. We also remark that the standard definition of a reduced set [MZ95] (in the context where $L$ is a free Lie algebra) is the condition that none of the $x'_j$ lie in the Lie subalgebra of $L$ generated by $\{x'_i\}_{i \neq j}$. We will see in the proof of Lemma 8.7 that this condition follows from our condition (8.1).

Lemma 8.6. A connected, graded, filtered, finite-type Lie algebra $J$ has a reduced set of generators.

Proof. We deduce by induction on dimension. Assume we have a reduced set of generators homogeneous with respect to dimension for the Lie subalgebra generated by $J_{s, < n}$. Since $J$ has finite-type, $J_{s, n}$ is a finite vector space. Therefore $J_{s, n} \cong J_{M, n}$ for some $M$.

We now deduce by induction on degree. Assume we have a reduced set of generators homogeneous with respect to dimension for the Lie subalgebra generated by $J_{s, < n} \oplus J_{m-1, n}$. Choose a basis $\{y_j\}$ for the finite vector space $J_{m, n} \oplus J_{m-1, n}$.

We finally deduce by induction on $j$. Assume we have a reduced set of generators $\{x_i\}$ homogeneous with respect to dimension, for the Lie subalgebra generated by $V := J_{s, < n} \oplus J_{m-1, n} \oplus \bigoplus_{j < k} \mathbb{F}\{y_j\}$, which we denote $\langle x_i \rangle$. More generally we will let $\langle S \rangle$ denote the Lie subalgebra of $J$ generated by the set $S$.

Consider $y_k$. If $\exists y \in \langle x_i \rangle_{s, n}$ such that $y_k - y \in J_{m-1, n}$ then since $J_{m-1, n} \subset \langle x_i \rangle$, $\exists x \in \langle x_i \rangle$ such that $y_k - y = x$. Therefore $y_k = x + y \in \langle x_i \rangle$, and $\mathbb{F}\{y_k\} \subset \langle x_i \rangle$. Thus $\{x_i\}$ is a reduced set of generators for the Lie subalgebra generated by $V \oplus \mathbb{F}\{y_k\}$.

If $\exists$ such a $y$ we claim that $\{x_i\}_{i \in I} \cup \{y_k\}$ is a reduced set of generators for the Lie subalgebra generated by $V \oplus \mathbb{F}\{y_k\}$. Clearly it is a set of generators; it remains to show that it is reduced. We need to check that each generator satisfies condition (8.1).

By assumption, $y_k$ satisfies the condition. By the induction hypothesis and since $y_k \in J_{s, n}$, all $x_i \in J_{s, < n}$ satisfy the condition. It remains to prove that each $x_i \in J_{m, n}$ satisfies condition (8.1).

Assume $\exists j$, and $\exists y \in \langle \{x_i\}_{i \neq j} \cup \{y_k\} \rangle_{s, n}$ such that $x_j \in J_{l, n} \setminus J_{l-1, n}$ (where $l \leq m$)
and \( x_j - y \in J_{l-1,n} \). For degree reasons, \( y \in J_{l,n} \subset J_{m,n} \) and since \( y_k \in J_{m,n} \) and \( J \) is connected, \( y = x + ay_k \), where \( x \in \langle \{ x_i \}_{i \neq j} \rangle_{*n} \) and \( a \in \mathbb{F} \). Therefore \( x_j - x - ay_k \in J_{l-1,n} \).

By assumption \( a \neq 0 \). Thus \( y_k - a^{-1}(x_j - x) \in J_{l-1,n} \subset J_{m-1,n} \). But \( a^{-1}(x_j - x) \in \langle x_i \rangle_{*n} \), which contradicts the assumption that \( y_k \) satisfies condition (8.1). Hence \( V \oplus \mathbb{F}\{y_k\} \) has a reduced set of generators.

So by induction, the Lie subalgebra generated by \( J_{*,<n} \oplus J_{m,n} \) has a reduced set of generators. Then by induction the Lie subalgebra generated by \( J_{*,\leq n} \) has a reduced set of generators. Finally by induction \( J \) has a reduced set of generators. \( \square \)

Let \( J \) be a Lie subalgebra of the semi-direct product \( L = L_0 \ltimes \mathbb{L}(L_1) \) such that \( J \cap L_0 = 0 \). \( \mathbb{L} \) is bigraded. Let \( L_j \) denote the component of \( L \) in degree \( j \). By the previous lemma, \( J \) has a reduced set of generators \( \{ x_i \} \) homogeneous with respect to dimension, where \( x_i = \sum_{j=0}^{n_i} x_i^{(j)} \), with \( x_i^{(j)} \in L_j \), \( x_i^{(n_i)} \neq 0 \) and \( n_i \geq 1 \). Let \( x'_i = x_i^{(n_i)} \).

Note that the \( x_i^{(j)} \) are homogeneous with respect to degree.

**Lemma 8.7.** There does not exist a nonzero \( f(y_1, \ldots, y_n) \in \mathbb{L}(y_1, \ldots, y_n) \) with \( |y_i| = |x_i| \) such that \( f(x'_1, \ldots, x'_n) = 0 \).

**Proof.** Assume such an \( f \) exists. Let \( (a_1, \ldots, a_j, \ldots, a_n) \) denote the list \( (a_1, \ldots, a_n) \) with \( a_j \) omitted. Since \( n_i \geq 1 \), \( x'_i \in L_{\geq 1} \cong \mathbb{L}(L_1) \). Since free graded Lie algebras satisfy the Nielsen property [MZ95, Theorem 14.1] and each \( x'_i \) is homogeneous with respect to degree, \( \exists 0 \neq g(y_1, \ldots, y_j, \ldots, y_n) \in \mathbb{L}(y_1, \ldots, y_j, \ldots, y_n) \) with \( |y_i| = |x_i| \) such that \( x'_j = g(x'_1, \ldots, x'_j, \ldots, x'_n) \) for some \( j \). Without loss of generality, we can assume that \( g(x'_1, \ldots, x'_j, \ldots, x'_n) \) is a sum of nonzero monomials in dimension \( |x_j| \) and degree \( n_j(= \deg(x'_j)) \).

By linearity,

\[
g(x_1, \ldots, x_j, \ldots, x_n) = \alpha + g(x'_1, \ldots, x'_j, \ldots, x'_n), \quad \alpha \in J_{n_j-1} \\
= \alpha + x'_j, \quad x'_j = x_j - \sum_{k=0}^{n_j-1} x_j^{(k)} \\
= \beta + x_j, \quad \beta \in J_{n_j-1}.
\]
Therefore $x_j - g(x_1, \ldots, x_j, \ldots x_n) \in J_{n-1}$. But this contradicts the assumption that \( \{x_i\} \) is reduced. \( \square \)

**Proof of Theorem 8.4.** Let \( J \subset L_0 \ltimes LL_1 \) be a Lie subalgebra such that \( J \cap L_0 = 0 \). By Lemma 8.6 \( J \) has a reduced set of generators \( \{x_i\} \).

Assume \( \exists \neq f(y_1, \ldots y_n) \in \mathbb{L}\langle y_1, \ldots y_n \rangle \) with \( |y_i| = |x_i| \) such that \( f(x_1, \ldots x_n) = 0 \). Without loss of generality assume that \( f(x_1, \ldots x_n) \) is a sum of nonzero monomials \( \{f_j(x_1, \ldots x_n)\}_{j \in J} \) in a fixed dimension.

Since \( f_j \in \mathbb{L}\langle y_1, \ldots y_n \rangle \), by the previous lemma \( f_j(x_1', \ldots x_n') \neq 0 \). Each of these is in a fixed degree \( M_j \).

By linearity and for degree reasons, \( f_j(x_1, \ldots x_n) = \alpha_j + f_j(x_1', \ldots x_n') \), with \( \alpha_j \in J_{M_j-1} \). Let \( \{f_k\}_{k \in K} \) be the subset of the monomials \( \{f_j\}_{j \in J} \) such that \( f_k(x_1', \ldots x_n') \) is in the highest degree, say \( M \). Then

\[
0 = f(x_1, \ldots x_n) = \sum_{j \in J}(\alpha_j + f_j(x_1', \ldots x_n')) \\
= \alpha + \sum_{k \in K} f_k(x_1', \ldots x_n'), \quad \alpha \in J_{M-1}.
\]

Therefore \( \sum_{k \in K} f_k(x_1', \ldots x_n') = 0 \). But this contradicts the previous lemma. \( \square \)
Chapter 9

An Algebraic Ganea Construction

In this section we apply our results to give an algebraic version of Ganea’s Fiber-Cofiber construction [Gan65, Rut71] (see Section 4.5). All of the isomorphisms in this chapter are algebra or Lie algebra isomorphisms.

9.1 The rational case

In the case when $R = \mathbb{Q}$ there is an exact correspondence between Lie algebras and rational spaces (see [FHT01] for a reference). Using this correspondence, Ganea’s construction gives a result about Lie algebras which we explain below. We will see that this result is a special case of our results.

Let $(L_0, 0)$ be a dgL with a Lie ideal $J \subset L_0$ such that $J$ is a free Lie algebra. Then as explained in [FHT01] the short exact sequence of Lie algebras

$$0 \to J \to L_0 \to L_0/J \to 0$$

models a fibration

$$F \xrightarrow{f} X \to B.$$  

In Ganea’s construction, we take the cofiber $X' = X \cup_f CF$, where $CF$ is the cone on $F$, and the fiber $F'$ of the fibration $X' \to B$. 

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Choose a set of generators $S$ for the Lie ideal $J$. Let $V_0$ be the free $R$-module with basis $S$. Let $V_1$ be an $R$-module such that there is a bijection $d : V_1 \to V_0$ where $d$ reduces dimension by 1. Let $L = (L_0 \amalg LV_1, d)$ be the dgL where $dL_0 = 0$ and $dV_1$ is give above. Then $L$ models $X'$ [FHT01]. Let $J' = \ker(L \to L_0/J)$. Then $J'$ models $F'$ [FHT01].

The long exact homology sequence splits into short exact sequences which can be collected to give the short exact sequence of Lie algebras

$$0 \to HJ' \to HL \to L_0/J \to 0.$$ 

By Ganea’s Theorem [Gan65] $F' \approx F \ast \Omega B \approx \Sigma(F \land \Omega B)$ which rationally is a wedge of spheres. So since $J'$ models $F'$, $HJ'$ should be a free Lie algebra.

Using our results and without using Ganea’s Theorem, we will see what this free Lie algebra is. In our terminology $UL$ is a dga extension of $((UL_0, 0), L_0)$. By Lemma 1.1 and Theorem B, as algebras $HUL \cong U\psi HL \cong UHL$ (since the map $\psi : UHL \to HUL$ is an isomorphism when $R = \mathbb{Q}$ [FHT01, Theorem 21.7(i)]), where $HL \cong L_0/J \ltimes \mathbb{L}(HL)_1$.

By Lemma 2.2 $HL$ fits into the split short exact sequence of Lie algebras

$$0 \to \mathbb{L}(HL)_1 \to HL \to L_0/J \to 0.$$ 

So we see that $HJ' \cong \mathbb{L}(HL)_1$.

**Remark 9.1.** In [FT88] Félix and Thomas also do this calculation using Ganea’s Theorem. They show that $HJ' \cong \mathbb{L}K$ where $K$ is the kernel of the canonical homomorphism $U(L/J) \otimes V_0 \to J/J^2$.

### 9.2 The construction

Let $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ containing $\frac{1}{p}$. For the cases when $R \neq \mathbb{Q}$ we do not have an exact correspondence between Lie models and spaces. However our results allow us to give an algebraic analogue to Ganea’s construction.
Let $(L_0,0)$ be a dgL with a Lie ideal $J \subset L_0$ such that $J$ is a free Lie algebra. Choose a set of generators $S$ for the Lie ideal $J$. Let $V_1$ be an $R$-module such that there is a bijection $d : V_1 \rightarrow R\{S\}$ where $d$ reduces dimension by 1. Let $L = (L_0 \oplus LV_1,d)$ be the dgL where $dL_0 = 0$ and $dV_1$ is given above. Then $[dV_1] = J \subset L_0$. In our terminology $UL$ is a dga extension of $((UL_0,0),L_0)$ which is free. By Lemma 1.1 and Theorems A and B, as algebras

$$HUL \cong U((HL)_0 \ltimes \mathbb{L}(HL)_1)$$

where $(HL)_0 \cong L_0/J$. By Lemma 2.2, $HUL \cong UL'$ as algebras where

$$0 \to \mathbb{L}(HL)_1 \to L' \to L_0/J \to 0$$

is a split short exact sequence of Lie algebras. If $R \subset \mathbb{Q}$ then $L' = \psi HL$ where $\psi$ is the natural map $\psi : UHL \to HUL$.

As seen in Section 9.1 this generalizes the algebraic analogue of Ganea’s construction obtained using rational homotopy theory.

### 9.3 Iterating the construction

Let $R = \mathbb{F}_p$ where $p > 3$ or $R \subset \mathbb{Q}$ containing $1/6$. As with Ganea’s construction (see Section 4.5) this algebraic construction can be iterated. This can be used to obtain successively closer approximations to a given Lie algebra.

Given the split short exact sequence of Lie algebras

$$0 \to \mathbb{L}(HL^{(n)})_1 \to L^{(n)}' \to L_0/J \to 0,$$

choose a set $S^{(n)}$ of generators for the Lie ideal $\mathbb{L}(HL^{(n)})_1$. Let $V^{(n+1)}$ be an $R$-module such that there is a bijection $d : V^{(n+1)} \rightarrow RS^{(n)}$ where $d$ reduces dimension by 1. Let

$$(L^{(n+1)},d^{(n+1)}) = (L^{(n)}', \mathbb{L}V^{(n+1)},d^{(n+1)}),$$
where $d^{(n+1)}L^{(n)} = 0$, and $d^{(n+1)}V^{(n+1)} \subset L^{(n)}$ is given above. Then $[d^{(n+1)}V^{(n+1)}] = \mathbb{L}(HL^{(n)})_1 \subset L^{(n)}$. By Lemma 1.1 and Theorems A and B, as algebras $HUL^{(n+1)} \cong UL^{(n+1)}$ where

$$0 \to \mathbb{L}(HL^{(n+1)})_1 \to L^{(n+1)} \to L_0/J \to 0$$

is a split short exact sequence of Lie algebras. By construction the lowest dimension in which $L^{(n)}$ and $L_0/J$ differ increases to $\infty$ as $n \to \infty$.

### 9.4 A more general construction

Ganea’s construction can be generalized to taking more general cofibers than $X \cup_f CF$ [Mat75, FT88] (see Section 4.5). We can do the same in our algebraic construction. Let $L = (L_0 \amalg LV_1, d)$ where $dL_0 = 0$ and $dV_1 \subset J \subset L_0$. Then $[dV_1] \subset J \subset L_0$. Since $[dV_1]$ is a subalgebra of the free Lie algebra $J$ it is also a free Lie algebra. So $UL$ is a free dga extension of $((UL_0, 0), L_0)$. Therefore

$$HUL \cong U((HL)_0 \ltimes \mathbb{L}(HL)_1), \text{ where } (HL)_0 \cong L/[dV_1].$$
Chapter 10

Topological Examples and Open Questions

In this chapter we use our results to analyze various topological examples. We conclude with some open questions.

10.1 Topological examples

All of the isomorphisms in this section are isomorphisms of algebras or Lie algebras.

The following spherical three-cone $Y$, illustrates our results.

Example 10.1. Let $R = \mathbb{Z}[\frac{1}{6}]$. Let $A = S^3 \vee S^3$ and let $Z = A \cup_{\alpha_1 \vee \alpha_2} (e^8 \vee e^8)$ where the attaching maps are given by the iterated Whitehead products $\alpha_1 = [\iota_a, \iota_b], \iota_a]$ and $\alpha_2 = [\iota_a, \iota_b], \iota_b]$ of the inclusions of the 3-spheres.

It is known (see Example 3.8) that $L_A \cong \mathbb{L}(x, y)$ where $|x| = |y| = 2$. It is also known (see [HL95, Example 4.1] or Example 4.7) that

$$H_*(\Omega Z; R) \cong U(L'_0 \Pi \mathbb{L}\langle \bar{w} \rangle),$$

where $L'_0 = \mathbb{L}(x, y)/J$,

with $J$ the Lie ideal of all brackets of length $\geq 3$ and $|\bar{w}| = 9$. By Lemma 4.13, $P_Z = \ldots$
\{2,3\}. Using [Ani89] or Theorem F,

\[ H_\ast(\Omega Z;R) \cong UL_Z \text{ where } L_Z \cong L^A_Z II \mathbb{L}(w) \]

with \( L^A_Z \cong L^0 \) and \( w = h_Z(\hat{\omega}) \) where \( \hat{\omega} \) is the adjoint of some map \( \omega : S^{10} \to Z \). By Theorem F there is a map \( \sigma_Z \) right inverse to \( h_Z \).

For \( i = 1,2 \) let \( Z_i \) be two copies of \( Z \). Let \( X = Z_1 \vee Z_2 \), \( W = S^{28} \vee S^{28} \) and let \( f = \beta_1 \vee \beta_2 \) where \( \beta_1 = [\omega_1, \omega_2, \omega_1] \) and \( \beta_2 = [\omega_1, \omega_2, \omega_2] \). Let

\[ Y = X \cup_f (e^{29} \vee e^{29}). \]

Now,

\[ L_X \cong L_{Z_1} II L_{Z_2} \cong L^A_{Z_1} II L^A_{Z_2} II \mathbb{L}(w_1,w_2). \] (10.1)

Therefore by Theorem G, \( f \) is a free attaching map. Thus \( Y \) satisfies the hypotheses of Theorem D.

By Corollary 1.8, if \( f \) is semi-inert then

\[ K'(z) = \tilde{H}_\ast(W)(z) + z[U L_X(z)^{-1} - U(L^X_Y)(z)^{-1}]. \]

\( L_X \) is given by (10.1) and \( L^X_Y \cong L_X/[h_X(\hat{\beta}_1), h_X(\hat{\beta}_2)] \). Since the Hurewicz images are \([w_1, w_2, w_1]\) and \([w_1, w_2, w_1]\) which are contained in \( \mathbb{L}(w_1, w_2) \) we have

\[ L^X_Y \cong L^A_{Z_1} II L^A_{Z_2} II \tilde{L} \text{ where } \tilde{L} = \mathbb{L}(w_1, w_2)/[R[[w_1, w_2, w_1], [w_1, w_2, w_2]]]. \]

Since \( (A II B)(z)^{-1} = A(z)^{-1} + B(z)^{-1} - 1 \) [Lem74, Lemma 5.1.10] it follows that

\[ K'(z) = \tilde{H}_\ast(W)(z) + z[U \mathbb{L}(w_1,w_2) - U \tilde{L}(z)^{-1}]. \]

Now \( U \mathbb{L}(w_1,w_2) \cong \mathbb{T}(w_1,w_2) \) and as \( R \)-modules \( \tilde{L} \cong R\{w_1, w_2, [w_1, w_1], [w_1, w_2], [w_2, w_2]\} \) and \( U \tilde{L} \cong S \tilde{L} \). Therefore

\[ K'(z) = 2z^{28} + z \left[ 1 - 2z^9 - \frac{(1 - z^{18})^3}{(1 + z^9)^2} \right] = z^{37}. \]
Recall that $L = (L_X \amalg L(e, g), d')$ where $d'e = [w_1, w_2], w_1$ and $d'g = [w_1, w_2], w_2$. Also recall (from Theorem C) that $(H^L)_0 \cong L^X_0$. Let $\bar{u} = [e, w_2] + [g, w_1]$ (with $|\bar{u}| = 37$). Then it is easy to check that

$$(H^L)_0 \times L(H^L) \cong (H^L)_0 \amalg L(\bar{u})$$

and $f$ is indeed semi-inert. Since $X = Z_1 \amalg Z_2$ we have the following commutative diagram.

$$\pi_* (\Omega X) \otimes R \cong \pi_* (\Omega Z_1) \otimes R \oplus \pi_* (\Omega Z_2) \otimes R \xrightarrow{h_X} H_* (\Omega X; R) \cong H_* (\Omega Z_1; R) \oplus H_* (\Omega Z_2; R)$$

Let $\sigma_X$ be the map corresponding to $\sigma_{Z_1} \oplus \sigma_{Z_2}$ under these isomorphisms. Since $\sigma_{Z_1}$ and $\sigma_{Z_2}$ are right inverses to $h_{Z_1}$ and $h_{Z_2}$ it follows that $\sigma_X = \sigma_{Z_1} \oplus \sigma_{Z_2}$ is right inverse to $h_X$. By Lemma 4.13, $P_Y = \{2, 3\}$. As a result by Theorem F,

$$H_* (\Omega Y; R) \cong UL_Y \text{ where } L_Y \cong L_Y^X \amalg L(u)$$

with $u = h_Y (\bar{u})$ for some map $\mu : S^{38} \to Y$. Furthermore $\Omega Y \in \prod S$ and there exists a map $\sigma_Y$ right inverse to $h_Y$.

Note that

$$L_Y \cong L_1^1 \amalg L_1^2 \amalg L_2 \amalg L \amalg L(u)$$

where $L_1^1 \cong L_1^2 \cong \mathbb{L}(x, y)/J_1$ and $L_2 \cong \mathbb{L}(w_1, w_2)/J_2$ with $J_1$ and $J_2$ the Lie ideals of all brackets of length $\geq 3$.

**Example 10.2.** An infinite family of finite CW-complexes constructed out of semi-inert attaching maps

The construction in the previous example can be extended inductively. By induction, we will construct spaces $X_n$ and maps $\omega_n : S^{\lambda_n} \to X_n$ for $n \geq 1$ such that $X_n$ is an $n$-cone constructed out of a sequence of semi-inert attaching maps. Given $\omega_n$, let $w_n = h_{X_n} ([\omega_n])$ and given $w^a_i$ and $w^b_i$, let $L_i = \mathbb{L}(w^a_i, w^b_i)/J_i$ where $J_i$ is the Lie ideal
generated by \([[[w_i^a, w_i^b], w_i^a]]\) and \([[[w_i^a, w_i^b], w_i^b]]\). That is \(L_i\) is the quotient of \(\mathbb{L}(w_i^a, w_i^b)\) where all brackets of length three are equal to zero.

Let \(R = \mathbb{Z}^{1/6}\). Begin with \(X_1 = S^3\) and \(\lambda_1 = 3\). Let \(\omega_1 : S^{\lambda_1} \to X_1\) be the identity map.

Given \(X_n\), let \(X_n^a\) and \(X_n^b\) be two copies of \(X_n\). For \(n \geq 1\), let

\[
X_{n+1} = X_n^a \cup X_n^b \cup f_{n+1} \left( e^{\kappa_{n+1}} \cup e^{\kappa_{n+1}} \right),
\]

where \(\kappa_{n+1} = 3\lambda_n - 1\) and \(f_{n+1} = [[[\omega_n^a, \omega_n^b], \omega_n^a] \vee [[[\omega_n^a, \omega_n^b], \omega_n^b]]\).

By the same argument as in the previous example, \(f_{n+1}\) is a semi-inert cell attachment and there exists a map

\[
\omega_{n+1} : S^{\lambda_{n+1}} \to X_{n+1}
\]

where \(\lambda_{n+1} = 4\lambda_n - 2\), such that

\[
H_\ast(\Omega X_{n+1}; R) \cong UL_{X_{n+1}} \text{ where } L_{X_{n+1}} = \left( \bigsqcup_{1 \leq i \leq n} L_i^2 \right) \square \mathbb{L}(w_{n+1})
\]

with \(L_i^2\) a copy of \(L_i\) and \(w_{n+1} = h_{X_{n+1}}(\omega_{n+1})\).

\[\square\]

**Example 10.3. Spherical 2-Cones**

A spherical 2-cone is an adjunction space (see Section 3.1) of the form

\[
\left( \bigsqcup_{j \in J_1} S^{m_j} \right) \cup_f \left( \bigsqcup_{j \in J_2} e^{n_j+1} \right).
\]

Equivalently it is the adjunction space of a cell attachment \(W_2 \xrightarrow{f_2} X_1\), where \(W_2 = \bigsqcup_{j \in J_2} S^{n_j}\) and \(X_1\) are finite-type wedges of spheres and \(f_2 = \bigsqcup_{j \in J_2} \alpha_j^{(2)}\). This is the situation studied by Anick in [Ani89]. Let \(R = \mathbb{F}\) where \(\mathbb{F} = \mathbb{Q}\) or \(\mathbb{F}_p\) with \(p > 3\), or let \(R \subset \mathbb{Q}\) be a subring containing \(1/6\). By the Hilton-Milnor Theorem \(H_\ast(\Omega X_1; R) \cong UL_{X_1}\) and \(L_{X_1} \cong \mathbb{L}V\) for some free \(R\)-module \(V\). \(X_1\) has Adams-Hilton model \(U(\mathbb{L}V, 0)\). Since \(L_{X_1} \cong \mathbb{L}V\) we can define a map \(\sigma_{X_1} : L_{X_1} \to \pi_\ast(\Omega X_1) \otimes R\) right inverse to \(h_{X_1}\) by choosing
inverse images of \( V \) and extending to a Lie algebra map. Let \( \underline{L}_2 = (L_{X_1} \amalg \langle y_j^{(2)} \rangle_{j \in J_2}, d') \)
where \( d'y_j^{(2)} = h_{X_1}(\hat{\alpha}_j^{(2)}) \).

If \( R = \mathbb{F} \), \([L_{X_1}^W] \subset L_{X_1} \) is a Lie subalgebra of a free Lie algebra and so is automatically a free Lie algebra. Since the Adams-Hilton model for \( X_1 \) has zero differential, by Lemma 1.1
\( H_*(\Omega X_2; \mathbb{F}) \cong \text{gr}(H_*(\Omega X_2; \mathbb{F})) \). Applying Theorem C(i) we get the following isomorphism
of algebras
\[
H_*(\Omega X_2; \mathbb{F}) \cong U(L_{X_2}^{X_1} \ltimes \mathbb{L}(H_{L_2})_{1} ) .
\]
Furthermore by Theorem E if \( \sigma_X h_{X_1} \hat{\alpha}_j^{(2)} = \hat{\alpha}_j^{(2)} \) then the canonical map
\[
UL_{X_2} \to H_*(\Omega X_2; \mathbb{F})
\]
is a surjection.

If \( R \subset \mathbb{Q} \) and \( L_{X_1}/[L_{X_1}^W] \) is \( R \)-free then we can apply Theorem D(i) to get that
\( H_*(\Omega X_2; R) \) is torsion-free and
\[
H_*(\Omega X_2; R) \cong U(L_{X_2}^{X_1} \ltimes \mathbb{L}(H_{L_2})_{1} ) .
\]
Furthermore since \( \exists \sigma_X \) right inverse to \( h_{X_1} \), then by Theorem F,
\[
H_*(\Omega X_2; \mathbb{Z}[P_{X_2}^{-1}]) \cong UL_{X_2}
\]
where \( \text{gr}(L_{X_2}) \cong L_{X_2}^{X_1} \ltimes \mathbb{L}(H_{L_2})_{1} \) and localized away from \( P_{X_2}, \Omega X_2 \in \prod \mathcal{S} \). This result is given in [Ani89].

In either case if \( f_2 \) is semi-inert then by part (ii) of Theorems C and D
\[
H_*(\Omega X_2; R) \cong U(L_V^X \amalg \mathbb{L}(K^{(2)}) )
\]
for some \( K^{(2)} \subset F_1 H_*(\Omega X_2; R) \). If \( R \subset \mathbb{Q} \) then by Theorem F(iii) \( L_{X_2} \cong L_V^X \amalg \mathbb{L}(H_{L_2})_{1} \)
and there exists a map \( \sigma_{X_2} \) right inverse to \( h_{X_2} \).

This last Lie algebra isomorphism proves a conjecture of Anick [Ani89, Conjecture 4.9] in the semi-inert case. \( \square \)
Example 10.4. Spherical 3-Cones

Consider the adjunction space $X_3$ of a cell attachment $W_3 \xrightarrow{f_3} X_2$ where $X_2$ is the spherical 2-cone given above, $W_3 = \bigvee_{j \in J_3} S_{n_j}$ is a finite-type wedge of spheres, and $f_3 = \bigvee_{j \in J_3} \alpha_j^{(3)}$. Let $R$ be a subring of $\mathbb{Q}$ containing $\frac{1}{6}$. Assume that $H_*(\Omega X_2; R) \cong ULX_2$ and that there exists a map $\sigma_{X_2}$ right inverse to $h_{X_2}$. Let $L_3 = (L_{X_2} \amalg \langle y_j^{(3)} \rangle_{j \in J_3}, d')$ where $d' y_j^{(3)} = h_{X_2}(\alpha_j^{(3)})$.

Assume $L_{X_2}/[L_{X_2}^W]$ is $R$-free and for any $p \in \bar{P}$, $[\bar{L}_{X_2}^W] \cap \bar{L}_{X_2}^{X_3} = 0$. By Example 10.3, $\text{gr}(L_{X_2}) \cong L_{X_2}^{X_3} \ltimes \mathbb{L}V^{(2)}$ for some free $R$-module $V^{(2)}$. It follows that for any $p \in \bar{P}$, $\text{gr}(\bar{L}_{X_2}) \cong \bar{L}_{X_2}^{X_3} \ltimes \hat{\mathbb{L}}V^{(2)}$. Then by Theorem G, for any $p \in \bar{P}$, $[\bar{L}_{X_2}^W]$ is a free Lie algebra. That is, $f_3$ is a free cell-attachment. Thus by Theorem D(i) we get that $H_*(\Omega X_3; R)$ is torsion-free and

$$\text{gr}(H_*(\Omega X_3; R)) \cong U(\bar{L}_{X_2}^{X_3} \ltimes \mathbb{L}V^{(3)}_1).$$

Furthermore since $\exists \sigma_{X_2}$ right inverse to $h_{X_2}$ then by Theorem F

$$H_*(\Omega X_3; \mathbb{Z}[P_{X_3}^{-1}]) \cong ULX_3$$

where $\text{gr}(L_{X_3}) \cong L_{X_3}^{X_3} \ltimes \mathbb{L}(HL_3)$ and localized away from $P_{X_3}$, $\Omega X_3 \in \prod S$.

If in addition $f_3$ is semi-inert then by part (ii) of Theorem D,

$$H_*(\Omega X_3; R) \cong U(\bar{L}_{X_2}^{X_3} \amalg \mathbb{L}K^{(3)}),$$

for some $K^{(3)} \subset F_1 H_*(\Omega X_3; R)$, and by Theorem F(iii), localized away from $P_{X_3}$ there exists a map $\sigma_{X_3}$ right inverse to $h_{X_3}$.

Example 10.5. Spherical $N$-cones

Let $W_k \xrightarrow{f_k} X_{k-1} \to X_k$ for $1 \leq k \leq N$ be a sequence of cell attachments (with adjunction space $X_k$) with $X_0 = \ast$, $W_k$ is a finite-type wedge of spheres and $X_N \cong X$. Assume $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. For each assume that $L_{X_{k-1}}/[L_{X_{k-1}}^W]$ is $R$-free and that for any $p \in \bar{P}$, $[\bar{L}_{X_{k-1}}^W] \cap \bar{L}_{X_{k-2}}^{X_{k-2}} = 0$. In addition for each assume that if $f_k$ is free then there exists a
Example 10.6. $W \to X \to Y$ where $X$ is coformal.

Let $W$ be a simply-connected finite-type wedge of spheres. Let $X$ be a simply-connected topological space such that $H_*(\Omega X; R)$ is torsion-free and has finite type. Let $R \subset \mathbb{Q}$ containing $\frac{1}{6}$ or $R = \mathbb{F}_p$ where $p > 3$. Call $X$ coformal if $X$ has a model (see Section 3.2) $UL(X) = U(L, 0)$ [NM78]. For example, when $R = \mathbb{Q}$, Neisendorfer and Miller [NM78] show that any compact $n$-connected $m$-dimensional manifold with $m \leq 3n + 1$, $n \geq 1$ and rank$(PH(M; \mathbb{Q})) \geq 2$ is coformal, where $P(\cdot)$ denotes the subspace of primitive elements.

Let $Y$ be the adjunction space of a cell attachment $W \to X$. For a model for $Y$ one can take $UL(Y) = U(L \amalg LV_1, d)$, where $dL = 0$ and $dV_1 \subset L$. Assume that $[dV_1] \subset L$ is a free Lie algebra. Since the differential in $L(X)$ is zero, by Lemma 1.1, $H_*(\Omega Y; R) \cong \text{gr}(H_*(\Omega Y; R))$.

By Theorems C(i) and D(i) if $L/[dV_1]$ is torsion-free then

$$H_*(\Omega Y; R) \cong U((HL(Y))_0 \ltimes \mathbb{L}(HL(Y)))_1$$

where $(HL(Y))_0 \cong L/[dV_1]$.

Example 10.7. $X$ is a finite-type CW-complex with only odd dimensional cells.

Let $X^{(n)}$ denote the $n$-skeleton of $X$. From the CW-structure of $X$, there is a sequence of cell attachments for $k \geq 1$.

$$W_{2k} \xrightarrow{f_{2k+1}} X^{(2k-1)} \to X^{(2k+1)}$$
where $W_{2k}$ is a finite wedge of $(2k)$-dimensional spheres, $X^{(2k+1)}$ is the adjunction space of the cell attachment and $X^{(1)} = \ast$. Assume $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Let $P_{X^{(n)}}$ be the set of implicit primes of $X^{(n)}$ (see Section 4.4). We will show by induction that

$$H_*(\Omega X^{(2k+1)}; \mathbb{Z}[P_{X^{(2k+1)}}^{-1}]) \cong UL_X^{(2k+1)}$$

where $L_X^{(2k+1)} \cong \mathbb{L}V^{(2k+1)}$ with $V^{(2k+1)}$ concentrated in even dimensions. In addition localized away from $P_{X^{(2k+1)}}$, $\Omega X^{(2k+1)} \in \prod S$ and there exists a map $\sigma_X^{(2k+1)}$ right inverse to $h_X^{(2k+1)}$.

For $k = 0$ these conditions are trivial. Assume they hold for $k-1$. Let $L = (L_X^{(2k-1)} \amalg \mathbb{L}K, d')$ where $K$ is a free $R$-module in dimension $2k$ corresponding to the spheres in $W_{2k}$. For degree reasons $L_{W_{2k}}^{(2k)} = d'(K) = 0$. So $f_{2k+1}$ is automatically free. Furthermore $L$ has zero differential so $H_* L = \mathbb{L}V^{(2k-1)} \amalg \mathbb{L}K$. Thus $f_{2k+1}$ is semi-inert. By Theorem F, $H_*(\Omega X^{(2k+1)}; \mathbb{Z}[P_{X^{(2k+1)}}^{-1}]) \cong UL_X^{(2k+1)}$, where

$$L_X^{(2k+1)} \cong L_X^{(2k-1)} \amalg \mathbb{L}K \cong L_X^{(2k-1)} \amalg \mathbb{L}K \cong \mathbb{L}(V^{(2k-1)} \oplus K).$$

Also by Theorem F, localized away from $P_{X^{(2k+1)}}$, $\Omega X^{(2k+1)} \in \prod S$, and there exists a map $\sigma_X^{(2k+1)}$ right inverse to $h_X^{(2k+1)}$.

Therefore by induction $H_*(\Omega X; \mathbb{Z}[P_X^{-1}]) \cong UL_X$, where $L_X \cong \mathbb{L}V$ and localized away from $P_X$, $\Omega X \in \prod S$. 

\[\square\]

## 10.2 Open questions

The results of this thesis naturally lead to the following open questions.

- Is there a dga extension $A$ which is free such that $\text{gr}(HA) \not\cong HA$ as algebras?

- If $H(A, d)$ is $R$-free and as algebras $H(A, d) \cong UL_0$ for some Lie algebra $L_0$ then is there a dga extension $(A \amalg TV, d)$ for which one cannot choose a Lie algebra $L_0$ $d'V \subset L_0$?
• If so, do any of these come from cell attachments, or have \( (\tilde{A}, \tilde{d}) = U(L, d) \)?

• Let \( R = \mathbb{Q} \) or \( \mathbb{F}_p \) where \( p > 3 \) and let \( (\tilde{A}, \tilde{d}) \) be a free dga extension of \( ((\tilde{A}, \tilde{d}), L_0) \). By Theorem A, as algebras \( H(\tilde{A}, \tilde{d}) \cong UL' \) where \( L' = (\mathbb{H}L_0) \otimes \mathbb{L}(\mathbb{H}L_1) \). Let \( A = \tilde{A} \amalg TV \) be a dga extension of \( \tilde{A} \) with induced map \( d' : V \to L' \) such that \( [d'V] \cap (\mathbb{H}L_0) = 0 \). By Theorem G, \( A \) is a free dgL extension. Do the relations which are obstructions to the semi-inertness of \( A \) necessarily come from the anti-commutativity and Jacobi relations (as in Examples 2.13 and 2.14)?

• Is there a finite CW-complex \( X \) such that \( H_*(\Omega X; R) \) is \( R \)-free and \( H_*(\Omega X; R) \cong UL_X \) as algebras but there does not exist a map \( \sigma_X \) right inverse to \( h_X \) even after localizing away from finitely many primes?

• One of the forms of the semi-inert condition, namely that \( \text{gr}_1(HA) \) is a free \( \text{gr}_0(HA) \)-bimodule, can be applied to more general differential graded algebras than those studied in this thesis. J.-M. Lemaire has suggested that the condition that the two-sided ideal

\[
(dV_1) \subset \tilde{A} \text{ is a free } \tilde{A}\text{-module}
\]  

(10.2)

be taken as the free condition for more general dga's. (For the dga's in this thesis, (10.2) follows from the free condition by Lemma 5.7. Can Theorems A and B be generalized by using these conditions?)
Bibliography


