Persistent homology of functions

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Persistent homology describes the homological features which persist as a single parameter changes.

Here, we take this parameter to be a threshold on the values of a function.

We consider the homology of the lower excursion sets (sublevel sets) of this function.
For example:

\[ y = f(x) \]
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\[ f^{-1}(-\infty, 1] \]
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\[ y = f(x) \]

\[ f^{-1}(-\infty, 2] \]
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\[ y = f(x) \]

\[ f^{-1}(-\infty, 3] \]
For example:

\[ y = f(x) \]

\[ f^{-1}(-\infty, 4] \]
Persistent homology of functions

On $\mathbb{R}$

Bottleneck Distance

Persistence Diagram

For example:

$$y = f(x)$$

birth

dead
Let $\mathcal{M}$ be a manifold and let $f : \mathcal{M} \rightarrow \mathbb{R}$.

This function gives an increasing filtration of $\mathcal{M}$ by sublevel sets

$$\mathcal{M}_{f \leq r} = \{x \in \mathcal{M} \mid f(x) \leq r\}.$$ 

[This induces an increasing filtration on $C_*(\mathcal{M})$.]
Let \( M \) be a manifold and let \( f : M \to \mathbb{R} \).

This function gives an increasing filtration of \( M \) by sublevel sets

\[
M_{f \leq r} = \{ x \in M \mid f(x) \leq r \}.
\]

[This induces an increasing filtration on \( C_*(M) \).]

For \( s \leq t \), the inclusion \( i^t_s : M_{f \leq s} \to M_{f \leq t} \) induces

\[
H_*(i^b_a) : H_k(M_{f \leq s}) \to H_k(M_{f \leq t}),
\]

whose image is the **persistent homology** from \( s \) to \( t \) of \( f \).
Let \( f : \mathcal{M} \rightarrow \mathbb{R} \) be a smooth function.
The points at which the differential vanishes are called critical points. The value of \( f \) at a critical point is a critical value.
We assume that the matrix of second partial derivatives (the Hessian) is non-singular at each critical point.
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For convenience, assume distinct critical values: $t_0 < t_1 < \cdots < t_k$. The index $p$ associated to $t_j$ is the number of negative eigenvalues of the Hessian.
A brief introduction to Morse theory

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Morse theory $\Rightarrow$ Varying $t$, $\mathcal{M}_{f \leq t}$ only changes (up to homotopy) if $t$ is a critical value. $\mathcal{M}_{f \leq t_j} \simeq \mathcal{M}_{f \leq t_{j-1}} + \text{a } p\text{-dimensional cell}$. So $\dim H_p$ increases by 1 (birth) or $\dim H_{p-1}$ decreases by 1 (death).
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Persistent homology $\Rightarrow$ pair the index $p$ birth critical values and the index $p + 1$ death critical values $\Rightarrow$ Persistence Diagram.
A Function

For example
Sublevel sets and $H_1$
Persistence Diagram of the function
The Bottleneck Distance

There is a useful metric on the space of Persistence Diagrams:

\[ d_B(D_p(f), D_p(g)) = \inf_{\eta} \sup_{x} \|x - \eta(x)\|_\infty, \]

where

- \( D_p(f) \) and \( D_p(g) \) are Persistence Diagrams of functions \( f \) and \( g \), respectively.
- The infimum is taken over all \( \eta \) that are upper semi-continuous and \( \eta(0) = 0 \).
- The supremum is taken over all \( x \) in the domain of \( f \) and \( g \).

This metric measures the closest correspondence between the persistence features of two functions, with smaller values indicating a closer match.
The following fundamental result bounds the bottleneck distance for persistence diagrams with the supremum norm.

**Theorem (Cohen-Steiner, Edelsbrunner, Harer)**

\[ d_B(D_p(f), D_p(g)) \leq \|f - g\|_\infty \]