

Research Statement

My most significant recent work is in the field of applied topology. While the early development of topology was firmly rooted in applications, many of its later advances have been driven by purely mathematical motivations. More recently there is a growing awareness that some of its powerful tools may be just what is needed to help address outstanding problems in diverse modern applications. I have also worked on problems in algebraic topology and algebra.

What is applied topology?

Algebraic topology is particularly useful for connecting local and global properties of mathematical objects. Often the local description is easy to understand, but one is particularly interested in global qualities that depend on how the local information fits together. Many powerful tools in algebraic topology have been developed for this purpose. Applied topology seeks to adapt these tools so that they may be used for the same purpose in an applied setting.

For example, data is preprocessed so that it may be considered to be a finite set of points on Euclidean space, a Riemannian manifold, an algebraic variety or a stratified space. The geometry of this space and insights from the application are used to construct a filtered simplicial complex from this data. Persistent homology provides an efficient algorithm for calculating and describing how the topology of this simplicial complex changes as one moves along the filtration. One then uses this summary to make inferences on the data.

Statistical topological data analysis

An impediment to the broader use of applied topology has been its incompatibility with statistical methods. Persistent homology has been proven to be stable, and heuristically, similar data produces similar topological summaries (barcodes and persistence diagrams), but this has been hard to quantify. For example, one would like to be able to calculate the average of a number of barcodes.

I have developed a new topological summary, the persistence landscape [7], which is more amenable to statistical analysis. Mathematically I have embedded the space of the barcodes into a larger L^p space and shown that the induced metric is topologically equivalent to the previously studied Wasserstein distance. In addition I have proved that this construction is stable. Furthermore, since this space is a separable Banach space, if we think of these summaries as random variables, they obey a Strong Law of Large Numbers and a Central Limit Theorem. Applying a linear functional we get a real valued random variable for which we can calculate confidence intervals and test hypotheses.

A student of mine, Brian Feister, has combined these techniques with discrete differential geometry to compare the outer cortical surfaces of autistic and control subjects from MRI data. Another student of mine, Luo Yixi, is using these ideas to study the persistent homology of random polynomials.

Presently I am working with collaborators on a number of follow up projects to this work. With Giseon Heo, Nikoleta Kovacev-Nikolic and Dragan Nikolic, I have used the persistence

landscape to analyze crystallographic protein structures [17]. With Pawel Dlotko and Ryan Lewis we have constructed and implemented efficient algorithms for calculating persistence landscapes [11].

With Peter Kim, Moo Chung, Zhi-Ming Luo and Gunnar Carlsson, I have previously combined topology and statistics in parametric situations [13], for (nonparametric) functions on Riemannian manifolds [8], and to brain imaging [16].

Foundations of applied topology

Algebraic topology has frequently undergone a process in which previous results were redeveloped from a more abstract point of view. This has both clarified key ideas and proofs and allowed previous results to be vastly generalized and applied in new settings. I have applied this process to some parts of applied topology.

With Jonathan A. Scott, I have redeveloped persistent homology from a categorical point of view [14]. The main objects of study are diagrams indexed by the poset of real numbers. Such diagrams have an interleaving distance, which we've shown generalizes the previously studied bottleneck distance. As a consequence we are able to greatly generalize previous stability results for persistence, extended persistence, and kernel, image and cokernel persistence. We describe a category of interleavings of such diagrams, and show that if the target category is abelian, so is this category.

Together with Vin de Silva, we have written up a second and third paper [10, 9] in which we consider more general indexing categories. Our general theorems specialize to useful results in previously studied settings such as multidimensional persistence and dynamical systems, and also suggest the right constructions in new settings, such as recent work on categorical Reeb graphs. Abstractly, we can apply persistence whenever we have a monoidal adjunction that allows us to measure homotopies. We also hope that these results will enable advances in applied topology to find applications in pure mathematics.

Topology and physics

Hard spheres are among the most well studied models of matter. A number of papers in statistical mechanics have explored the hypothesis that phase transitions are due to changes in the topology of the underlying configuration space. Surprisingly, the answers to a number of basic topological questions on the configuration space of hard spheres are unknown.

In work with Yuliy Baryshnikov and Matthew Kahle [1], we developed a Morse theory for studying the homotopy theory of this configuration space. The critical points and critical submanifolds in this theory correspond to mechanically balanced configurations of spheres. As an application, we find the precise threshold radius for such a configuration space to be homotopy equivalent to the configuration space of points. We are working on a followup paper analyzing the non-degeneracy and index of critical points and investigating the asymptotic properties of this configuration space.

I am also working with Rien van de Weijgaert and Pratyush Pranav on applying my persistence landscapes to cosmological data.

Topology and computer science

After many decades of exponential growth in the speed of single-thread execution, current advances in computation are more driven by increases in parallelism. However, the analysis of truly concurrent programs is very difficult. In a concurrent environment, the processes can access shared resources in various orders, which can result in very different executions.

Mathematically, concurrent programs can be described by a state space together with non-reversible execution paths. One approach to concurrency is a directed version of homotopy theory. In [5], I generalized the van Kampen theorem to these directed spaces. In [4], I studied pushouts in an undercategory of partially ordered spaces. More recently [6], I used work of Jacob Lurie on higher category theory to give a model for concurrency whose state space and execution spaces are simplicial sets.

This topic is also of interest in pure mathematics. For example, my work with Kris Worytkiewicz on a model category for local partial-ordered spaces [15] has been used in studying paths in stratified spaces.

Algebra and algebraic topology

In an algebra paper with Leah Gold [12], given a finite simple vertex-weighted graph, we construct a graded associative (non-commutative) algebra, whose generators correspond to vertices and whose ideal of relations has generators that are graded commutators corresponding to edges. We show that the Hilbert series of this algebra is the inverse of the clique polynomial of the graph. Using this result it is easy to recognize if the ideal is inert, from which strong results on the algebra follow. Noncommutative Gröbner bases play an important role in our proof. Our result has an interesting application to toric topology. Indeed, this algebra arises naturally from a partial product of spheres, which is a special case of a generalized moment angle complex, and we apply our result to the loop space homology of this space.

In earlier work [2, 3] I give new results on the cell attachment problem, which was perhaps first studied by J.H.C. Whitehead around 1940: If one attaches one or more cells to a topological space, what is the effect on the homology of the loop space, and on the homotopy-type? Much of this work involves a related question for differential graded Lie algebras.

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